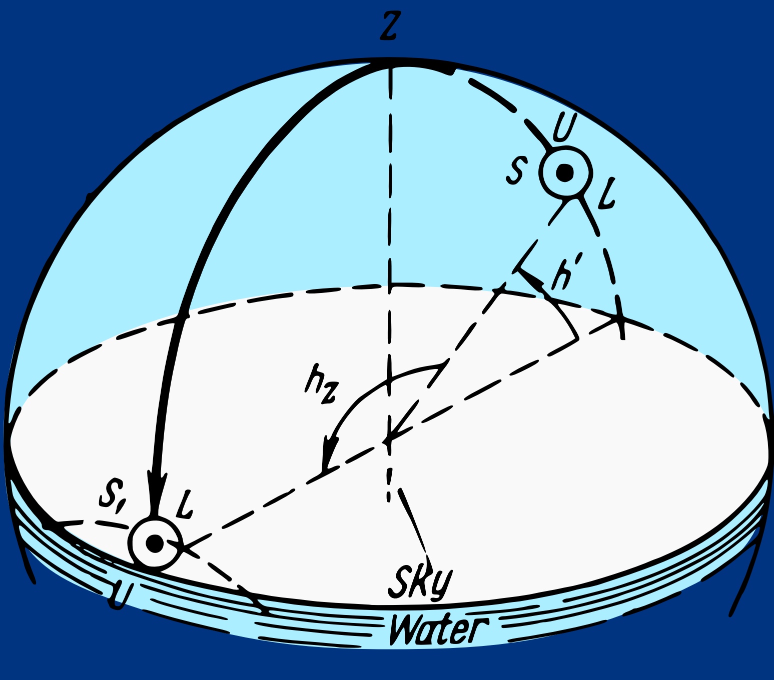
B. Krasavtsev, B. Khlyustin

NAUTICAL ASTRONOMY



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ЛЕНИНГРАД

B. KRASAVTSEV, B. KHLYUSTIN

NAUTICAL ASTRONOMY

TRANSLATED FROM THE RUSSIAN
BY
GEORGE YANKOVSKY

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TRANSLITERATION OF THE RUSSIAN ALPHABET

Russian	Transliteration	Russian	Transliteration
А а	a	Р р	r
Б б	b	С с	s
В в	v	Т т	t
Г г	g	У у	u
Д д	d	Ф ф	f
Е е	e	Х х	kh
Ж ж	zh	Ц ц	ts
З з	z	Ч ч	ch
И и	i	Ш ш	sh
К к	k	Щ щ	shch
Л л	l	Ы ы	y
М м	m	Э э	e
Н н	n	Ю ю	yu
О о	o	Я я	ya
П п	p		

На английском языке

CONTENTS

PART THREE. METHODS OF NAUTICAL ASTRONOMY

ABBREVIATIONS*

Aberration *aber*
Additional *add*
Altitude h
- apparent—h_{ap}
- computed—h_{com}
- corrected—h_{corr}
- measured—h_{meas}
- meridian—H
- observed—h_{ob}
- reduced—h_{red}
- true—h_{tr}

Angle
- parallactic—q
- wedge—γ

Argument *Arg*
Aries
- first point of Aries ♈

Astronomical unit AU
Atmospheric pressure B
Average av.
Azimuth A

Bearing
- compass—CB
- true—TB

Celestial body *body*
Comparison *com*
Comparison of watch and chronometer *ch-wat*
Correction Δ
- bearing—ΔB
- chronometer—u_{ch}
- compass—ΔK

---

* Where the same letters have been used as abbreviations of different words, their meanings can always be readily determined from the context.

Correction (*cont*)
    gyrocompass—ΔGC
    index—of sextant i
    instrument—s
    orthodromic—ψ
    watch—u_{wat}
    zero-point—Δ
Course made good CMG

Days d
Dead reckoning D.R.
Declination δ
Departure *Dep.*
Difference of latitude l
Difference of longitude DLo
Dip of horizon d
Distance of visible horizon D_h

Earth ⊕
East E
Eccentricity of earth's orbit e
Elongation *elong*
Epact Ep
Equation of time η
Equinox
    vernal— ♈
    autumnal— ♎
Error
    bisector—ε_b
    mean square—ε
    position—ε_{loc}
    random—δ_i
    systematic—Δ

Greenwich Gr (subscript: *gr*)

Height of eye e
Hour h
Hour angle t
    Greenwich—t_{gr}
    limiting—t_{lim}
    local—t_{loc}
        —of Aries $t^{♈}$
        —of vernal equinox point $t^{♈}$
    set—t_{set}
    sidereal—τ

Latitude
 ecliptic—β
 geographic—φ
 mean—φ_m
Leeway C
Libra
 first point of ♎
Longitude λ
 computed—λ_c
 ecliptic—λ
 —of perihelion of earth's orbit ω
 —of sun L
Lower *low*

Magnitude (stellar) *Mag.*
Mean of instants T_{av}
Mean of sextant reading sr_{av}
Minute m (superscript: min)
Moon ☾
 age of—$B_{☾}$
Most favourable *mf*

Nadir *n*
Node
 ascending—☊
 descending—☋
North N
North pole P_N
Number
 golden—*R*
 M—M
Nutation *nut*

Obliquity of the ecliptic ε
Observed *ob*

Parallax *p*
 annual—of a star π
 diurnal—*p*
 equatorial horizontal—p_0
Perihelion Per
Planet *pl*
Polar distance Δ
Precession *pr*

Quasi-difference Δ

Radius R
  —of the earth R_E
Rate
  daily—at temperature $t\omega_t$
  daily—at $t_0 = +18°$ C ω_0
  daily—of chronometer ω
Reading
  log—*lr*
    —of index correction *oi*
  sextant—*sr*
  watch—T_{wat}
Reduced *red*
Refraction
  mean astronomical—ρ_C
  terrestrial—ρ_1
  coefficient of—K
  true—ρ
Reverse compass bearing—*RCB*
Right ascension α
Rising (of a celestial body) *r*

Second s (superscript: sec)
Setting (of a celestial body) *s*
Solstice
  summer—*l*
  winter—*l'*
South. S
South pole P_S
Star ✳
Sun ☉
Sextant reading—*sr*

Tabulated tab (and T)
Temperature coefficient
  linear—α
  quadratic—β
Time
  apparent—t_{ap}
  chronometer—T_{ch}
  legal—T_{leg}
  local mean—T_{loc}
  local sidereal—S_{loc}
  mean or civil—T
  mean—on Greenwich meridian T_{gr}

Time (*cont*)
Moscow — T_{Mos}
—of moonrise T_r
—of moonset T_s
—of upper transit T_{tr}
ship — T_{sh}

sidereal S
sidereal on Greenwich meridian S_{gr}
starting T_{st}
tabulated — T_T
visible T_{vis}
zone T_z
Total *tot*
Termination *term*
Transit *tr*
lower — L.T.
upper — U.T.
True *tr*
True course *TC* (or *K*)
Twilight *twi*

Ursa Major UMa

Variation (chronometer rate) Δ
Venus ♀

West W

Zenith Z
Zenith distance z
meridian — Z
—of Greenwich Z_{gr}
Zone description ZD

INTRODUCTION

Nautical astronomy is a science that combines two areas of knowledge: navigation and astronomy.

All the sciences that contribute to navigation have one basic aim — to ensure safe, rapid and economical transportation from one point to another by sea. It is here that astronomy is linked with navigation.

Navigation, as we know, deals not only with the solution of general geographical problems that arise when travelling by sea (the shape and dimensions of the earth, cartography and its applications, etc.) but also with determining positions and instrument corrections from observations of terrestrial objects like beacons, signs and so forth.

The solution of these very same problems via the observation of celestial bodies is considered in nautical astronomy. Thus, here, celestial bodies are the objects of observation. Hence, nautical astronomy is intimately bound up with astronomy as such.

Let us take a look at the subject of astronomy, its subdivisions, and the place that is occupied by nautical astronomy.

Strictly speaking, *astronomy* is the science of celestial bodies. For a long time, astronomy dealt only with the movements of celestial bodies. However, at the present time astronomers investigate all observable mechanical, physical, chemical and biological processes occurring in the universe, so that we must define astronomy as the *science of the structure and development of the universe*. This multiplicity of objects of study has led to the subdivision of astronomy into a number of specialized fields.

Spherical astronomy studies methods of constructing coordinate systems of astronomical bodies on the surface of an auxiliary sphere, the variation of these coordinates due to a variety of causes, and also the principles of measuring time.

Practical astronomy studies methods of determining the coordinates of celestial bodies from observations, the methods of obtaining from astronomical observations geographical coordinates of the observer, true directions on the earth's surface, exact time, and so on. Practical astronomy also deals with instruments used for observations and with methods of operating them.

Depending upon the aims of such observations and also on the techniques and facilities employed, practical astronomy is subdivided into four independent areas:

(a) *observatory astronomy* (sometimes called *fundamental astronomy*), in which the observations are carried out by large stationary observatory instruments, by special high-precision methods, while the results of observations mainly serve basic scientific purposes and are used as starting data for the general solution of theoretical and practical problems;

(b) *geodetic astronomy* (or *field astronomy*), in which observations are carried out with precision instruments moved from point to point on land (under field conditions) and serve mainly the purely practical aims of determining the coordinates of a point and true directions;

(c) *nautical astronomy*, in which these very same problems are solved at sea with cruder instruments and with less accuracy;

(d) *aviation astronomy*, in which the very same problems are solved in application to aircraft transport; the accuracy is still lower than in nautical astronomy.

Celestial mechanics applies the laws of mechanics and the law of universal gravitation to the study of the true motions of astronomical bodies in space, their masses and shapes. This branch is closely tied in with theoretical astronomy.

Theoretical astronomy is engaged in studying methods of determining the apparent motions and positions of celestial bodies on the sphere on the basis of their actual motions in space (computation of ephemerides), and, conversely, methods of determining the true movements from the apparent positions of celestial bodies on the sphere and in determined time (computation of orbits).

Astrophysics investigates the physical and chemical processes that occur on celestial bodies and in the universe at large.

Stellar astronomy studies the distribution of stars (and matter) in space, their classification, and so forth; it likewise makes a study of all the processes that occur in stellar worlds.

Radio astronomy studies the radio emission of celestial bodies and the requisite techniques and facilities. This branch appeared after the Second World War and has been expanding rapidly.

Astrobiology deals with problems of life, the possibility of life and the conditions of life on celestial bodies, the planets of the solar system for example.

Cosmogony considers problems of the origin and development of celestial bodies, their systems and also of other accumulations of matter in space.

General (or *descriptive*) *astronomy* gives brief outlines of all the branches of astronomy for the purpose of general surveys of methods and results and for educational purposes.

All the above-mentioned branches of astronomy are closely related as to objects of observation and the techniques employed.

Nautical astronomy employs the general methods of *practical* and *spherical astronomy*, but the results of observations and computations serve the purposes of navigation. For this reason, the demands imposed on nautical astronomical observations and their treatment are quite different from those relating to "land" observations, namely: (1) the observations have to be made quickly and with sufficient accuracy from the deck of a moving ship; (2) the computations must be simple and handled rapidly and accurately by one person. These requirements have led to the construction of specialized instruments, methods and manuals that differ considerably from the "land" type. At present, the methods of nautical astronomy are so simplified and adapted to navigation that nautical astronomy is sometimes regarded as a component part of the former (celestial navigation)—a kind of astronomical check on the plotting of courses. However, this view is somewhat limited since nautical astronomy handles certain other navigational needs as well: determination of the natural lighting of the horizon by celestial bodies, the time service, and others. Therefore, the *subject of nautical astronomy may be defined as the application of astronomical knowledge to the needs of navigation.*

The basic problems handled by nautical astronomy are: determining positions at sea from observations of astronomical bodies and determining true directions to obtain compass corrections.

Astronomical determinations at sea are widely employed both in routine sailing in the open sea and, especially, in areas poorly equipped with radio facilities. At the present time, compass corrections in the open sea can still be made only by methods of nautical astronomy.

The methods of nautical astronomy are in certain respects inferior to radio techniques, but the former have a number of advantages, which include complete independence, secrecy of determinations, relatively high accuracy with simple and cheap apparatus, which has the added advantage of not requiring sources of electricity. At the same time, astronomical methods have a number of definite drawbacks: they are hampered by bad visibility and time-consuming computations, which sometimes restrict their application.

The course of nautical astronomy includes the following basic divisions:

(1) the principles of spherical astronomy, which studies the coordinates of celestial bodies and their variations and timekeeping;

(2) instruments of nautical astronomy, which include the sextant, the chronometer, the celestial globe and dipmeter;

(3) methods of determining compass corrections;

(4) methods of determining the position of a ship at sea or its coordinates.

In the U.S.S.R., the following tables are used for solving astronomical problems at sea: The Nautical Astronomical Almanac (MAE), Nautical Tables (MT-63) and special tables "TBA-57", "BAC-58", "Azimuth Tables of Celestial Bodies", and others.

PART ONE

THE PRINCIPLES OF SPHERICAL ASTRONOMY AND THE NAUTICAL ASTRONOMICAL ALMANAC (MAE)

CHAPTER 1

THE SPHERICAL COORDINATES OF CELESTIAL BODIES

SEC. 1. THE CELESTIAL SPHERE

When solving astronomical problems, it is frequently necessary to establish exact mathematical relationships between various directions in space and to analyze their variations.

The simplest way to obtain such relationships is by means of a so-called auxiliary sphere (see Appendix II). The auxiliary sphere is introduced for passing from directions and angles *in space* to points, lines and triangles *on the surface of a sphere*, which permits us to utilize the formulas of spherical trigonometry and thus to simplify the solution of such problems.

In astronomy, wide use is made of a special auxiliary sphere (called the **celestial sphere**) with systems of spherical coordinates constructed on it and with the indicated positions of celestial bodies. This auxiliary sphere is a purely mathematical construction and should in no way be identified with the actually observed vault of the heavens.

The significance of the celestial sphere is not confined to the solution of problems on finding angles in space; the sphere likewise gives a pictorial view of various movements of celestial bodies.

The centre of the celestial sphere, as an auxiliary mathematical construction, can obviously be located at any arbitrary point of space; however, its construction is much more pictorial and convenient if the centre is assumed at certain specific points like, say, the eye of the observer, the centre of the earth, or the centre of the solar system. It will vary accordingly. Incidentally, it is very simple to pass from one representation to another, for they actually represent the *same auxiliary sphere*. Representation of the celestial sphere with centre at the centre of the earth is inherited from the ancients and their view of the world system.

Let us first consider a construction of the celestial sphere on the assumption that its centre coincides with the observer's eye on the earth's surface, and then transfer it to an arbitrary point. In Fig. 1 we have the earth, P_NP_S is the earth's axis, points P_N and P_S are the north and south geographic poles, and *eq* is the earth's equator.

An observer is located at A, the geographic latitude of which is φ = arc eA. If we take the earth to be a sphere, then the radius AC will represent the plumb-line direction for the observer; the plane H, tangent to the earth's surface at A and perpendicular to the plumb line, will represent the plane of the true or mathematical horizon of the observer; the line NS, which lies in the plane of the geographic meridian, and the line EW, which is perpendicular to it,

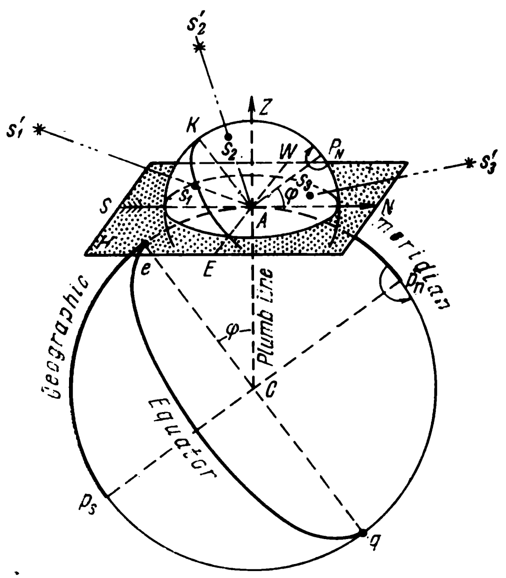

Fig. 1

determine the main directions or the points of the compass for this observer. Let us also assume that the straight lines As_1'; As_2'; As_3' . . . represent directions to celestial bodies s_1'; s_2'; s_3' . . . located in space at different distances from the observer.

Let us now construct a sphere of arbitrary radius with centre at A and draw, through its centre, lines and planes parallel to the corresponding lines on the earth. From the figure it will be seen that, where the planes of the true horizon, the geographic meridian and the plane parallel to the equator intersect the sphere, they form great circles. In this way the celestial sphere produces a geometric representation of the sky.

As constructed, the line P_NA, which is parallel to the earth's axis, forms with the plane H an angle equal to the geographic latitude φ. On the surface of the sphere, we can obtain the points s_1,

s_2, s_3 and so forth, which are projections on this sphere of the apparent directions from the centre of the sphere to celestial bodies. If such a sphere is put into motion in the direction from E to W (as indicated by the arrow) with the velocity of the earth's rotation, then the diurnal motion of the celestial bodies for a given observer will be completely reproduced.

However, as we have already pointed out, to construct a celestial sphere and solve problems there is no necessity to put the centre

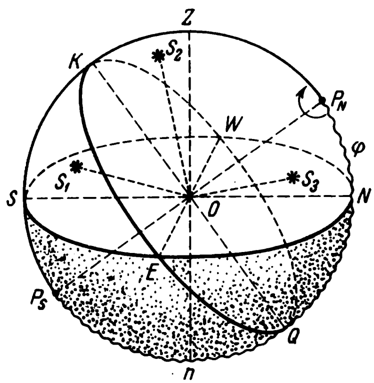

Fig. 2

in the eye of the observer. All constructions may be carried out at an arbitrary point O of space. Through O (Fig. 2) draw lines and planes *parallel* to the corresponding directions and planes on the earth. It is clear that the points and lines obtained on the sphere will have the same configuration as those seen by an observer at A (see Fig. 1). *This auxiliary sphere of arbitrary radius with centre at an arbitrary point of space and with indicated basic lines and locations of celestial bodies is called a celestial sphere.**

In Fig. 2, let the diameter ZOn be drawn parallel to the plumb line AC (see Fig. 1); its intersection with the sphere yields two points: the **zenith** Z, which represents the uppermost point of the sphere (above the observer's head), and the **nadir** n, a point opposite the zenith.

If the sphere is cut by a plane parallel to the true horizon, we get (on the surface of the sphere) a great circle NESW, which is called the **celestial horizon**. This circle divides the sphere into two parts:

* It should be borne in mind that spherical astronomy deals with a *conventional representation of the sphere* (Fig. 2) which does not correspond exactly to its perspective but is more convenient.

the visible hemisphere, that *above the horizon* with the zenith, and that located *below the horizon*.

If the sphere is cut by a plane parallel to the geographic meridian of the observer, we get, on the surface of the sphere, a great circle $P_N n P_S Z$, called the **observer's meridian** or the **local meridian.**

The diameter $P_N P_S$, parallel to the earth's axis, is called the **celestial axis** and is an imaginary axis round which occurs the apparent diurnal rotation of the sphere. The points of intersection of the celestial axis with the celestial sphere are called the **celestial poles,** the closest to the north pole of the earth being called the north pole P_N, and the closest to the south pole of the earth, the south pole P_S.

During the diurnal rotation of the celestial sphere, the celestial poles remain stationary. The celestial pole located in the part above the horizon is termed the *elevated* pole, that below the horizon, the *depressed* pole.

The intersection of the plane of the true horizon with the plane of the observer's meridian defines the **noon line** NS, whose intersection with the sphere yields the points N and S.

Intersection of the plane parallel to that of the earth's equator *eq* with the sphere yields the great circle KQ, which is known as the **celestial equator**. Its plane is perpendicular to the celestial axis $P_N P_S$. The celestial equator divides the sphere into two parts: *northern* and *southern* hemispheres, in accord with the names of the poles.

Intersection of the plane of the celestial equator with the plane of the true horizon defines the line EW, which is perpendicular to the upper meridian. Intersection of the celestial equator with the celestial horizon yields points E and W on the sphere.

The observer's meridian divides the sphere into two parts: *eastern* (E) and *western* (W).

The directions NS and EW divide the plane of the true horizon into four quadrants: NE, SE, SW and NW.

From Figs. 1 and 2 it will be seen that for northern geographic latitude of the position of the observer, the north pole of the celestial sphere P_N will be the elevated pole; and for an observer located in a southern latitude, P_S will be the elevated pole, so that the *name of the elevated pole is always in accord with the name of the geographic latitude of the observer*.

From the construction of the sphere it will be seen that no matter which pole is elevated (N or S), the closest point to P_N on the horizon will be N, and the closest point to P_S will be S. It should also be noted that the celestial axis (points P_N and P_S) divides the observer's meridian into two parts: the *upper branch* $P_N Z P_S$, on which the zenith point is located, and the *lower branch* $P_N n P_S$ with the nadir (the wavy line in Fig. 2).

If through the centre of the sphere we draw lines parallel to directions towards celestial bodies, we get on its surface the so-called *apparent places* of the bodies s_1, s_2, s_3, etc., which from now on will simply be called celestial bodies.

Let us now introduce systems of auxiliary circles.

The great circles on the celestial sphere (Fig. 3), whose planes pass through the plumb line, are called **vertical circles**. Each vertical circle passes through the points Z and n, and the plane of any vertical circle is perpendicular to the plane of the true horizon.

The vertical circle that passes through the place of a given celestial body on the sphere is called the *vertical circle of this body*. When

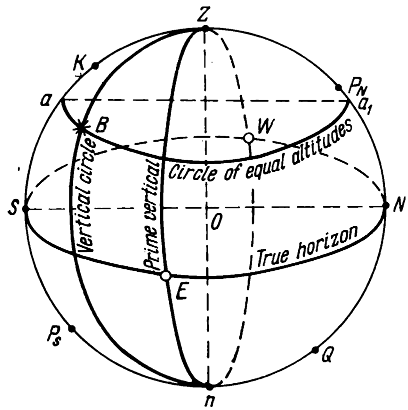

Fig. 3

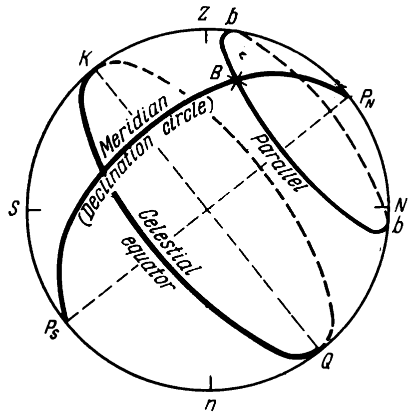

Fig. 4

using this term, however, we shall mean not the entire vertical circle (360°) but only that half of it (from Z to n) which includes the celestial body; thus, the vertical circle of some body B is the arc ZBn.

The vertical circle that passes through the points E and W is called the **prime vertical**, which is divided into east and west parts by the line Zn.

Small circles on the sphere with planes parallel to the celestial horizon are called **parallels of altitude**: that one which passes through the place of a given celestial body on the sphere is called the *parallel of altitude of the given body* (aa_1 in Fig. 3).

The great circles of the sphere whose planes pass through the celestial axis are called **celestial meridians** or **declination circles** (Fig. 4). Thus, every meridian passes through both poles, P_NP_S, and the plane of every meridian is perpendicular to the plane of the celestial equator KQ.

The meridian passing through the place of a given celestial body on the sphere from pole to pole and passing through the body is called the *meridian* or *declination circle of the body.* Thus, the meridian of some body B is the arc $P_N B P_S$.

The small circles of the sphere whose planes are parallel to the celestial equator are called *parallels of declination* (by analogy with terrestrial parallels of latitude). The one which passes through the place of a given celestial body on the sphere is called the *parallel of declination of that body* (sometimes called daily parallels, since celestial bodies describe parallels in their daily motion: bb_1 in Fig. 4).

Of the infinitude of meridians on the sphere, one is of special significance: this is the meridian that passes through the points Z and n and is called the observer's meridian.

At the same time, this meridian is the *vertical circle that passes through the celestial poles* and is called the *principal vertical circle.* Since this particular meridian (and vertical circle) occupies for the given observer a very definite and invariable position, it will serve as the *basic meridian and vertical circle* for coordination of celestial bodies on the sphere.

SEC. 2. COORDINATES OF CELESTIAL BODIES ON THE CELESTIAL SPHERE

In geography and navigation, the coordinates of various points of the earth's surface are found from their relationship to two mutually perpendicular great circles—the equator and the Greenwich meridian, which occupy very definite positions on the surface of the earth. The same method is employed for developing coordinate systems of celestial bodies on the celestial sphere: systems are chosen of two mutually perpendicular great circles that occupy very definite positions on the celestial sphere. Such systems of circles are:

(1) the celestial horizon and the principal vertical circle (observer's meridian);

(2) the celestial equator and the observer's meridian;

(3) the celestial equator and the meridian that passes through a definite point of the sphere, the first point of Aries (♈).

These circles serve as the basis for three systems of coordinates: the **horizon system and two equatorial systems.**

Spherical astronomy makes use of two other systems of coordinates: an ecliptic system with the ecliptic as the basic circle, and a galactic system with the basic circle close to the centre of the Milky Way. However, these systems are not used in nautical astronomy. The ecliptic system will be briefly discussed in Sec. 14.

I. HORIZON COORDINATE SYSTEM

In this system, the principal direction relative to which the construction is carried out is the direction of the plumb line; the basic circles are the celestial horizon and the principal vertical circle (observer's meridian). The place of any point of the sphere is determined relative to these circles by two horizon coordinates: **azimuth** and **altitude**.

(1) *The azimuth* (A) *of a celestial body is the arc of the celestial horizon which lies between the observer's meridian and the vertical circle of the body.*

This arc (for instance ND in Fig. 5) measures the corresponding central angle A and, hence, also the spherical angle A at the zenith

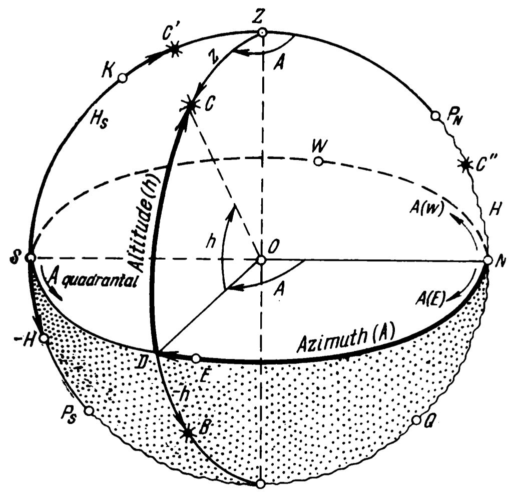

Fig. 5

Z, so that either of these angles may be called the azimuth of the body C.

For this reason, the *spherical angle at the zenith between the observer's meridian and the vertical circle of the celestial body is also called the azimuth.*

There are several systems of reckoning azimuth, depending on the starting point, the direction and the limits used for reckoning. Nautical astronomy makes use of three systems of reckoning azimuth: *semicircular* (0°-180°), *circular* (0°-360°), and *quadrantal* (0°-90°).

(a) In *semicircular* reckoning, the azimuth (azimuth angle) is measured by the arc of the celestial horizon from the lower branch

of the meridian (points N or S) in the direction of E or W to the vertical circle of the celestial body from 0° to 180°. Since in the intersection of the horizon on the lower branch of the meridian we have a point of the compass of the same name as the latitude, the *first letter of the azimuth designation in semicircular reckoning is always the same as the latitude name of the observer*. The second letter of the designation depends on the half of the sphere (east or west) in which the celestial body is located. To take an example, a celestial body C is written in the semicircular system as follows (Fig. 5): A = N 105°E. If the observer is situated in a southern latitude, the azimuth of a body will be written as follows: A = S 105°E, and so forth. The semicircular system of reckoning azimuth is used in solving spherical triangles by means of formulas and tables of logarithms, and also with the aid of special tables or instruments.

(b) In *circular* reckoning, the azimuth is measured by the arc of the celestial horizon from the point N in an easterly direction to the vertical circle of the celestial body, reckoning from 0° to 360°. For instance, A = 105°. As may be seen, this computation of azimuths coincides with that of the true bearings in navigation. Circular computation is used for determining compass corrections.

(c) In the *quadrantal* system of computing, azimuth is measured by the arc of the horizon from the point N or S towards E or W to the vertical circle of the celestial body, reckoning from 0° to 90°, similar to the quadrantal reckoning of compass points in navigation. For example, A = 75°SE (Fig. 5). Quadrantal computation is used in one of the formulas of the line of position method. Sometimes *amplitude* is used: the arc from E or W to the vertical circle of the celestial body.

Nautical astronomy requires frequent conversion from one system of calculating azimuth to another, and so it is necessary to learn, to convert azimuths rapidly and accurately. By way of illustration the azimuth of celestial bodies are written as follows in the three systems:

Semicircular		Circular		Quadrantal
1. N 118°W	=	242°	=	62°SW
2. S 145°W	=	325°	=	35°NW
3. S 95°E	=	85°	=	85°NE

(2) *The altitude* (h) *of a celestial body is the arc of* its *vertical circle from the celestial horizon to the place of the body on the sphere*. This arc measures the central angle h; for this reason, the *altitude is also the vertical angle with the centre of the sphere between the plane of the true horizon and the direction to the body.*

For example, the altitude of a celestial body C is the arc DC or the central angle DOC. Altitude is written thus: $h = 47°$.

If the body is above the celestial horizon, then its altitude is considered positive (C); but if it is located below the horizon (B), then its altitude is considered negative.

From the definition it follows that no celestial body can have an altitude numerically greater than 90°. The zenith point has an altitude of +90°, the nadir point, —90°. The altitude of any point on the horizon is 0°.

The altitude, as a coordinate of a celestial body, may be replaced by the *arc of the vertical circle of the body from the zenith to the place of the body*. This arc is called the **zenith distance**, and is denoted by z. Thus, the zenith distance of a body C is arc ZC, etc.

The zenith distance of a celestial body is measured from 0° to 180°. If the body is above the horizon (C), then $z < 90°$, if it is below the horizon (B), then $z > 90°$. For any point of the horizon, $z = 90°$; for the zenith point, $z = 0°$; and for the nadir point, $z = 180°$.

As will be seen from the figure, the *altitude and zenith distance always complement one another to* 90°:

$$\left.\begin{aligned} z &= 90° - h \\ \text{or}\quad h &= 90° - z \end{aligned}\right\} \qquad (1.1)$$

In these formulas, the altitude should be taken with its sign. Thus, if $h = 47°$, then $z = 43°$; if $h = -29°$, then $z = 90° - (-29°) = 119°$; and so forth.

If a celestial body lies on the observer's meridian (C' or C''), then its altitude is called the *meridian altitude* and is designated by H; its zenith distance is then called the *meridian zenith distance* (Z).

To the meridian altitude H and the meridian zenith distance Z we add the designations N or S: to the meridian altitude, according to the point of the horizon above which this altitude is measured; to the meridian zenith distance, the reverse. For example, in Fig. 5, the celestial body C' has a south meridian altitude $H = 65°$S or $Z = 25°$N.

The horizon coordinates A and h that we have just considered are fully sufficient to specify the position of a point on the sphere. A single coordinate defines the position of some one circle of the sphere: azimuth, the position of the vertical circle; altitude, the position of the parallel of altitude.

II. FIRST EQUATORIAL COORDINATE SYSTEM

In this system, the principal direction is that of the celestial axis, and the basic circles are the celestial equator KQ and the observer's meridian (Fig. 6). The position of any point of the sphere

in this system is defined by two equatorial coordinates: the **hour angle** and **declination**.

(1) *The hour angle* (t) *of a celestial body is the arc of the equator reckoned from the upper meridian of the observer westwards to the meridian of the body.*

Due to the fact that this arc ($KWQD$, Fig. 6) measures the angle at the centre of the sphere or the spherical angle for the elevated pole,

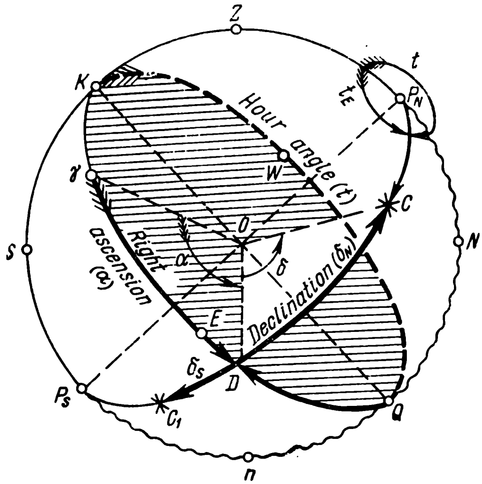

Fig. 6

the *hour angle is also defined as the spherical angle for the elevated pole between the upper meridian of the observer and the meridian of the celestial body.*

An hour angle reckoned westwards can have any value from 0° to 360° and is called the **west** or **ordinary hour angle**. If the west hour angle is greater than 180°, then its complement to 360° is called the **east hour angle**. For example, a celestial body (Fig. 6) has t = 255°W =105°E. The east hour angle can never be more than 180°.

The hour angle (east or west, but less than 180°) is used in the solution of spherical triangles and is therefore called the *practical hour angle* (east or west, respectively). Obviously, for a celestial body situated on the upper meridian, the hour angle t = 0°; for the lower meridian, t = 180°; for point W—t = 90°W; for point E—t = 270°W = 90°E.

(2) *The declination* (δ) *of a celestial body is the arc of the meridian* (*declination circle*) *of the body from the celestial equator to the place of the body.*

Due to the fact that this arc measures the central angle δ (see Fig. 6) in the plane of the meridian, the *declination of a celestial body is also defined as the angle between the plane of the celestial equator and the direction from the centre of the sphere to the body.* If the celestial body is located in the northern hemisphere, then N is prefixed to its declination, if it is in the southern hemisphere, S is prefixed. From the definition of declination it follows that it is measured from 0° to 90°. The declination of any point of the equator is equal to 0°, the declination P_N is 90°N, while the declination P_S is 90°S. The declination of a celestial body C (Fig. 6) is written as $\delta = 55°$N; that of celestial body C_1, as $\delta = 35°$S.

For this equatorial coordinate, in place of the declination we can take the **polar distance** (Δ) of the celestial body; this is the arc of the meridian of the body reckoned always from the elevated pole to the place of the body. The polar distance is measured from 0° to 180°. Thus, for any point of the equator $\Delta = 90°$, for the elevated pole, $\Delta = 0°$, for the depressed pole, $\Delta = 180°$.

The declination δ and the polar distance Δ are mutually complementary to 90°, that is,

$$\delta = 90° - \Delta$$

or

$$\Delta = 90° - \delta \tag{1.2}$$

The declination and latitude of a place may be of the same name (N or S) or of opposite (or contrary) names.

In nautical astronomy, *declination of the same name as latitude is considered positive* (+); *declination of name contrary to latitude is considered negative* (—). Hence, sometimes south declination is negative and sometimes north declination.

For example, for a body C (see Fig. 6): $\Delta = 90° - 55° = 35°$; for C_1 we get $\Delta = 90° - (-35°) = 125°$.

Thus, the position of any point of the sphere is defined by two coordinates: t and δ. The hour angle taken alone defines the position of the meridian (declination circle), the declination, the position of the parallel of the celestial body.

III. SECOND EQUATORIAL COORDINATE SYSTEM

In this system, the basic circles are the celestial equator and the meridian of the vernal equinox (or first point of Aries), which is denoted by the symbol ♈ of the constellation Aries.

The point ♈ occupies a very definite place (independent of the observer) on the celestial equator; therefore the choice of the meridian of this point as the reference origin is extremely convenient, especially when reckoning time.

In this system, the place of a celestial body is determined by two equatorial coordinates: **right ascension** and **declination** (Fig. 6).

The right ascension (α) *of a celestial body is the arc of the equator from the vernal equinox* (Υ) *to the meridian of the body reckoned from* 0° *to* 360° *in a direction opposite to the reckoning of west hour angles* (counter to the diurnal rotation of the sphere). For example, a celestial body C has $\alpha = 65°$.

In modern Soviet and foreign manuals (for instance, in nautical astronomical almanacs), right ascension of stars is replaced by the so-called **sidereal hour angle** (S.H.A. or τ) which is the complement of α up to 360°, that is,

$$\tau = 360° - \alpha$$

Obviously, the *sidereal hour angle* τ *is the arc of the equator from the point* Υ *to the meridian of the body reckoning towards west hour angles.*

For instance, for a body C we have $\alpha = 65°$, $\tau = 295°$.

The quantity τ is not generally considered a special coordinate but only an auxiliary quantity; however, when it is introduced for stars, the coordinate α is no longer needed. Right ascension (or the quantity τ) defines the position of the meridian of a celestial body on the sphere.

The second coordinate of this system is the *declination*, δ, which was considered in the first equatorial coordinate system.

Thus, the second equatorial system differs from the first only in the position of the initial or prime meridian.

To pass from the first equatorial system to the second, and conversely, all that is needed is to know the position, on the equator, of the vernal equinox Υ, which is defined at each instant by its west hour angle (Fig. 6, the arc $KW\Upsilon$) denoted by t^{Υ}. From the figure it will be seen that the arc KWD, which is equal to the west hour angle of the celestial body C, and the arc ΥD, equal to its right ascension, are together equal to the arc $KW\Upsilon$, that is, to the hour angle of the first point of Aries, hence

$$t^{\Upsilon} = t + \alpha \tag{1.3}$$

This formula may be used to pass from coordinates of the first equatorial system to the second, provided t^{Υ} is known.

Later on we shall learn that **sidereal time** is measured by the quantity t^{Υ} and so the latter may be obtained for any time by a chronometer.

In concluding this examination of systems of spherical coordinates, let us touch on the question of **units of measurement of the coordinates.**

Spherical coordinates are *arcs* of great circles and so may be measured with the same units as arcs and angles, that is, in *degrees* and *radians*. For conversion from degrees to radians, use Tables 38 MT-63 or general rules (see Appendix III, 7).

Time units are also used. Here, the unit is an interval of time of 24 hours during which the earth completes one rotation of 360°; thus the circle contains 24 hours. Whence we get the following relations: $24^h = 360°$; $1h = 15°$, $1m = 15'$, $1s = 15'' = 0'.25$ or $1° = 4m$; $1' = 4s$, etc. The rules for conversion from one system to another are given in Sec. 33.

Time units are sometimes used to measure the quantities α, t, t^{Υ} and sometimes also geographic longitude.

For example: $\alpha = 220° = 14h\ 40m$; $t^{\Upsilon} = 110° = 7h\ 20\ m$, etc.

SEC. 3. THE RELATIONSHIP BETWEEN THE GEOGRAPHIC LATITUDE OF THE OBSERVER'S POSITION AND THE SPHERICAL COORDINATES OF POINTS OF THE SPHERE

The spherical coordinates of certain points of the celestial sphere may be related to the geographic coordinates of the observer. For the longitude of a place, this problem is rather involved and will be considered later on. However, the relationship between geographic latitude and the spherical coordinates of the points P_N and Z is readily seen from an inspection of Fig. 1. Indeed, the angles eCA and KAZ are equal as corresponding angles of the parallel lines eC and KA. Hence, the arc KZ is equal to the latitude φ of the place, the arc ZP_N is equal to $90° - \varphi$, the arc P_NN is equal to φ, and so forth. But the arc P_NN (see Fig. 2) is the altitude of the elevated pole, the arc ZK is the zenith distance of the point K, and the arc ZP_N is the zenith distance of the elevated pole (P_N), hence:

(1) the *altitude of the elevated pole*, which is equal to the zenith distance of point K of the equator, *is equal to the geographic latitude of the observer*:

$$h_p = Z_K = \varphi \tag{1.4}$$

(2) the *zenith distance of the elevated pole is equal to the complement of the latitude* (*to* 90°), i.e.,

$$Z_p = 90° - \varphi \tag{1.5}$$

These arcs may also be expressed in terms of declination. Indeed, the arc KZ is the declination of the zenith, while the arc P_NZ is the polar distance of the zenith, hence:

(1) the *declination of the zenith is equal to the latitude of the place* and has the same name, that is,

$$\delta_Z = \varphi \tag{1.6}$$

(2) the *polar distance of the zenith is equal to the complement of the latitude* (*to* 90°) or

$$\Delta_Z = 90° - \varphi \tag{1.7}$$

These relations are very important in constructing the sphere and especially in determining geographic latitude.

SEC. 4. REPRESENTATIONS OF THE CELESTIAL SPHERE USED IN NAUTICAL ASTRONOMY

The representation most frequently used in spherical and nautical astronomy is the one we have already considered: the celestial sphere is depicted on the plane of the observer's meridian (see Fig. 2 et al.), that is, when the meridian of the observer coincides with the plane of the drawing.

However, in a number of cases it is more convenient to use other representations of this same auxiliary celestial sphere: (a) in the plane of the celestial equator; (b) in the plane of the horizon; (c) in the plane of an arbitrary meridian. These representations may be spatial and plane. In addition, frequent use is also made of a representation of the sphere with centre at the earth's centre. Sometimes the sphere is given with the centre at the centre of the solar system. All of these representations are actually only different aspects of one and the same auxiliary mathematical sphere. Let us consider some of the more typical ones.

Representation of the sphere in the plane of the celestial equator is shown in Fig. 7. The circle $KWQE$ represents the celestial equator with pole P_N at the centre, the straight line KQ is the observer's meridian at latitude φ_N, the lower branch of the meridian is indicated by a wavy line; the straight lines P_ND and P_NB are meridians of celestial bodies C_1 and C_2; the ellipse NESW is the celestial horizon, and the part of the ellipse $ZC_1\,n$ is the vertical circle of C_1.

From an examination of the figure we conclude that this representation of the sphere is convenient for reckoning hour angles and right ascension of bodies and is not convenient for measuring the horizon coordinates A and h. This factor determines its use.

Let us now consider a representation of the celestial sphere that is somewhat different from the foregoing types. In this representation (Fig. 8), the centre of the sphere is placed at the centre of the earth, while the celestial sphere itself is concentric with the earth, which is taken to be a sphere. Here, the celestial sphere remains

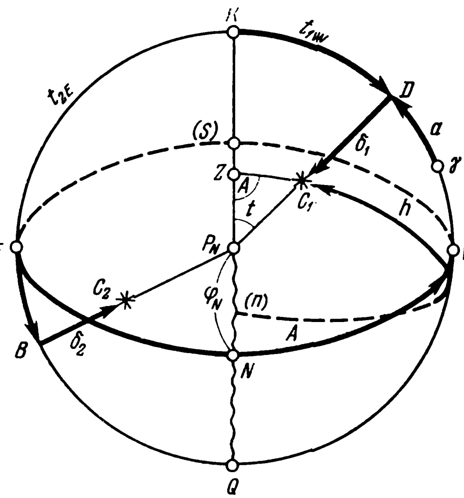
K
t_{1W}
D
t_{2E}
a
(S)
δ_1
γ
Z
A
t
C_1
h
E
P_N
W
C_2
φ_N
(n)
A
B
δ_2
N
Q

Fig. 7

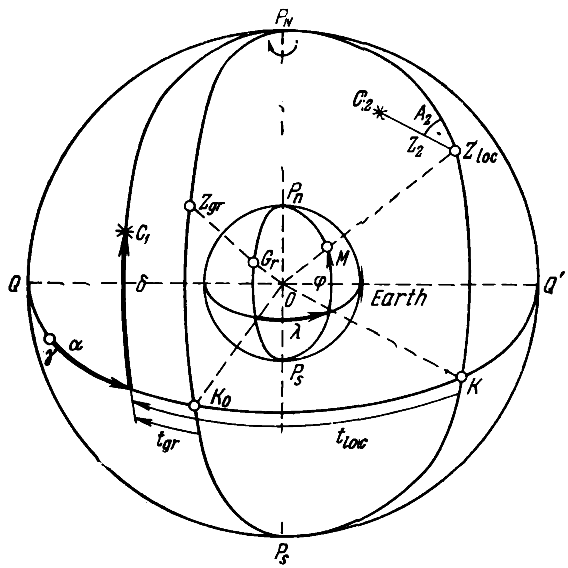
P_N
C_2
A_2
Z_2
Z_{loc}
P_n
Z_{gr}
C_1
M
Gr
Q
δ
φ
O
Earth
Q'
λ
α
γ
P_s
K
Ko
t_{gr}
t_{loc}
P_s

Fig. 8

an auxiliary mathematical construction, but, unlike earlier representations, it holds for all observers on the earth.

The representation is constructed in the plane of an arbitrary celestial meridian, and the radius of the sphere is likewise arbitrary. To obtain the basic circles and directions on the celestial sphere, it is necessary to continue the basic planes and directions on the earth to intersection with the sphere; then we get $P_N P_S$ as the celestial axis, QQ' as the celestial equator, and C_1, C_2 as the places of the celestial bodies.

Let point M be the position of some observer on the surface of the earth, whose geographic coordinates are φ and λ; $p_n M p_s$ is the geographic meridian of the observer M; $p_n Gr p_s$ is the initial or prime meridian (Greenwich meridian). On the celestial sphere we accordingly get Z_M (or Z_{loc}) as the zenith of the observer M; $P_N Z_M P_S$ as the observer's meridian (its upper branch); Z_{gr} is the zenith of Greenwich; $P_N Z_{gr} P_S$ is the upper branch of the Greenwich meridian; this celestial meridian is taken as the initial one. Hour angles are reckoned along the celestial equator $Q'Q$ from the indicated upper branches of the meridians (points K and K_0). Thus, for the celestial body C_1 we have t_{loc}, the local hour angle, which corresponds to the meridian of the position of the observer M, and t_{gr}, which is the Greenwich hour angle corresponding to the initial meridian. From Fig. 8 it is evident that $t_{loc} - t_{gr} = \lambda$.

Let γ be the place of the first point of Aries on the equator, then the arc α is the right ascension of C_1; it is obvious that for all observers, α will be the same. From the figure it will be seen that the *equatorial coordinate systems* are absolutely analogous to the *system of geographic coordinates*, this makes possible simple conversion from the celestial coordinates of points of the sphere to the geographic coordinates of the place. In this figure, the coordinates of the horizon system are represented at the zenith of the observer without indicating the horizon. Thus, for the celestial body C_2 the observer M has a zenith distance $z_2 = 90° - h_2$ and azimuth A_2; the arc $Z_{loc} C_2$ is the vertical circle of the body. This representation of the sphere may readily be reduced to ordinary representation for a given observer if we draw a plane through the centre perpendicular to the plumb line $Z_{loc} O$ of the observer.

Representation of the sphere with centre at the earth's centre is widely used in practical astronomy when considering ways of determining geographic coordinates.

Representation of the sphere in the plane of the horizon of the observer is also shown in Fig. 21.

C H A P T E R 2

CONVERSION FROM ONE SYSTEM OF SPHERICAL COORDINATES TO OTHER SYSTEMS

SEC. 5. CONSTRUCTING THE CELESTIAL SPHERE

When solving astronomical problems it is frequently necessary to pass from one coordinate system to another, that is, to transform the coordinates of one system to other systems. Transformation of coordinates is carried out in a variety of ways:

(a) graphically (approximately) by means of a drawing of the celestial sphere;

(b) graphically (more precisely) by means of special grids or instruments;

(c) analytically, by the solution of the spherical triangle to the required accuracy.

In this section, we shall consider the graphical solution of a problem by construction of the celestial sphere.

Constructing the celestial sphere consists in representing it for the latitude of the observer and indicating on the sphere the equatorial or horizon coordinate system of the given body. The sphere thus obtained not only permits us to get an approximation of the coordinates of another system, but also pictorially illustrates a number of astronomical phenomena (we shall make wide use of this later on).

The sphere is constructed in the plane of the observer's meridian by a conventional drawing; it is also done as if the observer were viewing the celestial sphere from the outside.

Here are some pointers that should be remembered when making the drawing:

(1) On the half of the sphere visible to the observer, draw the circles as solid lines; on the opposite half (and inside the sphere) as dashed lines.

(2) All great circles (with the exception of the observer's meridian) are depicted as ellipses which are carefully drawn freehand.

(3) When laying off the arc values roughly, keep to within 5° in the scale of angles and arcs. For this purpose, the observer's meridian should be taken as the arc measure, the construction being made from the middle of the drawing where the arcs are less distorted.

(4) In order to indicate a body, it is, of course, necessary to know any two of its coordinates, even if they belong to different systems. Each specified coordinate defines the position of only one circle of the sphere, for instance, declination defines parallels, altitude defines the position of the parallel of equal altitude, and the like.

Example 1. Construct the celestial sphere for $\varphi = 55°$S and indicate a body with $h = 50°$ and $A = 70°$NE. From the drawing find the equatorial coordinates of the body t and δ.

Draw a circle of arbitrary radius and a vertical diameter; we get the points Z and n (Fig. 9). Draw the celestial horizon. Since the azimuth of the NE quadrant

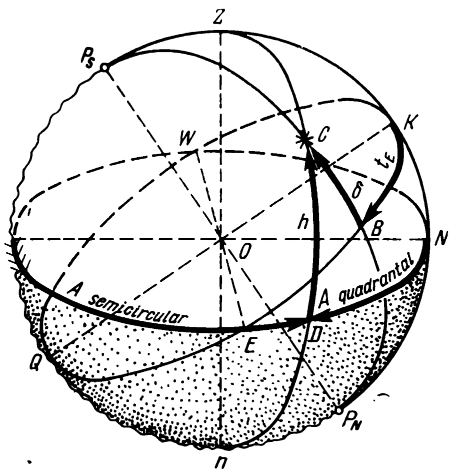

Fig. 9

is given, for the body to be on the side facing the reader, point N must be on the right of the drawing, and point S on the left side. Since the latitude is south, the elevated pole must be P_S and at a distance from the point S of the horizon 55° upwards along the meridian. Then draw the celestial axis P_SP_N, the celestial equator KQ; the points E and W, where E will be in the hemisphere facing the reader. Mark with a wavy line the lower branch of the meridian P_SnP_N. Indicate the celestial body: from the point N lay off A along the horizon eastwards arc N$D = 70°$ and through D draw the vertical circle of the body ZDn. Along this vertical circle lay off an arc of $50° = h$ from the horizon (from point D) towards the zenith and we obtain the place of the celestial body on the sphere, the point C.

To obtain from the drawing the equatorial coordinates of the body, draw through C the meridian P_SCP_N and mark the point B as the intersection of this meridian with the equator.

Then arc BC, which is approximately equal to 30°, will represent the declination of the celestial body C (it will be southwards), that is, $\delta = 30°$S; but the arc of the equator KB, equal approximately to 45°, will represent the east hour angle of the body, i.e., $t = 45°$E (or 315°W).

In the examples given below it is required to construct the sphere from the given latitude φ, to indicate the celestial body on the basis of its coordinates, and from a freehand sketch to give a rough approximation (to within 5°) of the desired coordinates.

Examples

No.	Given	Find
2	$\varphi = 55°N$, $\delta = 60°N$, $t = 135°W$	h, A
3	$\varphi = 20°N$, $h = 25°$, $A = 50°NE$	δ, t
4	$\varphi = 45°S$, $t = 60°E$, $A = 75°NE$	h, δ
5	$\varphi = 50°N$, $h = 30°$, $t = 50°W$	δ, A
6	$\varphi = 30°S$, $h = 20°$, $\delta = 15°N$ celestial body is in eastern hemisphere	A, t
7	$\varphi = 25°S$, $\delta = 10°S$, body on western vertical circle	h, t
8	$\varphi = 50°N$, $\delta = 20°S$, body on horizon in western part	t, A
9	$\varphi = 40°N$, hour angle of first point of Aries $t^{\Upsilon} = 30°W$; right ascension of body $\alpha = 60°$; $\delta = 20°N$	t, h, A
10	$\varphi = 70°N$, $h = 35°$, $A = 80°NW$, $\tau = 345°$	δ, t, t^{Υ}
11	$\varphi = 0°$, $h = 30°$, $A = 45°SE$, $t^{\Upsilon} = 240°$	δ, t, τ
12	$\varphi = 60°N$, $\delta = 10°S$, $t = 120°E$, $t^{\Upsilon} = 30°$	h, A, α

SEC. 6. SPECIAL GRIDS FOR TRANSFORMATION OF COORDINATES (FUNDAMENTALS)

From what has been said about conversion from one set of coordinates to another by means of a drawing of the celestial sphere, it follows that a more exact solution of the problem is possible by means of a cartographical projection of the sphere on the plane of the observer's meridian.

Indeed, let us imagine the circle of an observer's meridian drawn in the plane of the paper (Fig. 10*a*), with a grid of the vertical circles and parallel of altitude, say, at intervals of 5° (Fig. 10 has intervals of 30°) in a projection convenient for this purpose, for example, in an azimuthal equidistant (Postel) projection or in a stereographic projection. On another sheet of paper, a similar grid in the same projection is constructed for the celestial meridians and parallels (Fig. 10*b*). Label each grid with the appropriate letters as in Fig. 5 and Fig. 6. Both grids are absolutely identical, only one represents the horizon coordinate system, while the other represents the equatorial system.

If the horizon grid has been drawn on a transparent sheet of tracing paper and if we now indicate the celestial body on the equato-

rial grid from the data $\delta = 61°N$; $t = 135°W$ ($\varphi = 50°N$) and superimpose the horizon grid on it at the centre (Fig. 10*b*), then after turning point Z of the horizon grid through an angle of $90° - \varphi$ relative to the point of the elevated pole (P_N), we readily obtain from the horizon grid the values $A = 23°NW$ and $h = 27°$. The accuracy of the coordinates obtained depends on the grid scale.

The reverse problem of finding equatorial coordinates from the horizon coordinates of the body is solved in similar fashion.

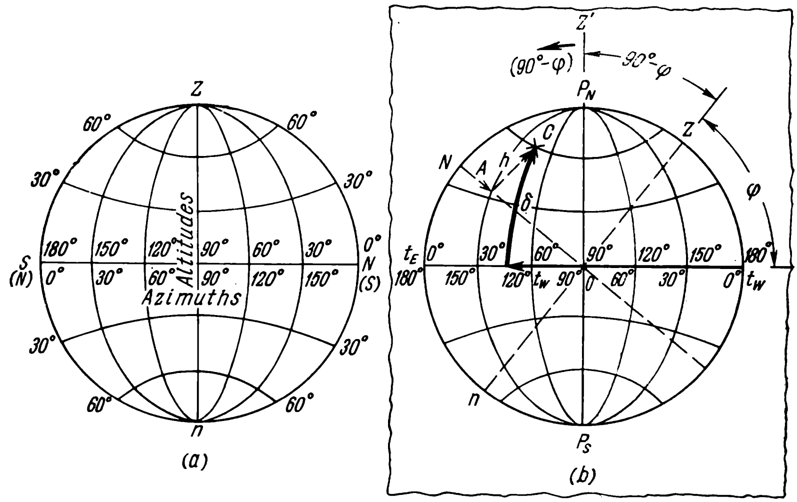

Fig. 10

For the foregoing solution, a number of authors (for instance, Professor Wulf, Kohlschütter, and others) have published 20-to-25-cm-diameter grids that yield coordinates accurate to about 1°-3°. However, this problem may be solved in a still more simple fashion by means of a single grid and a piece of tracing paper. Indeed, from Fig. 10*b* we see that if the diameters $P_N P_S$ and Zn are superposed, the two grids coincide. Conversion from one system of coordinates to another is effected by rotating the diameter Zn through an angle of $90° - \varphi$. Hence, we may confine ourselves to a single grid and a piece of tracing paper pinned at the centre O. Using the grid as an equatorial grid, we indicate the position of a body C on the tracing paper on the basis of t and δ and at the point P_N we prime Z (Z'). Then turn Z' with the sheet through an angle of $90° - \varphi$ relative to the point P_N (leftwards in Fig. 10*b*). The grid now represents the horizon coordinate system and enables us to find the coordinates A and h of the body C. A grid constructed on this or a similar

principle in an azimuthal equidistant projection with the places of the stars indicated is known as *Kavraisky's planisphere.* Later, this same principle was utilized for the construction of a special instrument "Vega" (German instrument called ARG, Astronomische Rachengerät) in which a fine-grid is viewed by microscope and is turned by a special device (see Sec. 108).

SEC. 7. THE ASTRONOMICAL TRIANGLE OF A CELESTIAL BODY AND ITS SOLUTION

In various astronomical problems, a more exact solution of problems in conversion from one coordinate system to another is required, and this should obviously be done analytically, by calculation. For this purpose, we need strict formulas relating the coordinates.

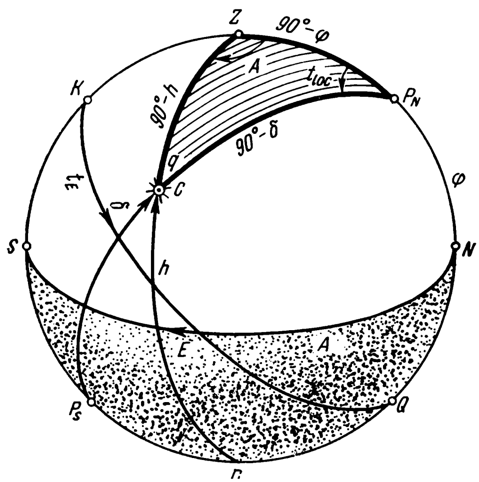

Fig. 11

Construct the celestial sphere for a given latitude and on it draw the vertical circle and meridian of a body C (Fig. 11); on the sphere we get a spherical triangle ZP_NC, whose vertices are the *zenith*, the *elevated pole*, and the *place of the celestial body*. This triangle is formed by the arcs of great circles: the *observer's meridian*, the *vertical circle of the body, and its meridian.*

In astronomy, this is a very important triangle and is variously called the **parallactic**, **navigational**, or **astronomical** triangle of the

celestial body. From Fig. 11 we immediately see that the sides of this triangle are:

side ZP_N is equal to colatitude $(90°-\varphi)$;
side P_NC is equal to codeclination $(90°-\delta)$;
side ZC is equal to coaltitude $(90°-h)$.

Its angles are:

the angle at the zenith Z is equal to the azimuth A *always in semicircular reckoning*;

the angle at the pole P is equal to the local hour angle t_{loc}, *which is always practical*, that is, less than 180° (in the figure it is east).

The angle at the body C is called the **parallactic** angle q and is hardly ever used in nautical astronomy.

The astronomical triangle of a celestial body relates the horizon coordinates of the body to its equatorial coordinates and, what is particularly important, to the geographic coordinates of the observer (the longitude of the observer enters implicitly into the local hour angle t_{loc}, see Fig. 8, body C_1). If at least three elements of this triangle are known, it may be solved by the ordinary formulas and rules of spherical trigonometry to the requisite degree of accuracy. In nautical astronomy, computations are performed to within 0′.1-1′.0, and sometimes to 0°.1, that is, with the use of five-place and four-place tables, and sometimes with the aid of calculating charts. In the general case, if the sides and angles are arbitrary, the astronomical triangle of a body will be oblique; if one of the angles is equal to 90° (for example, the body is on the first vertical circle; $A = 90°$), then the triangle will be *right-angled*; but if one of the sides is equal to 90° (for instance, the body is on the horizon; $h = 0$), then the triangle is *quadrantal*. Accordingly, its solution may be general or particular; the latter will be considered in Chapter III.

THE GENERAL CASE FOR SOLVING THE ASTRONOMICAL TRIANGLE

When solving the astronomical triangle, that is, when seeking the unknown elements from known elements, it is necessary to abide by the general rules of spherical trigonometry for solving such triangles, namely: (a) make a drawing of the triangle (sometimes outside the sphere) and label the knowns and unknowns; (b) choose and write down the formulas relating the knowns and unknowns; simplify these formulas and express the unknowns explicitly; (c) investigate the signs of the formulas abiding by the general trigonometry rules and remarks made below. The aim of the investigation is to establish the sign and magnitude of the desired coordinate and to determine which tables (for sums or differences) should be

used in solving by nonlogarithmic formulas and by tables of logarithms; (d) perform the computations in a specific sequence using four- and five-place tables of logarithms (MT-43, MT-53, MT-63); (e) check your computations.

When investigating the signs, bear in mind the following:

1. *The latitude* (φ) *is always considered positive*, since it is the basis for constructing the sphere. The latitude is always numerically less than 90°; for this reason, all its trigonometric functions have the *plus* sign.

2. *The declination* (δ) *may be of the same name as the latitude*; then it should be considered positive; or it has a *contrary name* to the latitude; then it is negative; the declination is always less than 90°. For this reason, if the declination and the latitude are of the same names, all trigonometric functions of declination will be positive; but if the declination and latitude are of opposite names (the fourth quadrant trigonometrically), then cos δ and sec δ will be positive, and the other functions, negative.

3. *The altitude* (h) *may be positive* or *negative*, but numerically it is always less than 90°. Therefore, if h is positive, then all its trigonometric functions are positive; if it is negative, then cos h and sec h are positive, and all the other functions are negative.

4. *The azimuth* (A) in semicircular reckoning *cannot be greater than* 180°, that is, it may be in the first and second trigonometric quadrants. If $A < 90°$ (first quadrant), then all its trigonometric functions are positive; if $A > 90°$ (second quadrant), then sin A and cosec A are positive, and the other functions are negative.

It should be noted that the azimuth always enters into the astronomical triangle of a body in semicircular units; therefore, if the azimuth is given in quadrantal or circular units, it must first be converted to semicircular reckoning, and only then can the formula be investigated.

5. *The hour angle* (t) in the astronomical triangle of a body may be west or east, *but not greater than* 180°, which means in the first or second trigonometric quadrants; therefore, if a west hour angle greater than 180° is indicated, it must first be converted to east, less than 180°. If the hour angle, west or east, is less than 90°, then all its trigonometric functions are positive; if the hour angle $t > 90°$, then sin t and cosec t are positive, and all the other functions are negative.

6. *Parallactic angle* (q) *may be either less or greater than* 90°.

7. After investigating the signs of the formulas, note the following: if as a result of investigation the signs of the first and second terms of the formula are different, then the sign of the left-hand side of the formula (the function being computed) will be the same as the sign of the greater term; if the signs of the terms are the same,

then the sign of the desired function will coincide with it. If the triangle is right-angled or quadrantal, the formulas for the desired quantities will be single-termed and their signs will be found immediately after investigation.

8. When computing any part of the astronomical triangle by a formula that has been investigated as to sign, one should remember:

(a) when computing altitude, that the altitude is never greater than 90° and that a negative altitude indicates reduced height;

(b) when computing azimuth, that the azimuth computed from the astronomical triangle of a celestial body is always in semicircular units, which numerically may be less or more than 90°. Compute the numerical value of the azimuth according to the investigation, and then give it a name; here, the first letter of the name of the azimuth will always be the same as that of the latitude, the second will either be the same as the practical hour angle that enters into the triangle (west or east), or, if the hour angle is not given, it will depend on whether the body is in the western or eastern half of the sphere.

To check the correctness of computations, we can use:

(a) intermediate checking of the values of the chosen logarithms;

(b) approximate checking of the results obtained by constructing the celestial sphere (or on the basis of the relations between the sides and angles of the triangle, or in other ways);

(c) exact checking of the final results by special *check formulas* which relate the obtained results to the given data. Checking will be illustrated in examples.

Let us consider the principal cases of solving the astronomical triangle that occur in nautical astronomy.

I. Given: φ, δ and t; to find h and A.

Depict the astronomical triangle in the general form (Fig. 12) and label the knowns and unknowns. Applying the formulas of the cosine of a side and of the four adjacent parts,* we get

$$\cos(90^\circ - h) = \cos(90^\circ - \varphi)\cdot\cos(90^\circ - \delta) + \\ + \sin(90^\circ - \varphi)\cdot\sin(90^\circ - \delta)\cdot\cos t$$

and

$$\cot A\cdot\sin t = \cot(90^\circ - \delta)\cdot\sin(90^\circ - \varphi) - \cos(90^\circ - \varphi)\cdot\cos t$$

Simplifying and isolating the unknowns, we finally get

$$\sin h = \sin\varphi\cdot\sin\delta + \cos\varphi\cdot\cos\delta\cdot\cos t \tag{2.1}$$

$$\cot A = \tan\delta\cdot\cos\varphi\cdot\operatorname{cosec} t - \sin\varphi\cdot\cot t \tag{2.2}$$

* See Appendix III.

To check, apply the formula of sines:

$$\frac{\sin A}{\sin(90°-\delta)} = \frac{\sin t}{\sin(90-h)}$$

or

$$\sin A \cdot \cos h = \sin t \cdot \cos \delta$$

From now on, when computing the azimuth, we will also make use of this formula in the form

$$\sin A = \sin t \cdot \cos \delta \cdot \sec h \tag{2.3}$$

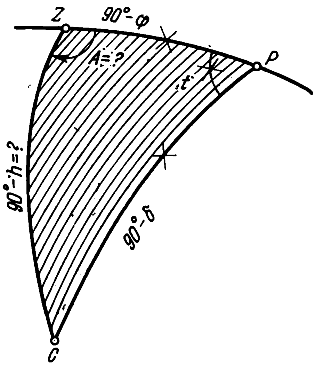

Fig. 12

Example 2. Given: $\varphi = 52°19'.7N$
$\delta = 14°43'.8S$
$t = 293°29'.6W$

Find h and A of the celestial body.

Since the hour angle is greater than 180°, we convert it to the east hour angle; $t = 66°30'.4E$. Investigate the signs of formulas (2.1) and (2.2):

$$\sin h = \overset{+}{\sin}\,\varphi \cdot \overset{-}{\sin}\,\delta + \overset{+}{\cos}\,\varphi \cdot \overset{+}{\cos}\,\delta \cdot \overset{+}{\cos}\,t \qquad (-\mathrm{I}+\mathrm{II})$$

$$\cot A = \overset{-}{\tan}\,\delta \cdot \overset{+}{\cos}\,\varphi \cdot \overset{+}{\operatorname{cosec}}\,t - \overset{+}{\sin}\,\varphi \cdot \overset{+}{\cot}\,t \qquad (-\mathrm{I}-\mathrm{II})$$

We place the plus sign above all the functions φ; since δ and φ have contrary names, that is, δ is negative, we put minus above $\sin\delta$ and $\tan\delta$, and plus above $\cos\delta$; since the hour angle $t < 90°$, we put the plus sign above all its functions. We finally get $\sin h$ = —I term +II term and $\cot A$ = —I term —II term.

Arrange the following form and perform the computations with the aid of table 5a MT-63 (see top of p. 50).

By investigation, $\cot A = -\mathrm{I} - \mathrm{II}$, that is, it is negative; this can occur only if the azimuth $A > 90°$; therefore, the arc A' chosen from the tables by the logarithm of $\cot A'$ will be $180° - A$. After computing the azimuth A = 117°28′.4, we give it the name N-E, since the latitude is north and the hour angle is east.

$\varphi = 52°19'.7$N	sin	9.89846	cos	9.78614	cos	9.78614	sin	9.89846
$\delta = 14°43'.8$S	sin	9.40528	cos	9.98549	tan	9.41980	—	
$t = 66°30'.4$E	—		cos	9.60058	csc	0.03758	cot	9.63816
	I	9.30374	II	9.37221	I	9.24352	II	9.53662
	Arg	0.06847	β	9.16393	*Arg*	0.29310	α	0.17875
			$\sin h$	8.53614			$\cot A'$	9.71537

$h = 1°58'.2$ $\qquad$ $180° - A = 62°33'.6$

$A = \text{N } 117°26'.4\text{E}$

When computing the altitude we see that the logarithm II > I and, hence, II term > I term in absolute value, and since $\sin h = -\text{I} + \text{II}$, the computed altitude will be positive.

Check:

(a) Intermediate checking is possible for the second and third lines of the scheme; indeed,

$$\log \sin \delta - \log \cos \delta = \log \tan \delta$$

and

$$\log \cos t + \log \operatorname{cosec} t = \log \cot t$$

Performing the indicated operations we get

$$9.41979 \approx 9.41980 \text{ and } 9.63816 = 9.63816$$

(b) Applying the checking formula (2.3), we get

$\sin A$	9.94817	$\sin t$	9.96242
$\cos h$	9.99974	$\cos \delta$	9.98549
	9.94791	=	9.94791

which shows that the computations are correct.

Formula $\sin^2 \frac{z}{2}$ (haversine z or hav z).

For computing h, formula (2.1) does not always ensure sufficient accuracy, especially when working with four-place tables. For this reason, for altitudes greater than 30°, use is sometimes made of another formula in which in place of $\sin x$ we apply the more precise function $\sin^2 \frac{x}{2}$ *. In formula (2.1), replace h by $90° - z$ and apply the trigonometric formula $\cos x = 1 - 2 \sin^2 \frac{x}{2}$; we get

$$\cos z = \sin \varphi \cdot \sin \delta + \cos \varphi \cdot \cos \delta \cdot \cos t$$

* See Appendix IV.

or

$$1-2\sin^2\frac{z}{2}=\sin\varphi\cdot\sin\delta+\cos\varphi\cdot\cos\delta\left(1-2\sin^2\frac{t}{2}\right)$$

Taking into account that $\sin\varphi\cdot\sin\delta+\cos\varphi\cdot\cos\delta=\cos(\varphi-\delta)$, we obtain

$$1-2\sin^2\frac{z}{2}=1-2\sin^2\frac{\varphi-\delta}{2}-2\cos\varphi\cdot\cos\delta\cdot\sin^2\frac{t}{2}$$

and, finally,

$$\sin^2\frac{z}{2}=\sin^2\frac{\varphi-\delta}{2}+\cos\varphi\cdot\cos\delta\cdot\sin^2\frac{t}{2} \tag{2.4}$$

When performing computations with this formula, no investigation is required since both terms are always positive; when working with logarithms, always use tables for sums (α). The values of $\log\sin^2\frac{x}{2}$, in Table 5a and б, MT-63, are given in a special side column. It should be borne in mind that for φ and δ of contrary names, the formula will contain: $\varphi-(-\delta)=\varphi+\delta$, which is a sum; for φ and δ of same name, we have the difference $\varphi-\delta$, the smaller value being subtracted from the greater.

Example 3. *Given:* $\varphi=52°12'.5$N, $\delta=12°22'.6$N, $t=11°52'.7$W. Determine h.

$t=11°52'.7$	$\sin^2$	8.02965		
$\varphi=52°12'.5$	cos	9.78731		
$\delta=12°22'.6$	cos	9.98979		
$\varphi-\delta=39°49'.9$	I	7.80675	$\sin^2$	9.06459
	Arg	1.25784	α	0.02335
			$\sin^2\frac{z}{2}$	9.08794
$h=49°2'.1$			$z=40°57'.9$	

II. Given φ, h, and A; to find δ and t.

Indicate in Fig. 13 the known and unknown quantities; apply the cosine formulas of a side and four adjacent parts:

$$\cos(90°-\delta)=\cos(90°-\varphi)\cdot\cos(90°-h)+$$
$$+\sin(90°-\varphi)\cdot\sin(90°-h)\cdot\cos A$$

and

$$\cot t\cdot\sin A=\cot(90°-h)\cdot\sin(90°-\varphi)-\cos A\cdot\cos(90°-\varphi)$$

Simplifying we get

$$\sin\delta = \sin\varphi \cdot \sin h + \cos\varphi \cdot \cos h \cdot \cos A \tag{2.5}$$

$$\cot t = \tan h \cdot \cos\varphi \cdot \operatorname{cosec} A - \cot A \cdot \sin\varphi \tag{2.6}$$

We leave it to the student to solve this problem by the indicated formulas. Formula (2.3) may be used as a check.

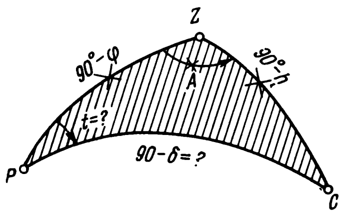

Fig. 13

In addition to these basic cases of conversion from one system of coordinates to another, there are other possible variants.

III. Given: φ, δ and h; to find t.

Using the cosine formula of a side, we get formula (2.1) from the astronomical triangle PZC.

Solving (2.1) for $\cos t$, we get

$$\cos t = \sin h \cdot \sec\varphi \cdot \sec\delta - \tan\varphi \cdot \tan\delta \tag{2.7}$$

This formula may be reduced to the function $\sin^2\frac{t}{2}$; replacing $\cos t$ in it by $1 - 2\sin^2\frac{t}{2}$ and simplifying, we obtain

$$\sin^2\frac{t}{2} = \frac{\cos(\varphi-\delta)}{2\cos\varphi \cdot \cos\delta}\left[1 - \frac{\sin h}{\cos(\varphi-\delta)}\right] \tag{2.8}$$

The foregoing examples, of course, do not exhaust all possible solutions. In each specific case, one should seek the solution that is best in the sense of accuracy and simplicity of computation.

The following examples are for independent solution by the student.

Examples

No.	Given			Find
15	$\varphi = 24°1'.9$N,	$\delta = 74°45'.6$N,	$t = 143°20'.2$W	h, A
16	$\varphi = 79°53'.4$S,	$\delta = 2°22'.4$N,	$t = 163°7'.0$E	h, A
17	$\varphi = 51°52'.4$S,	$\delta = 7°28'.9$N,	$h = 2°3'.7$, the	t, A
	body is in the western hemisphere			
18	$\varphi = 0°30'.7$N,	$h = 68°43'.1$,	$A = 39°11'.2$SE	t, δ
19	$\varphi = 59°25'.3$N,	$t = 104°1'.7$W,	$A = 67°19'.4$NW	h, δ

CHAPTER 3

APPARENT DIURNAL MOTION OF CELESTIAL BODIES

SEC. 8. GENERAL CHARACTERISTICS OF THE DIURNAL MOTION OF CELESTIAL BODIES. CONDITIONS FOR THE RISING AND SETTING OF BODIES. THEIR PASSAGE THROUGH THE ZENITH AND SO FORTH

Direct observations of stars with the unaided eye on a clear night will soon convince us that *all the stars are in constant motion across the sky, their mutual relationships remaining substantially unchanged*; this motion is such that in approximately 24 hours they occupy the same positions, the same altitude and the same azimuth. Such motion

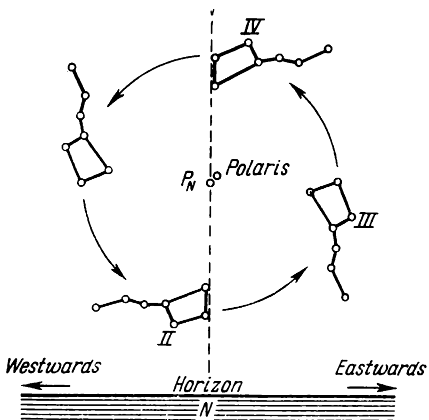

Fig. 14

is called the **apparent diurnal motion of celestial bodies.** For most stars, this motion is from east to west, but there are some (that never set) which at times move from east to west, and at other times, from west to east.

Fig. 14 shows the diurnal motion of the constellation Ursa Major which never sets in the latitudes of the Soviet Union. In the figure, the observer is looking at the heavens in the northerly direction. During 24 hours, the constellation describes a small circle about the celestial pole, near which is located Polaris. In the autumn,

Ursa Major occupies position I in the evening, position II at midnight, position III in the morning, and position IV in the daytime, when it is not visible. The constellation moves from east to west from III to I; and (below) from west to east in position I, II, III.

This apparent motion of celestial bodies is known to be due to the earth's rotation on its axis, which takes place uniformly *from west to east* at a constant angular velocity.

If we disregard the reasons for this phenomenon, and consider it purely geometrically, then it occurs as if the earth were stationary (Figs. 8 and 1), while the celestial sphere with all its stars rotated about the celestial axis with the same angular velocity and just as uniformly but in a direction reverse to that of the actual rotation of the earth, that is, *from east to west*. Therefore, the apparent diurnal motion of celestial bodies is called *retrograde motion*, as distinct from *forward motion*, i.e., the earth's rotation on its axis.

If we take it that the sphere rotates about the celestial axis, then each body on the sphere will, in its diurnal motion, move round the celestial axis parallel to the equator and will describe its own *parallel*.

On this assumption, all the planes and circles associated with an observer on the earth, namely: the plumb line, and true horizon, the observer's meridian and the prime vertical, must be considered stationary. Therefore, various bodies and points of the sphere located on the cc_1 parallel (Fig. 15) will pass, for instance, through the zenith point in the diurnal rotation of the sphere. All bodies will invariably cross both the upper and lower branches of the stationary meridian of the observer, etc.

Let Fig. 15 be a sphere constructed for some observer at latitude φ_N.

The small circles of the sphere aa_1, bb_1, cc_1, etc., represent parallels of bodies with declinations of different magnitude and name.

During the diurnal motion along a parallel, a body is constantly varying its altitude; the altitude is greatest when the body passes across the upper branch of the observer's meridian (points a_1, b_1, c_1, d_1, . . .) and least when crossing the lower meridian (points a, b, c, d, . . .). These passages are called meridian passages or **transits** (sometimes **culminations**): the *upper* transit (U.T.) in the former case, and the *lower* transit (L.T.) in the latter case.

If the parallel of diurnal motion of some celestial body crosses the horizon (for instance, dd_1; ee_1), then the body will **rise** and **set**.

The points of intersection of the parallel of the body with the horizon of the observer are called *rising points* (in the eastern hemisphere, for instance r) and *setting points* (in the western hemisphere).

Some bodies do not set (aa_1, Fig. 15); some do not rise at all (ff_1); some just graze the horizon at the instant of lower transit (bb_1).

In diurnal motion, bodies can cross the prime vertical in the above-horizon part (parallel dd_1) or in the below-horizon part (ee_1), or not cross the prime vertical at all (bb_1, aa_1). At upper transit, the celestial body C (parallel cc_1) only touches the prime vertical

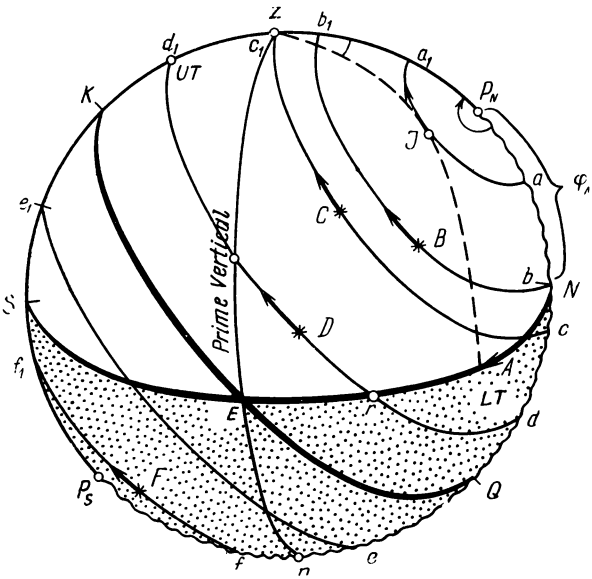

Fig. 15

and passes through the zenith. It is obvious that these peculiarities in the diurnal motion of various bodies are associated with the place of the body on the sphere and the inclination of the celestial axis to the horizon, that is, with the numerical relationships between the declination of the bodies and the latitude of the observer's position.

I. CONDITIONS OF RISING AND SETTING OF CELESTIAL BODIES

From Fig. 16, which depicts a sphere projected on the plane of the observer's meridian, it will be seen that arc $Qd = \delta_D$ is less than arc $QN = 90° - \varphi$ and the parallel of the celestial body D crosses the celestial horizon. The parallel of B with declination δ_B touches the horizon, therefore $\delta_B = 90° - \varphi$; the parallel of F is similarly located; the parallel of C passes above the horizon, obviously $\delta_C > 90° - \varphi$. Hence, for the parallel of the body to cut the

horizon, that is, for the body to rise and set, it is necessary that its declination be less than the colatitude $90° - \varphi$, irrespective of the name of the declination; thus, the *condition of the rising and setting of the celestial body* will be

$$|\delta| < 90° - \varphi \tag{3.1}$$

If, here, δ and φ are of the same name (dd_1), then the greater part of the parallel of the body will lie above the horizon and the smaller

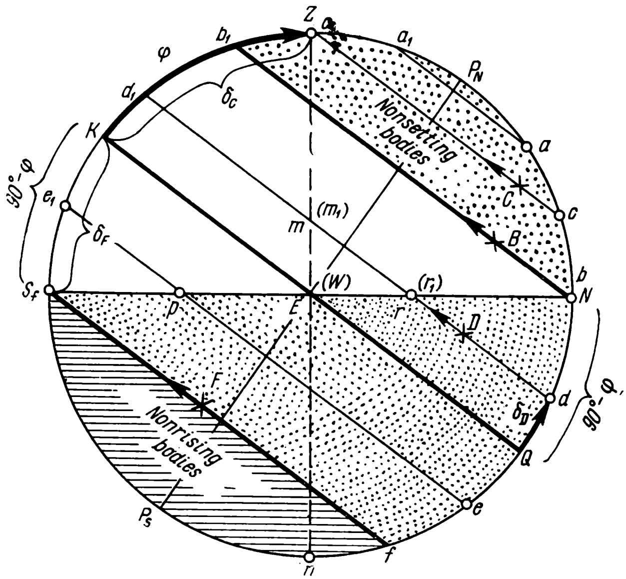

Fig. 16

part in the subhorizon part of the sphere ($rd_1 > rd$); but if δ and φ are of opposite names (ee_1), then we have the converse: the smaller portion of the parallel of the body will lie in the above-horizon part, and the greater portion will be in the subhorizon part ($e_1p < pe$).

When the declination is $\delta = 0°$, the body moves along the equator, and the above-horizon part of its diurnal path will be equal to the part below the horizon, since the equator is divided in half by the horizon.

If $\delta = 90° - \varphi$ and is of the same name as φ (the parallel bb_1), then the body does not set at all and only touches the horizon (at the point N) at lower transit. But if δ and φ are of opposite names (the parallel ff_1), the celestial body does not rise at all and only touches the horizon (at the point S) at the instant of upper transit.

For $\delta > 90° - \varphi$, the bodies will not touch the horizon at all and their entire diurnal path will lie above the horizon, if φ and δ are of the same name, or below the horizon, if φ and δ are of opposite names. The former are called *nonsetting* bodies, the latter, *nonrising* (invisible) bodies. For this reason, an observer at, say, the latitude of Leningrad ($\varphi = 60°$ N) cannot see celestial bodies whose $\delta_S > 30°$.

II. CONDITION FOR CELESTIAL BODIES INTERSECTING THE PRIME VERTICAL

From Fig. 16 it is readily seen that the prime vertical will be crossed by parallels of those celestial bodies whose declination (irrespective of the name) is less than the latitude of the observer; for instance, arc Kb_1, arc Kd_1 and arc Ke_1 are less than arc KZ; thus the *condition of intersection of the prime vertical by a body* is

$$|\delta| < \varphi \tag{3.2}$$

If the δ of the body is of contrary name to φ, then the prime vertical will be intersected in the subhorizon part of the sphere (the parallels ff_1, ee_1), but if δ and φ are of the same name, then the intersection will be in the above-horizon part (the parallels dd_1, bb_1).

III. THE SEQUENTIAL PASSAGE OF A BODY THROUGH THE QUADRANTS OF THE HORIZON

In the case of celestial bodies intersecting the above-horizon portion of the prime vertical, the azimuths will be located in all four quadrants of the horizon. Thus, for a body D moving along the parallel dd_1 (Fig. 16), from point r (rising) to intersection of the east part of the prime vertical (point m), the azimuth will be in the NE quadrant; in motion from the point m to the upper transit d_1, in the SE quadrant; from the upper transit to the west part of the prime vertical (point m_1), in the SW quandrant, and, finally, from the west vertical circle to the point r_1 (setting), in the NW quadrant.

In southern latitudes, the sequence of passage through the quadrants of the horizon by bodies that intersect the above-horizon portion of the prime vertical will be different, namely, rising in the SE and then the NE, NW quadrants and setting in the SW quadrant.

For bodies whose declinations are of contrary names to the latitude (see ee_1 in Figs. 15 and 16), the azimuths will lie only in two quadrants: SE and SW for north latitude, and NE, NW for south latitude.

But if the declination of the body is of the same name as the latitude, and is greater than the latitude (parallel aa_1), then this body does not intersect the prime vertical; its upper transit will occur between the zenith and the pole. For these bodies, the azimuths

will not go beyond two quadrants of the horizon: NE and NW for north declination of the body and SE and SW for south declination.

The position J (see Fig. 15) of such a body, when it is farthest away in azimuth from the observer's meridian is called its **elongation**. Elongation is easterly and westerly depending on the hemisphere in which it occurs. The azimuths of such bodies will be the same at the instants of upper (a_1) and lower (a) transits: N for δ_N or S for δ_S.

IV. CONDITION FOR PASSAGE OF A BODY THROUGH THE ZENITH

If the declination of a body is of the same name as the latitude and is exactly equal to the latter (parallel cc_1), then at upper transit the body will pass through the zenith and in so doing will only touch the prime vertical without crossing it. Thus, the *condition for a body passing through the zenith* is that $\delta = \varphi$ and that they have the same name.

In the examples given below, use a hand drawing of the celestial sphere and trace the diurnal motion of the indicated celestial bodies*, determine the phenomena associated with this motion, and answer the questions:

(a) Will the given body rise and set? If so, note the above-horizon portion of the parallel and write the azimuths for its rising and setting in quadrantal and semicircular units.

(b) Will the body cross the above-horizon part of the prime vertical? If so, give the approximate values of h and t on the prime vertical.

(c) Will the body pass through the zenith?

(d) For bodies transiting between the zenith and the pole, determine the approximate values of h and A at the instants of elongations.

(e) Compute the meridian altitude H and the zenith distance Z of a body at the instant of its upper transit (this is done after reading Sec. 9, Item IV).

Examples: 1. $\varphi = 43°$N. (a) $\delta = 15°$N.
(b) ✳ Spica
(c) ✳ Dubhe
(d) ✳ Achernar

2. $\varphi = 20°$S. (a) $\delta = 20°$S
(b) ✳ Rigel
(c) ✳ Arcturus
(d) ✳ Canopus

3. $\varphi = 56°$N. (a) $\delta = 34°$S
(b) $\delta = 56°$N
(c) $\delta = 59°$N
(d) ✳ Vega

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* Declinations of stars are given in Table 5, Sec. 31.

SEC. 9. SOME PROBLEMS ASSOCIATED WITH THE DIURNAL MOTION OF CELESTIAL BODIES

Let us consider the solution of problems for various particular positions of celestial bodies in their diurnal motions. The solutions reduce to an approximate construction of the sphere and the solution of the astronomical triangle in the given particular case.

1. TRUE RISING (SETTING) OF A BODY

It is required to determine the azimuth and hour angle of a body on the celestial horizon (rising or setting) if we know the latitude of the place and the declination of the body.

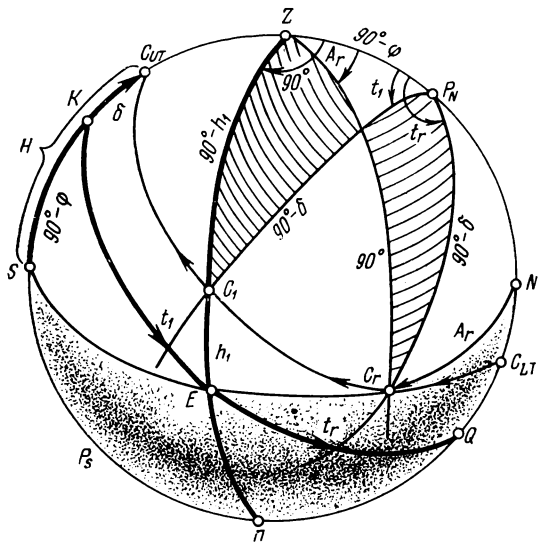

Fig. 17

Construct the sphere for a body lying on the celestial horizon, for example, when rising (Fig. 17), denote the body by C_r and find the approximate values of A_r and t_r.

For this position of the body, the astronomical triangle P_NZC_r turns into a quadrantal triangle ($ZC_r = 90°$), since $h = 0$.

From the cosine formula of the side P_NC_r we get

$$\cos(90° - \delta) = \cos 90° \cdot \cos(90° - \varphi) + \sin 90° \cdot \sin(90° - \varphi) \cdot \cos A_r$$

whence

$$\cos A_r = \sin \delta \cdot \sec \varphi \tag{3.3}$$

By the same formula, but for the side ZC_r, we get

$$\cos 90^\circ = \sin\varphi\cdot\sin\delta + \cos\varphi\cdot\cos\delta\cdot\cos t_r$$

whence

$$\cos t_r = -\tan\varphi\cdot\tan\delta \tag{3.4}$$

The signs in (3.3) and (3.4) are investigated according to the general rules; the investigation shows in what quadrant (I or II) the unknowns t and A will lie; the hour angle of rising (t_r) will always be east, of setting (t_s), west. From the computed hour angles t_r and t_s we can obtain the time of rising and setting of the given celestial body (see Sec. 47).

To check the calculations, you can use the sine formula:

$$\sin t_r\cdot\cos\delta = \sin A_r\cdot\sin 90^\circ$$

or

$$\cos\delta = \sin A_r\cdot\operatorname{cosec} t_r \tag{3.5}$$

Example 4. $\varphi = 42^\circ 37'.4\text{S}$; $\delta = 12^\circ 10'.8\text{N}$, the body is in the western part of the celestial horizon (setting). Determine t_s and A_s.

(a) Investigate the signs of formulas (3.3) and (3.4)

$$\overset{+}{\cos}\, t_s = -\overset{+}{\tan}\,\varphi\cdot\overset{-}{\tan}\,\delta \qquad (t_s < 90^\circ)$$

(b) $$\overset{-}{\cos}\, A_s = \overset{+}{\sec}\,\varphi\cdot\overset{-}{\sin}\,\delta \qquad (A_s > 90^\circ)$$

$\varphi = 42^\circ 37'.4\text{S}$	tan	9.96393	sec	0.13323
$\delta = 12^\circ 10'.8\text{N}$	tan	9.33414	sin	9.32425
	cos	9.29807	cos	9.45748

$t_s = 78^\circ 32'.5\text{W}$ $\qquad 73^\circ 20'.2 = 180^\circ - A_s$

$A_s = \text{S } 106^\circ 39'.8\text{W} = 73^\circ 20'.2\text{NW} = 286^\circ 39'.8$

Check:

cos δ	9.99011	sin A	9.98137
		cosc t	0.00874
	9.99011		9.99011

II. PASSAGE OF BODY ACROSS PRIME VERTICAL

It is required to determine the altitude and hour angle of a body on the prime vertical from known φ and δ.

Extending the parallel of celestial body C, in Fig. 17, to intersection with the prime vertical at the point C_1, we approximately get h_1 and t_1.

From the astronomical triangle $P_N ZC_1$, which in this case becomes right-angled ($A = 90°$), and using the cotangent formula, we get

$$\cot A_1 \cdot \sin t_1 = \cot (90° - \delta) \cdot \sin (90° - \varphi) - \cos (90° - \varphi) \cdot \cos t_1$$

whence for $A = 90°$ or $270°$ we obtain

$$\cos t_1 = \cot \varphi \cdot \tan \delta \qquad (3.6)$$

From the computed hour angle t_1 we can obtain the time of passage of the body across the prime vertical (Sec. 47).

From the cosine formula of the side $P_N C_1$ we obtain

$$\cos (90° - \delta) = \cos (90° - h_1) \cdot \cos (90° - \varphi) + \\ + \sin (90° - h_1) \cdot \sin (90° - \varphi) \cdot \cos A_1$$

whence for $A = 90°$ or $270°$ we obtain

$$\sin h_1 = \sin \delta \cdot \operatorname{cosec} \varphi \qquad (3.7)$$

Formulas (3.6) and (3.7) are investigated as to sign by the general rules; the investigation shows the quadrant that t_1 will be in and the sign of h_1.

These same results may be obtained by using the mnemonic rules of Napier* for a right triangle:

$$\cos t_1 = \cot \; [90° - (90° - \varphi)] \cdot \cot (90° - \delta)$$

and

$$\cos (90° - \delta) = \sin [90° - (90° - h_1)] \cdot \sin [90° - (90° - \varphi)]$$

or

$$\cos t_1 = \cot \varphi \cdot \tan \delta$$

and

$$\sin h_1 = \sin \delta \cdot \operatorname{cosec} \varphi$$

In MT-53, Tables 21a and 21б were computed from formulas (3.6) and (3.7). In these tables, the values of t_1 and h_1 are given to within 0'.1 from the arguments φ and δ (of same name). In MT-63 only Table 21 is based on formula (3.7).

Example 5. $\varphi = 71°19'.5$N; $\delta = 8°41'.6$N; $A = 90°$. Determine t_1 and h_1.

$$\overset{+}{\cos t_1} = \overset{+}{\cot \varphi} \cdot \overset{+}{\tan \delta} \qquad (t_1 < 90°)$$

$$\overset{+}{\sin h_1} = \overset{+}{\sin \delta} \cdot \overset{+}{\operatorname{cosec} \varphi} \qquad (h_1 > 0°)$$

---

* See Appendix III.

$\varphi = 71°19'.5$	cot	9.52891	cosc	0.02349
$\delta = 8°41'.6$	tan	9.18441	sin	9.17940
	cos	8.71332	sin	9.20289
	$t_1 = 87°2'.3E$		$h_1 = 9°10'.8$	

From Tables 21a and 21б we get $t_1 = 87°.0E$, $h_1 = 9°.2$.

III. TRANSIT (MERIDIAN PASSAGE) OF A CELESTIAL BODY. RELATIONSHIPS BETWEEN H, δ, AND φ

For the instant of upper transit of a celestial body, $t = 0°$, $A = 180°$, and $q = 0°$. An inspection of Fig. 17 shows that we can establish an important relationship between the coordinates H, δ and φ. For the body C in the position of upper transit C_{UT} we have arc $CS = H$; arc $KC = \delta$ and arc $KS = 90° - \varphi$; taking this into consideration, we obtain

$$\left.\begin{aligned} H &= 90° - \varphi_N + \delta_N \\ \text{or} \quad \varphi_N &= 90° - H + \delta_N \end{aligned}\right\} \tag{3.8}$$

If the declination is of contrary name to φ, then it obviously must be subtracted; that is,

$$\left.\begin{aligned} H &= 90° - \varphi_N - \delta_S \\ \text{or} \quad \varphi_N &= 90° - H - \delta_S \end{aligned}\right\} \tag{3.9}$$

These relations are widely used in the practical part of the course.

Example 6. Determine H from the given quantities of Examples 1, a and 3, a.

(1) $H = 90° - \varphi_N + \delta_N = 90° - 43° + 15° = 62$ southwards

(2) $H = 90° - \varphi_N - \delta_S = 90° - 56° - 34° = 0°$

Examples

No.	Given		Find
7	$\varphi = 29°37'.6S$,	$\delta = 39°47'.2N$, body on horizon in east	t, A
8	$\varphi = 59°57'.6N$,	$\delta = 7°11'.3N$, body on horizon in west	t, A
9	$\varphi = 55°19'.3S$,	$\delta = 18°1'.1S$, body on east prime vertical	h, t
10	$\varphi = 4°21'7N$,	$\delta = 2°49'.4S$, body on west prime vertical	h, t

SEC. 10 PECULIARITIES IN THE APPARENT DIURNAL MOTION OF CELESTIAL BODIES FOR AN OBSERVER AT THE EQUATOR AND AT THE POLES

I. OBSERVER AT THE EQUATOR ($\varphi = 0°$)

In this case (Fig. 18), the celestial poles P_N and P_S coincide with N and S of the horizon, the celestial axis coincides with the meridian NS, and the equator coincides with the prime vertical. For this reason:

1. The parallels of diurnal motion of all bodies are perpendicular to the horizon and are divided in half by the latter; hence,

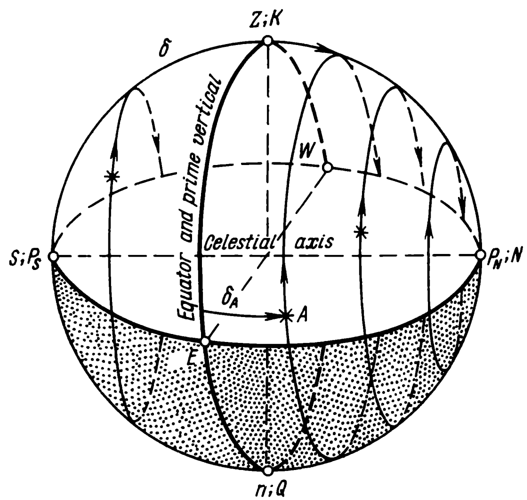

Fig. 18

(a) all celestial bodies (without any exceptions) rise and set ($\delta < 90°$);

(b) all bodies spend the same time above the horizon as below it.

2. No body crosses the prime vertical in its diurnal motion, since $|\delta| > \varphi$; hence, the azimuth of any body may lie only in two halves of the horizon: N for north declination of the body and S for south declination.

3. A body with $\delta = 0°$ moves along the prime vertical in its diurnal motion, and its azimuths will be: east from rising to upper transit, and west from upper transit to setting.

4. At the instant of upper transit, all bodies have $Z = \delta$ and $h = 90° - \delta$.

5. The azimuths of rising (setting) of bodies are equal to their polar distances.

II. OBSERVER AT POLES ($\varphi = 90°$ N OR S)

In this case (Fig. 19):

1. The elevated pole coincides with the zenith point, the depressed pole with the nadir, and the celestial axis with the plumb line, hence:

(a) there is no observer's meridian (Fig. 19, in the plane of an arbitrary meridian);

(b) there are no N, E, S, W points of the horizon; for the north pole, all directions are southwards; for the south pole all directions are northwards.

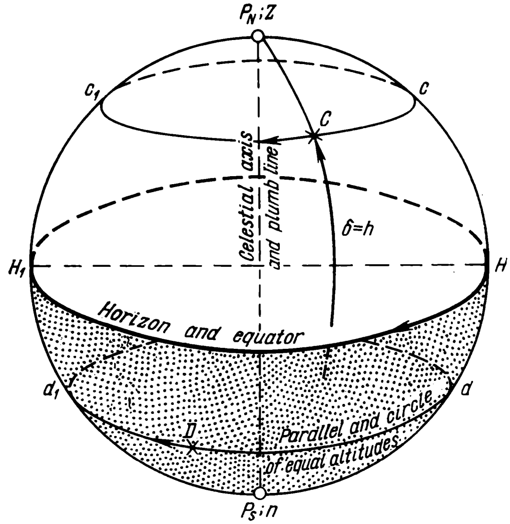

Fig. 19

2. The celestial equator coincides with the true horizon, the parallels of declination with the parallels of altitude, the meridians with the vertical circles.

3. In their diurnal motion, all bodies describe parallels of altitude (h = const). There is no upper and lower transit.

4. The altitude of a body is always equal to its declination.

5. Bodies do not rise and do not set ($\delta > 0°$).

6. The observer never sees bodies whose declination and latitude are of opposite names; but bodies whose declination is of the same name as the latitude are always above the horizon.

Thus, change in latitude of the observer affects the nature of diurnal motion of bodies: as the latitude increases, the angle of

inclination of the parallels to the horizon diminishes becoming zero at the poles. These peculiarities in the motions of bodies exert a great influence on a number of astronomical and climatic phenomena.

SEC. 11. CHANGES IN THE COORDINATES OF BODIES DUE TO THEIR APPARENT DIURNAL MOTION

Above it was noted that in the diurnal motion of the sphere every celestial body changes its place relative to the planes of the horizon

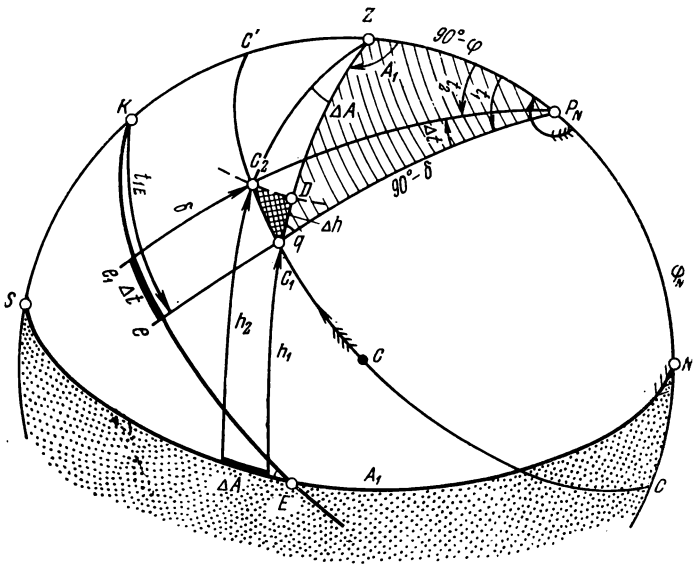

Fig. 20

and the meridian of the observer, which do not take part in the rotation of the sphere.

The diurnal rotation of the sphere, which is a reflection of the earth's rotation, is quite uniform* and so the hour angles of bodies connected with the sphere (reckoned from the observer's meridian) will likewise vary uniformly in proportion to the angle of rotation of the sphere. Hence, the *west hour angle of any body and any point on the sphere will, due to diurnal motion, continuously and uniformly increase from* 0° *to* 360°.

* If we neglect certain fluctuations mentioned in the section dealing with timekeeping.

As we have already discovered, the diurnal motions of bodies occur along parallels with the property $\delta = \text{const}$; therefore, the *declinations of bodies (due to their diurnal motion) will not vary.*

The first point of Aries (Υ), relative to which we reckon right ascensions (or the quantities τ) of bodies, is fixed in the sphere and moves with it; therefore, *α and τ of bodies will not vary due to diurnal motion.*

As has been noted, the coordinates of the horizon system (altitude and azimuth of a body) will undergo continual change as the sphere rotates. Since the rotation of the sphere may be evaluated by the uniformly varying hour angle of the body, we shall consider that the altitude and azimuth of a body are functions of its hour angle.

Let us assume that for an observer with latitude φ a body C with declination δ (Fig. 20) is in motion along the parallel CC'. At some instant the body occupies the position C_1, its altitude will be h_1, the azimuth A_1 and the hour angle t_{1E}. Upon rotation of the sphere through some angle Δt, the hour angle will take on the value $t_2 = t_{1E} - \Delta t$, the altitude $h_2 = h_1 + \Delta h$, the azimuth $A_2 = A_1 + \Delta A$, and the declination will remain without change. Let us derive formulas for the differential variations of altitude and azimuth of a body, or the quantities Δh and ΔA.

I. VARIATION OF ALTITUDE

The relationship $\Delta h = f(\varphi, A\ldots)\,\Delta t$ may be obtained geometrically, by means of a drawing of the sphere, or analytically, by differentiating the formula relating the indicated coordinates. In nautical astronomy, both methods are used in such problems, and so let us determine Δh by both methods. Later on we shall use some one method, mainly the analytical method.

(1) *Geometric solution.*

For sufficiently small increments Δt, the triangle C_1C_2D (Fig. 20) will be small and may be taken as a plane triangle. Here, the angle $C_1 = 90° - q$, the side $C_1D = \Delta h$, and the side C_1C_2 may be expressed in terms of the increment Δt. Indeed, the arc of the parallel C_1C_2 will be $\cos\delta$ times less than the arc of the equator ee_1^* equal to $-\Delta t$ (where the minus sign indicates that the east hour angle is diminishing), that is, $C_1C_2 = -\Delta t \cdot \cos\delta$.

From the right triangle C_1C_2D we get

$$\Delta h = C_1C_2 \cdot \cos(90° - q) = -\Delta t \cdot \cos\delta \cdot \sin q \qquad (3.10)$$

Applying the sine formula, from the astronomical triangle PC_1Z we get

$$\sin q \cdot \cos\delta = \sin A_1 \cdot \cos\varphi \qquad (3.11)$$

* See Appendix II.

Putting (3.11) into (3.10), we finally obtain in the general form

$$\Delta h = -\cos\varphi \cdot \sin A \cdot \Delta t \tag{3.12}$$

From this same triangle C_1C_2D we can also obtain geometrically the variation in azimuth [see formula (3.18)]; this problem is left to the student.

(2) *Analytical solution.*

From the astronomical triangle of the body P_NZC_1 we get a formula that relates the variable h and the independent variable t:

$$\sin h = \sin\varphi \cdot \sin\delta + \cos\varphi \cdot \cos\delta \cdot \cos t$$

where φ and δ are constants.

Differentiating this formula with respect to h and t, we get

$$\cos h \cdot dh = -\cos\varphi \cdot \cos\delta \cdot \sin t \cdot dt$$

whence

$$dh = -\cos\varphi \cdot \frac{\cos\delta}{\cos h} \cdot \sin t \cdot dt \tag{13.13}$$

We reduce this formula to A and φ, which are more convenient for analysis; to do this, we replace the second factor by the sine formula:

$$\frac{\cos\delta}{\cos h} = \frac{\sin A}{\sin t}$$

After substitution into expression (3.13) and replacement of the differentials by finite increments (this is permissible for very small values of Δt), we obtain

$$\Delta h = -\cos\varphi \cdot \sin A \cdot \Delta t \tag{3.14}$$

This formula permits computing the change in Δh for the change in Δt expressed in the same units. In practical work, it is common to express the arc Δh in minutes of arc and Δt in minutes or seconds of time on the basis of the relation $15' = 1$ m or $1' = 4s$. Hence, formula (3.14) will take the form

$$\Delta h' = -15\cos\varphi \cdot \sin A \cdot \Delta t_{\min} \tag{3.15}$$

or

$$\Delta h' = -0.25\cos\varphi \cdot \sin A \cdot \Delta t_{\text{sec}} \tag{3.16}$$

These formulas have been used in the compilation of special tables 15a, 15б, 15в, MT-63, which are used for checking measurements of altitudes and for deriving their accuracy.

Formula (3.14) written in the form of a derivative

$$\frac{dh}{dt} = -\cos\varphi \cdot \sin A \tag{3.17}$$

is *the rate of change of altitude of a body* for a given latitude and a given portion of the diurnal circle, or the rate of motion of the body in altitude.

Example 7. Determine the change in altitude at

$$\varphi = 57°.5\text{N for } A_{\odot} = 42°, \qquad \Delta t = +2m.5\,(150\text{s})\text{W}.$$

(a) Using a slide rule and Table 6a, MT-63,

$$\Delta h' = -0.25 \cdot 0.537 \cdot 0.669 \cdot 150 = -13'.5.$$

(b) Using Table 15a, MT-63:

$$\Delta h' = -5'.4/1\text{m} \cdot 2\text{m}.5 = -13'.6.$$

Let us analyze formulas (3.14) and (3.17).

(a) Let us determine in what latitudes the altitudes of the body vary most; to do this, put A = const in formula (3.14). For $\varphi = 0°$, $\cos\varphi = 1$ and $\Delta h = -\sin A\,\Delta t$; but if $\varphi = 90°$, then $\Delta h = 0$, that is, *the greatest change in altitude for the observer will be at the equator, and the least change at the pole*, where celestial bodies do not alter altitude, they move in parallels of altitude.

A similar conclusion may be drawn from an analysis of formula (3.17): on the equator the rate of motion of a body in altitude is greatest; at the poles it is zero. It should be pointed out that we will have the highest rate of motion in altitude for a body moving along the celestial equator for an observer at latitude $\varphi = 0$. Indeed, $\cos\varphi = 1$; $\sin A = 1$ and $\frac{dh}{dt} = -1$ or $\Delta h = -\Delta t$; the altitude varies exactly as the hour angle.

(b) Analyzing formula (3.14) for a given latitude, we find that variation of altitude will be nonuniform and will depend exclusively on the body's azimuth. For $A = 90°$ or $270°$, $\sin A = \pm 1$ and $\Delta h = \pm \cos\varphi \cdot \Delta t$; hence, *on the prime vertical the variation in altitude will be greatest and will be proportional to the time*, which means that the altitude varies uniformly.

For the azimuth of a body not equal to 90° (270°), but close to these values, $\sin A$ varies very slowly; for this reason, the variation in altitude will be practically uniform close to the prime vertical as well.

From this peculiarity of motion of celestial bodies there follows an important practical consequence: if several altitudes of a body are measured close to the prime vertical and if the instants are clocked, the arithmetical mean of the altitudes will correspond to the mean instant (or the mean hour angle).

At $A = 0°$ (180°), $\sin A = 0$ and $\Delta h = 0$. Therefore, *on the observer's meridian the altitude of the body does not change and the body moves parallel to the horizon. Near the meridian, the altitude of the body varies nonuniformly* due to the nonuniform variation of $\sin A$

for small azimuths. Thus, for a large interval of time (Δt), the mean altitude will not correspond to the mean instant (hour angle), and theoretically it is not permissible to compute the arithmetical mean [see formula (3.27)].

(c) *For a body that does not cross the prime vertical, the greatest variation in altitude will be at elongation*, where from formula (3.10) we have $\Delta h = \pm \cos\delta \cdot \Delta t$; here Δh will vary in *uniform* proportion to Δt.

(d) Analyzing formula (3.17), we can determine the maximum and minimum altitude in diurnal motion. Putting $\frac{dh}{dt} = 0$ or $-\cos\varphi \times \sin A = 0$, we find that the extremal values of this function will occur at $A = 0°$ and $180°$. From an analysis of the second derivative [formula (3.25)] it follows that $\frac{d^2h}{dt^2} < 0$ for $A = 180°$ and $\frac{d^2h}{dt^2} > 0$ for $A = 0°$; hence, the altitude will be greatest at upper transit. Thus the analysis confirms what we derived graphically.

II. VARIATION OF AZIMUTH

To obtain variation of azimuth, let us apply the analytical method. To do this, take the formula

$$\cot A \cdot \sin t = \cos\varphi \cdot \tan\delta - \sin\varphi \cdot \cos t$$

which relates A and t, and differentiate it with respect to these variables, holding the other quantities constant:

$$-\frac{\sin t}{\sin^2 A}\, dA + \cot A \cdot \cos t \cdot dt = \sin\varphi \cdot \sin t \cdot dt$$

whence

$$dA = \frac{(\cot A \cdot \cos t - \sin\varphi \cdot \sin t)\sin^2 A}{\sin t}\, dt$$

Putting $\sin A$ in brackets in the numerator and noting that the expression obtained, $\cos A \cdot \cos t - \sin\varphi \cdot \sin t \cdot \sin A$ is, by the cosine formula, equal to $-\cos q$, we get

$$dA = -\cos q \cdot \frac{\sin A}{\sin t}\, dt$$

Substituting, in accord with the sine formula, $\frac{\sin A}{\sin t} = \frac{\cos\delta}{\cos h}$ and passing to finite increments, we obtain

$$\Delta A = -\cos q \cdot \cos\delta \cdot \sec h \cdot \Delta t \tag{3.18}$$

For the purpose of analysis, it is more convenient to transform this formula so as to eliminate q; to do this, we write (by the formula of five parts)

$$\cos q \cdot \sin(90^\circ - \delta) = \sin(90^\circ - h) \cdot \cos(90^\circ - \varphi) - \\ - \cos(90^\circ - h) \cdot \sin(90^\circ - \varphi) \cdot \cos A$$

whence

$$\cos q \cdot \cos \delta = \cos h \cdot \sin \varphi - \sin h \cdot \cos \varphi \cdot \cos A$$

Substituting and dividing by $\cos h$, we finally get

$$\Delta A = -(\sin \varphi - \tan h \cdot \cos \varphi \cdot \cos A) \cdot \Delta t \qquad (3.19)$$

In the expressions (3.18) and (3.19), the minus sign indicates that with increasing practical hour angle (less than 180°), the azimuth of the body (in semicircular reckoning) will decrease, and vice versa.

Formulas (3.18) and (3.19) may also be written in the form of derivatives:

$$\frac{dA}{dt} = -\cos q \cdot \cos \delta \cdot \sec h \qquad (3.20)$$

or

$$\frac{dA}{dt} = -(\sin \varphi - \tan h \cdot \cos \varphi \cdot \cos A) \qquad (3.21)$$

These expressions represent the *rate of change of the azimuth of a body* under different conditions.

For computing the variation of azimuth, expression (3.19) may be represented in the form

$$\Delta A^\circ = -0.25(\sin \varphi - \tan h \cdot \cos \varphi \cdot \cos A)\,\Delta t_{\min} \qquad (3.22)$$

Formula (3.22) was used to compile Table 15r, MT-63, which gives change of azimuth in one minute of time. To obtain ΔA for the interval ΔT, the tabulated value is multiplied by ΔT. This may be done with a slide rule.

Example 8. Determine the change of azimuth in 5 minutes at $\varphi = 18^\circ$N for $h = 53^\circ$, $A = 150^\circ$ (30°SE).

(1) Using the slide rule, we get

$$\Delta A^\circ = +0.25(0.309 + 1.33 \cdot 0.951 \cdot 0.866) \cdot 5 = +1^\circ.76$$

(in 5 minu es)

(2) From Table 15r, MT-63, we obtain

$$\Delta A = 0.3^\circ/\text{m} \cdot 5\text{m} = 1^\circ.5$$

An investigation of the formulas obtained shows that:

(a) In the general case the azimuth of a body varies *nonuniformly.* In special cases, however, its variation is sometimes uniform; indeed, for $h = 0°$ or $A = 90°$, $\Delta A = -\sin\varphi\cdot\Delta t$, and for $\varphi = 90°$, $\Delta A = -\Delta t$, which means that at rising, setting, and on the prime vertical the azimuth varies in proportion to the hour angle (i.e., the time), while at the poles the azimuths of bodies vary just as their hour angles do.

(b) The rate of change of azimuth of a body will be different over different sections of the diurnal path and also at different latitudes.

At a given latitude, i.e., for $\varphi =$ const, the *highest rate of change of the azimuth of a body will be near its upper transit.* Indeed, for $A = 180°$ and $h = H$, $\tan H$ will have its greatest value and $\frac{dA}{dt}$ will be a maximum. For upper transit ($q = 0$), from (3.18) we get

$$\Delta A° = \cos\delta\cdot\sec H\cdot\Delta t° = 0.25\cos\delta\cdot\sec H\cdot\Delta t_{\min} \tag{3.23}$$

this same formula is also used to compute ΔA near transits.

The smallest rate of change of azimuth will be between the prime vertical and the rising of the body, and if it does not cross the prime vertical, then near the instants of elongations. Indeed, investigating the signs of (3.19), we see that the second term has a plus sign between rising and the prime vertical and in the formula we get a difference; in all other cases both terms have a minus sign and are summed. For the time of elongations $q = 90°$ and $\Delta A = 0$.

In Fig. 21, constructed in the plane of the horizon NESW, we see that to identical sections of the diurnal path of a body a_1a_2, a_2a_3, a_4a_5, a_6a_7, . . . there correspond different changes of azimuth on the horizon, and $\Delta A_1 > \Delta A_4 > \Delta A_3$.

Thus, near upper transit the azimuths of celestial bodies vary with maximum speed, and if it is required to obtain the greatest difference of azimuths during one and the same interval of time (as in the lines of position method), the body should be observed closer to its upper transit.

(c) When the observer's latitude changes, the nature of the variation of azimuth also changes.

As has already been pointed out, for all bodies, irrespective of their position, at the poles ($\varphi = 90°$), $\Delta A = -\Delta t$.

At the equator ($\varphi = 0°$) we have

$$\Delta A = \tan h\cdot\cos A\cdot\Delta t \tag{3.24}$$

For all practical purposes, this expression likewise holds true for small latitudes where $\varphi \neq 0$. At rising and setting of a body, $h = 0$ and $\Delta A = 0$; near the instants of transits, the altitude of the body will be great and ΔA may be many times greater than Δt.

As may be seen, *in low latitudes azimuth varies extremely nonuniformly*: either very rapidly or hardly at all for a long time. This is important to bear in mind in practical work.

(d) The azimuths of bodies crossing the prime vertical increase from 0° to 360° continuously, though nonuniformly. For bodies not crossing the prime vertical, the azimuths (after lower transit)

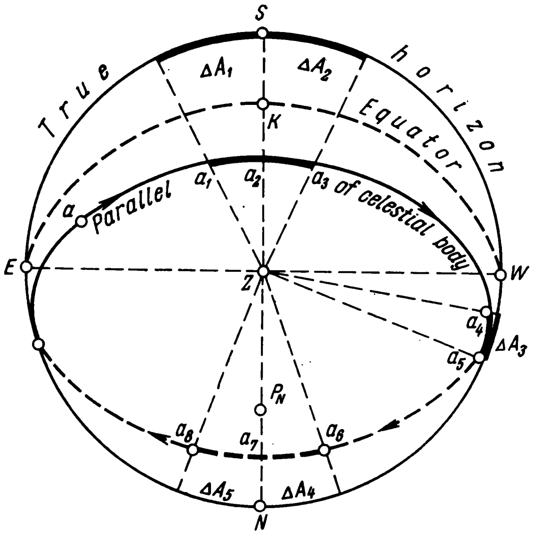

Fig. 21

increase from 0° to A_{max} at the instant of elongation and then diminish to 0° at the time of upper transit. A similar change occurs in the west half.

Not in all cases are formulas (3.14) and (3.19) sufficiently exact, since they give increments of altitude and azimuth only for very low values of Δt.

When more exact values of the increments Δh and ΔA for finite Δt are needed, one should apply Taylor's formula for determining the increments of functions:

$$f(x+\Delta x)-f(x)=\Delta f=f'(x)\cdot\Delta x+\frac{f''(x)}{2!}\cdot\Delta x^2+\frac{f'''(x)}{3!}\cdot\Delta x^3+\ldots (*)$$

Let us consider the increment in altitude as the only one of practical value and let us confine ourselves to the first two terms of the expression(*):

$$\Delta h=\frac{dh}{dt}\cdot\Delta t+\frac{1}{2}\frac{d^2h}{dt^2}\Delta t^2+\ldots$$

We already have the first derivative:

$$\frac{dh}{dt} = -\cos\varphi \cdot \sin A$$

The second derivative $\frac{d^2h}{dt^2}$ is obtained by differentiating the first with respect to h and A (as functions of t):

$$\frac{d^2h}{dt^2} = -\cos\varphi \cdot \cos A \frac{dA}{dt}$$

Substituting the earlier obtained $\frac{dA}{dt}$, we get

$$\frac{d^2h}{dt^2} = \cos\varphi \cdot \cos A \cdot \cos\delta \cdot \cos q \cdot \sec h \tag{3.25}$$

or

$$\frac{d^2h}{dt^2} = (\sin\varphi - \cos\varphi \cdot \tan h \cdot \cos A) \cdot \cos\varphi \cdot \cos A \tag{3.26}$$

Thus, a more exact expression for the increment in altitude is

$$\Delta h = -\cos\varphi \cdot \sin A\Delta t + \frac{1}{2}\cos\varphi \cdot \cos A \cdot \cos q \cdot \cos\delta \cdot \sec h\Delta t^2 \tag{3.27}$$

If $\frac{dh}{dt}$ is the rate of change of altitude, then the second derivative $\frac{d^2h}{dt^2}$ is the *acceleration of the body in altitude*, or the acceleration in change of altitude. The acceleration will be greatest for A, q close to 180° or 0°, that is, on the observer's meridian or near it. It is to be noted that in low latitudes, for high altitudes of a body, the acceleration is considerable and cannot be ignored. For instance, under these circumstances and for Δt = 5-10 minutes the second term may be of the order of 3′ and more.

The above-mentioned peculiarities in the variation of coordinates of different systems show that each of these systems is convenient for solving specific practical problems; thus:

(1) hour angles varying uniformly are convenient for measuring time;

(2) horizon coordinates vary markedly with the latitude and hour angle, and are also readily obtainable from observations; for this reason, these coordinates are convenient for practical observations when obtaining geographic coordinates and directions;

(3) the coordinates α and δ do not depend on the diurnal motion (connected with the sphere, as it were), and therefore are convenient for compiling star catalogues and maps, lists of apparent positions of stars and in the study of diverse movements of celestial bodies over the sphere.

SEC. 12. THE APPARENT DIURNAL MOTION OF CELESTIAL BODIES EXPLAINED

Above we considered the diurnal motion of bodies as if the earth and the observer on it were stationary, while the celestial sphere with its bodies was in uniform rotation about the earth, and that this rotation of the sphere accounted for the variations in azimuth, altitude and the hour angle that we actually observe.

Even in ancient times there were thinkers that doubted the correctness of the then accepted doctrine of a stationary earth in space,

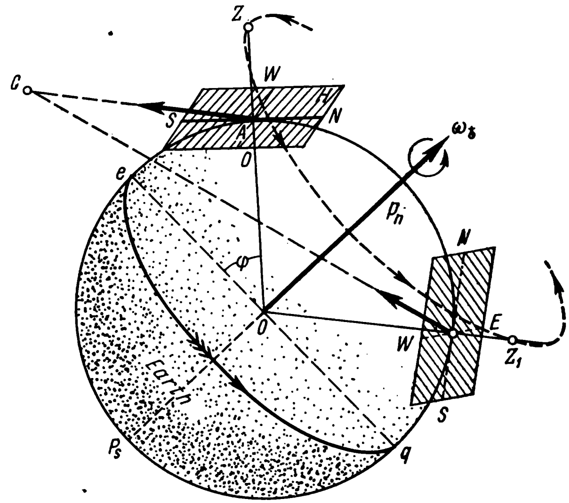

Fig. 22

but it was only in the sixteenth century that Copernicus explained the apparent diurnal motion of celestial bodies (rising and setting, variations in azimuth and hour angle) *as due to the rotation of the earth about its axis.* Modern science is in possession of a wealth of unquestionable and cogent proofs of the earth's rotation, as, for instance, the variation of the force of gravity with change in latitude, deviation eastwards of a freely falling body, the Foucault pendulum, the behaviour of the axis of a free gyroscope, the ellipsoidal shape of the earth, and others.

We will not dwell on these proofs, which are not included in the course of nautical astronomy, and will only demonstrate how variations in azimuth, altitude and the hour angle of a celestial body can be explained by the earth's rotation.

Referring to Fig. 22, let the vector of angular velocity of the earth ω_{δ} be laid off along the earth's axis of rotation from its centre O.

The vector is laid off towards P_N since the rotation of the earth occurs counterclockwise as viewed from P_N. The quantity $\omega_♁$ is equal to 0.000073 rad/s, or $\frac{360°}{24h}$ or $\frac{900'}{1h}$.

The plumb line OZ of an observer A fixed on the earth will describe a conical surface as the earth rotates. The noon line NS connected to the plumb line will likewise describe a conical surface about the earth's axis, which is to say that the true horizon H will be constantly changing its position in space, as a result of which the direction

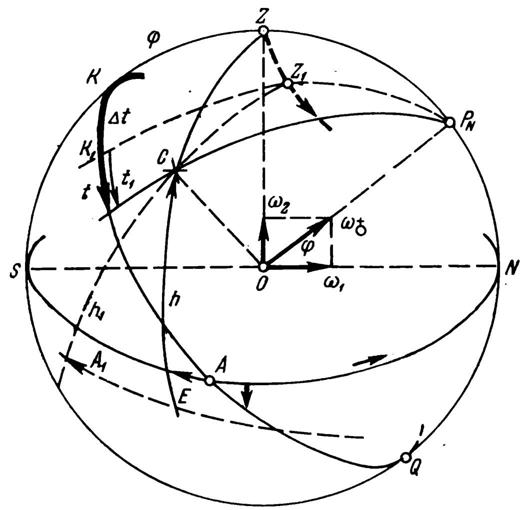

Fig. 23

towards some fixed body C (in Fig. 22 the azimuth changed from south to north in 12 hours) will also change.

To analyze the motion of the horizon, consider an auxiliary celestial sphere which we will now consider fixed with centre at the centre of the earth (Fig. 23). Through its centre draw a plane parallel to the plane of the horizon H (see Fig. 22); we obtain the celestial horizon.

The vector of angular velocity of the earth $\omega_♁$ will now be directed along the celestial axis at an angle φ to the horizon. Decomposing it into two components: ω_1 directed along the meridian NS and ω_2 along the plumb line OZ, we get (from the drawing)

$$\omega_1 = \omega_♁ \cdot \cos\varphi \tag{3.28}$$

and

$$\omega_2 = \omega_♁ \cdot \sin\varphi \tag{3.29}$$

The first formula shows that due to the rotation of the earth the plane of the true horizon is in constant rotation about the meridian with a velocity ω_1, i.e., the eastern half of the horizon is continuously tilting while the western half is continuously rising. The vector ω_1 is called the **horizontal component** of terrestrial rotation.

The second formula shows, in addition, that the plane of the horizon is also in rotation about the plumb line with a velocity ω_2 in the direction indicated by the arrow. The vector ω_2 is called the **vertical component** of terrestrial rotation.

On the equator, when $\varphi = 0$, $\omega_1 = \omega_{\oplus}$ and $\omega_2 = 0$, that is, on the equator we only have dip of the horizon without rotation about OZ. For $\varphi = 90°$, $\omega_1 = 0$, $\omega_2 = \omega_{\oplus}$ which means that at the poles, the horizon does not tilt but only rotates in a single plane about the plumb line. Obviously, in the first case, the altitudes of bodies should change with maximum rapidity, while in the second case they should not change at all.

Variation of azimuth is similar. Thus, the rotation of the earth accounts for the diurnal motion of celestial bodies just as well as the imaginary rotation of a sphere about a fixed earth does; however, it is mathematically more convenient to introduce a fixed auxiliary "celestial sphere".

CHAPTER 4

APPARENT ANNUAL MOTION OF THE SUN

SEC. 13. A CHARACTERISTIC OF THE APPARENT MOTION OF THE SUN. THE ECLIPTIC

From direct observations of the diurnal motion of the sun (astronomical symbol ⊙) and stars during the year or at least for several weeks, one notices the following:

1. A star rises and sets every day at the same points on the horizon; the rising point and setting point of the sun are in constant motion: for north latitudes they move northwards in the spring and southwards in the autumn.

2. The meridian altitude of any star remains the same from day to day; the meridian altitude of the sun is constantly changing: the altitudes of the sun in winter and summer differ markedly. For example, at Leningrad at the end of December the sun rises just slightly above the horizon at noon with maximum altitude about 6°.5; a half year later, at the end of June, its maximum altitude is 53°.5.

3. For one and the same hour at night, the stellar sky does not remain constant and is continually changing. A northern observer in the evening sees the constellation Orion near the eastern horizon in the middle of November, while in the middle of January, it is in the south rather high above the horizon, and in the middle of March, in the west near the horizon. Thus, at night one sees different sections of the sky with different constellations.

From these and other facts it may be concluded that in addition to its diurnal motion with the entire celestial sphere and with the "fixed" stars, the *sun has its own apparent motion* over the celestial sphere.

The first two peculiarities in the apparent motion of the sun indicate that the sun approaches the equator and recedes from it, which is to say that it changes its declination ($\delta_{\odot}$); the third factor indicates that the sun is also in motion along the equator, which means that its right ascension ($\alpha_{\odot}$) is undergoing change. Numerous observations have shown these phenomena to recur periodically, every year. Whence the name, the **apparent annual motion of the sun.**

If $\alpha_{\odot}$ and $\delta_{\odot}$ are determined for every day of the year and if these positions of the sun are plotted on the celestial sphere, they will all be found to lie on a large circle whose plane is inclined to that of the equator at a constant angle ε; this large circle, which represents the apparent path of the annual motion of the sun among the stars

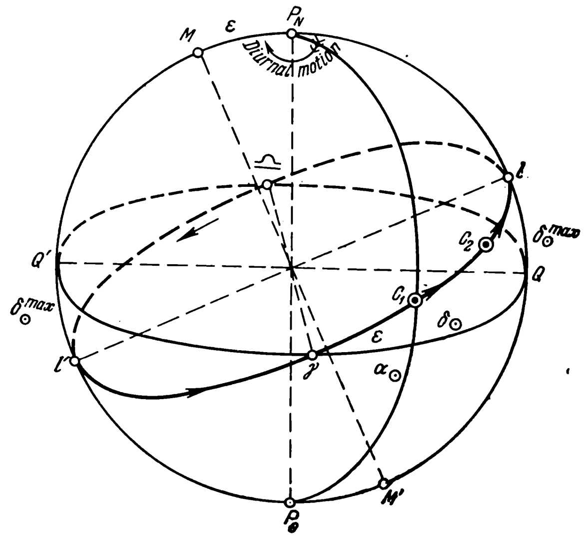

Fig. 24

is called the **ecliptic*** and the angle ε, equal to 23°27′, is called the **obliquity of the ecliptic** (Fig. 24).

The diameter MM', perpendicular to the plane of the ecliptic is called the axis of the ecliptic, while points of its intersection with the sphere M and M' are the poles of the ecliptic.

The pole of the ecliptic closest to the north celestial pole P_N is called the **north pole of the ecliptic** (M), the other pole, closest to P_S, is the **south pole of the ecliptic** (M'). The north pole of the ecliptic is situated in the constellation Draco in the direction from Polaris to Vega; its coordinates are $\alpha = 270°0'$; $\delta = 66°33'$N.

In Fig. 24, constructed on the plane of the meridian $P_N Q P_S Q'$, the great circle $\Upsilon\, l \,\text{♎}\, l'$ is the ecliptic; $Q'Q$ is the equator; the arrow at the pole indicates the direction of diurnal rotation of the sphere. The apparent annual motion of the sun in the ecliptic is *counter* to the diurnal rotation of the sphere, it is *direct motion* as indicated by arrows on the ecliptic.

* This circle is called the ecliptic because eclipses of the moon and sun can occur only when the moon is located on this circle or near it.

As has been pointed out, the sun makes a complete revolution around the ecliptic in roughly one year, that is, 365 days; hence, the sun covers about 1° a day along the ecliptic. This figure varies, since, as we shall see later on, the motion of the sun along the ecliptic is not quite uniform.

Since the equator divides the ecliptic in half, half of the year the sun is in the northern hemisphere (♈ l ♎) and has north declination, while the other half of the year it lies in the southern hemisphere (♎ l' ♈) with south declination.

The points of intersection of the ecliptic and the equator (**equinoctial**) are called **equinoctial points: the vernal equinox,** ♈ (or the first point of Aries), in which the sun passes from the southern hemisphere into the northern hemisphere, and the **autumnal equinox,** ♎ (or the first point of Libra), in which the sun passes from the northern hemisphere into the southern hemisphere, i.e., changes its declination from north to south. These points are called equinoctial points because on the day that the sun (moving in the ecliptic) reaches these points, its declination $\delta = 0°$, the portion of the sun's diurnal circle above the horizon will be equal to that below the horizon and for the inhabitants of the earth day is equal to night.

The first of these points, ♈, is called the vernal (or March) equinox, and the second, ♎, the autumnal (or September) equinox, because the sun in the northern hemisphere of the earth crosses the first in the spring, about March 21, and the second in the autumn, about September 23.

In the northern hemisphere points of the ecliptic l and l', 90° from the equinoctial points, are called *solsticial points*: *summer* (or June) *solstice* l when the sun is in the north, and *winter* (or December) *solstice* l' when the sun is in the south. The sun passes through the first point in summer, about June 22, the second in winter, about December 22. They are called solsticial points because the sun appears to move parallel to the equator. At these points, the sun's declination is a maximum, arc $Ql = Ql' = \varepsilon = 23°27'$N or S— north at point l and south at point l'.

From the foregoing and on the basis of Fig. 24 we can now form a table of positions of the sun on the sphere (Table 1).

Table 1

Date	Point of ecliptic	$\delta_{\odot}$	$\alpha_{\odot}$
21 March	Vernal equinox (♈)	0°	0°
22 June	Summer solstice (l)	23°.5N	90°
23 September	Autumnal equinox (♎)	0°	180°
22 December	Winter solstice (l')	23°.5S	270°

The apparent annual path of the sun passes through a belt of the sphere which includes 12 well-observed constellations, so that the sun spends about a month in each one. In antiquity this belt, which occupies about 8° either side of the ecliptic, was called the **belt** or **ring of the Zodiac.** The moon and most of the planets also move in this belt.

The zodiacal constellations have special symbols, which also indicate the months. The sun passes through them in the following order (♈ stands for March, the others in that order).

Spring	Summer
Pisces the Fishes ♓ Aries the Ram ♈ Taurus the Bull ♉	Gemini the Twins ♊ Cancer the Crab ♋ Leo the Lion ♌
Autumn	Winter
Virgo the Virgin ♍ Libra the Balance ♎ Scorpio the Scorpion ♏	Sagittarius the Archer ♐ Capricornus the Goat ♑ Aquarius the Water Bearer ♒

Numerous exact observations have established the mean duration of a revolution of the sun along the ecliptic relative to ♈. This is the so-called **tropical* year** equal to 365.2422 mean solar days or 365d5h48m46s..

A tropical year is thus the elapsed time between two successive passages of the centre of the sun through the vernal equinox (♈).

The joint motions of the sun (its annual motion in the ecliptic and its diurnal motion about the celestial axis together with the entire sphere) produce the total apparent motion of the sun in a spiral, the intervals between the turns of which are the diurnal variations in the declination of the sun, which diminish as the declination increases, that is, as the sun recedes from the equator (Fig. 25).

A complete circuit of the sun relative to the observer's meridian will obviously be longer than a circuit of a fixed star by the amount of diurnal displacement of the sun—1° or, to be more exact, 3m56s.56; therefore, according to a clock that indicates the rotation of the sphere (the point ♈) the sun's transit will lag each day by that amount.

* Tropic comes from the Greek meaning "turn". At the extreme parallels (tropics) the sun turns in its apparent annual motion. The parallel of its diurnal motion ceases to recede from the equator and begins to approach it.

The extreme parallels described by the sun in its diurnal motion during the days of the summer (l) and winter (l') solstice and which are distant from the equator KQ by the amount of greatest declination of the sun, $\delta_{max} = 23°27'$ are called the **Tropic of Cancer** (the northernmost parallel of latitude a_1a) and the **Tropic of Capricorn** (the southernmost, b_1b) according to the constellations. The parallels of terrestrial latitude for $\varphi = \varepsilon = 23°27'$N and S have the same names.

The obliquity of the ecliptic (ε) may be computed from the formula (see Fig. 24) $\tan \varepsilon = \tan \delta_{\odot} \operatorname{cosec} \alpha_{\odot}$ and may also be obtained

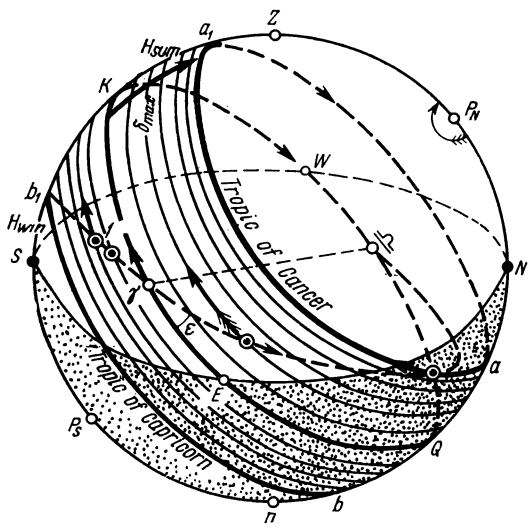

Fig. 25

by measuring the meridian altitude of the sun or its meridian zenith distance at the instant of upper transit during the days of the summer and winter solstices; then, irrespective of any knowledge of the latitude, we will have (see Fig. 25)

$$\varepsilon = \delta_{\odot\, max} = \frac{H_{summer} - H_{winter}}{2} \tag{4.1}$$

However, if the latitude is known, it is sufficient to observe H or Z on one of the days of the solstice and then we have

$$\varepsilon = \delta_{\odot\, max} = \varphi - Z \tag{4.2}$$

SEC. 14. ECLIPTIC COORDINATES

The ecliptic, being a great circle of the sphere, serves as a basis for the fourth, so-called ecliptic system of coordinates. The other fundamental circle in this system is that which passes through the poles of the ecliptic and the vernal equinox (Fig. 26). The circles that pass through the poles of the ecliptic are called **circles of latitude.** Hence, the second fundamental circle in this system is the

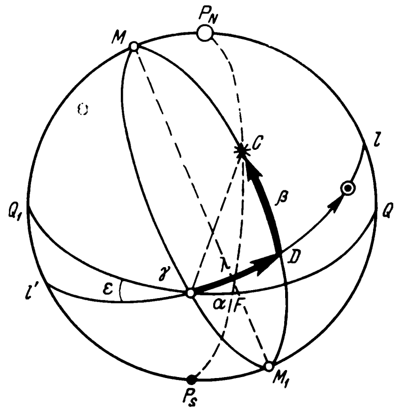

Fig. 26

circle of latitude of Aries. The position of any point of the sphere in this system is defined by the ecliptic coordinates: **ecliptic latitude** and **longitude.**

The ecliptic latitude of a celestial body (β) *is the arc of the circle of latitude of the body from the ecliptic to the place of the body.*

The latitude may be *north* or *south* and is reckoned from 0° to 90°.

The ecliptic longitude of a body (λ) *is the arc of the ecliptic from the vernal equinox* (γ) *to the circle of latitude of the body*, which is always reckoned in the direction of the sun's proper motion from 0° to 360°.

From the foregoing it follows that the latitude of the sun is always 0°, since the sun is always located on the ecliptic. The longitude of the sun, designated by L, is constantly though not uniformly increasing by approximately 1° per 24 hours.

Conversion from coordinates α, δ to λ and β is done by the formulas of spherical trigonometry as applied to the triangles $\gamma\ CD$ and $\gamma\ FC$ which connect these two systems (Fig. 26). Ecliptic coordinates are used chiefly in theoretical astronomy for computing planetary orbits and ephemerides.

SEC. 15. GEOGRAPHIC (CLIMATIC) ZONES. SEASONS

The amount of thermal energy received from the sun by various sections of the earth's surface depends mainly on the angle of incidence of the solar rays on the earth's surface, i.e., on the concentration of rays on a given area of the earth's surface. Indeed, despite the fact that the total number of hours that the sun is above the horizon during the year is roughly the same for all latitudes, the polar regions are appreciably colder than the equatorial regions. The angle of incidence of the rays depends on the altitude of the sun above the horizon, while the altitude, in turn, is dependent on the angle of tilt of the equator to the horizon (see Fig. 25) and on the declination of the sun, that is, on the relationship of φ and $\delta_{\odot}$. Therefore, from the viewpoint of astronomy, the climatic belts of the earth are based on peculiarities in the diurnal motion of the sun during the year and are distinguished by the following features.

The **torrid zone** includes regions of the earth's surface in which the sun during the year passes through the zenith at noon twice (on the boundaries of the zone, once).

The **temperate zone** includes regions where the sun is never in the zenith; yet every day the sun rises and sets.

The **frigid zone** is characterized by the fact that during some days the sun never sets, during others it never rises.

Proceeding from these features of the zones and applying to the sun the conditions of passage of a body through the zenith ($\varphi = \delta_{\odot}$), rising and setting ($\delta_{\odot} < 90° - \varphi$), and nonsetting and nonrising ($\delta_{\odot} \geqslant 90° - \varphi$), we obtain the following boundaries of the five climatic or geographic zones: torrid, between latitudes 23°27′N and 23°27′S, two temperate zones between 23°27′ and 66°33′N and S, and two frigid zones between 66°33′ and the poles.

At a given place, the altitude of the sun above the horizon changes appreciably during the year. Thus, in medium latitudes the altitude of the sun at noon varies by 46°54′ during the year. As a result, the amount of heat energy received by a given point of the earth's surface increases and decreases, thus producing the *changing seasons* of the year. The generally accepted astronomical signs of the beginning and end of the seasons are the relationship of the magnitude and sign of the declination of the sun and the latitude of the place. *Spring* and *summer* occur when $\delta_{\odot}$ and φ are of the same name, and *autumn* and *winter*, when they are of different names.

On the basis of the foregoing we get the following dates which specify the seasons for the northern hemisphere of the earth:

spring from 21.03 to 22.06; summer from 22.06 to 23.09;
autumn from 23.09 to 22.12; winter from 22.12 to 21.03.

In the torrid zone, the change of season means something quite different from what it does in medium and high latitudes; for that reason, in the torrid zone one distinguishes the rainy season and the dry season.

SEC. 16 THE DIURNAL AND ANNUAL MOTION OF THE SUN FOR OBSERVERS IN DIFFERENT LATITUDES

As we already know, the difference in phenomena associated with the diurnal motion of some body depends on the relationship between the latitude (φ) of the observer and the declination (δ) of the body. Since the declination of the sun varies from 23°.5N to 23°.5S during the year, it is obvious that phenomena associated with the apparent diurnal motion of the sun are constantly undergoing change during the year for any observer on the earth.

First of all, note that the following will be common to all terrestrial observers, irrespective of their latitude: on the days of the vernal and autumnal equinox, 21 March and 23 September, day equals night, and for all inhabitants the sun will rise near the point E and will set near W on the horizon; the reason is that during these days $\delta_{\odot} \approx 0°$ and the diurnal motion of the sun is roughly along the equator.*

In addition, from 21 March to 23 September, when the declination of the sun is north, sunrise (if it occurs in the given latitude) will be in the NE quadrant and sunset in the NW quadrant of the horizon. During the remaining time, from 23 September to 21 March, when the sun has south declination, sunrise will be in the SE quadrant and sunset in the SW quadrant of the horizon.

I. FOR AN OBSERVER ON THE EQUATOR; $\varphi = 0°$ (FIG. 27)

1. Day is always equal to night since the planes of all parallels are perpendicular to the plane of the horizon and are divided by the latter into two equal parts.

2. On equinox days, 21.03 and 23.09, $\delta_{\odot} = 0° = \varphi$ and the sun passes through the zenith at noon.

3. After 21.03 the declination of the sun becomes north and increases, the parallel of diurnal motion of the sun gradually recedes

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* The value $\delta_{\odot} = 0°0'$ is an instantaneous value, but during the equinoctial 24-hour period δ changes by 23'.4 in autumn and by 23'.7 in spring. As a result, the sun does not move exactly along the equator but along a curve inclined to the equator.

from the equator (which is also the prime vertical) towards P_N and the zenith distance at the instant of upper transit (which is always equal to the declination) $Z = \delta_{\odot}$ or $H = 90° - \delta_{\odot}$ also increases, reaching a maximum on the day of summer solstice, 22.06. On this day, at the instant of upper transit, $Z = \delta_{max} = 23°27'$N and the meridian altitude H will be a minimum, equal to 66°33′N. On this day, the diurnal motion of the sun describes the tropic of Cancer (a_1a).

After 22.06 the declination of the sun begins to diminish and the parallel of diurnal motion of the sun again approaches the equator;

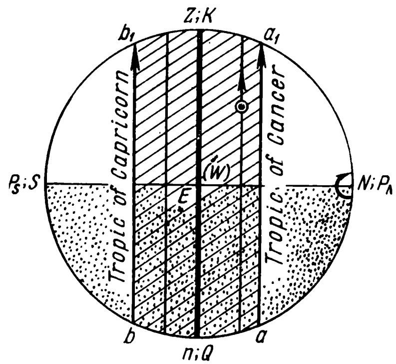

Fig. 27

the zenith distance Z diminishes at upper transit, and the meridian altitude H increases.

After 23.09 we observe the very same shift in the parallel of diurnal motion of the sun, but this time in the direction of P_S with greatest recession from the equator, 23°27′S, on 22.12.

4. At sunrise and sunset the azimuth of the sun is numerically equal to the polar distance of the sun $(90° - \delta_{\odot})$.

5. The sun in its diurnal motion never crosses the prime vertical because we always have $\delta_{\odot} > \varphi$. Therefore, during the 24-hour period, the azimuth of the sun will be only in two quadrants of the horizon: NE (before upper transit) and NW (following transit) for north declination (from 21.03 to 23.09) and in SE (before transit) and SW (following transit) for south declination of the sun (from 23.09 to 21.03). On equinox days, from sunrise to transit, the azimuth of the sun will be east, at the instant of transit A instantly changes 180° and becomes west.

II. FOR AN OBSERVER BETWEEN THE TROPICS IN THE TORRID ZONE, $\varphi < 23°27'$N OR S (FIG. 28)

1. The sun rises and sets every day since $|\delta_\odot| < 90° - \varphi$.
2. Twice a year, $\delta_\odot$ will definitely be equal to the latitude of any observer in the torrid zone, and on such days the sun at noon will pass through the zenith of this observer; as φ increases, the time intervals between these days will become ever less and, finally, when $\varphi = \delta_{\odot max} = 23°27'$ the observer will see the sun in his zenith only once a year, namely on 22.06 or 22.12.
3. For $\delta_\odot < \varphi$ and of the same name, the sun in its diurnal motion crosses the portion of the prime vertical above the horizon and is

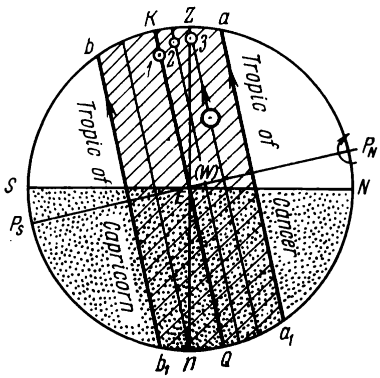

Fig. 28

observed in all four quadrants of the horizon. But if $\delta_\odot > \varphi$ and of the same name, then the sun does not cross the portion of the prime vertical above the horizon. In this latter case, the sun will be observed in only two quadrants; and from sunrise to elongation the azimuth will increase up to A_{max}, then decrease to 0° during transit, after which it again increases up to elongation and again diminishes. The rate of change of azimuth in low latitudes is extremely nonuniform: from sunrise to moments close to transit, the azimuth changes but slightly (by 5° to 10°); near transit, however (points *1*, *2*, *3* in Fig. 28), it begins to change rapidly and during a small time interval it varies by tens of degrees. This peculiarity must be allowed for when taking observations in low latitudes.

4. At the extreme parallels of this zone, that is, on the tropics, the longest day and the longest night (during the days of summer and winter solstices) are about 13.5 hours long.

III. FOR OBSERVERS IN THE TEMPERATE ZONE, BETWEEN PARALLELS 23°27′N (S) AND 66°33′N (S) (SEE FIG. 25)

1. The sun is never in the zenith because we always have

$$\delta_{\odot} < \varphi$$

2. The sun definitely rises and sets, since

$$|\delta_{\odot}| < 90° - \varphi \tag{4.3}$$

3. In diurnal motion, the sun always crosses the prime vertical; hence, for declination of the same name as latitude, the apparent azimuths of the sun will in the course of the day lie in all four quadrants of the horizon; when the names are different, the azimuths will lie in two quadrants.

IV. FOR OBSERVERS IN THE POLAR ZONE, BEYOND THE ARCTIC CIRCLE, $\varphi > 66°33'$N (S) (FIG. 29)

1. One may observe a nonsetting (when $\delta_{\odot}$ and φ are of the same name) or a nonrising sun (when $\delta_{\odot}$ and φ are of different names), since in these latitudes the condition $\delta_{\odot} > 90° - \varphi$ is fulfilled.

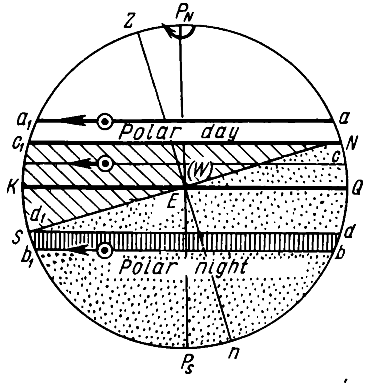

Fig. 29

2. The number of days with nonsetting or nonrising sun or, to put it differently, the length of the "*polar day*" or "*polar night*" increases with increasing latitude of the observer.

The polar day is an interval of time expressed in days during which the sun does not drop below the horizon (parallels aa_1-cc_1); the polar night is that period during which the sun does not rise above the horizon (parallels dd_1-bb_1).

3. For $\delta_\odot$ and φ of the same name, the apparent azimuths of the sun will be located in all four quadrants of the horizon during a 24-hour period.

V. FOR AN OBSERVER AT THE POLE; $\varphi = 90°$N OR S (FIG. 30)

1. Half the year is day and the sun has a declination of the same name as the latitude; half the year is night and $\delta_\odot$ and φ are of different names.

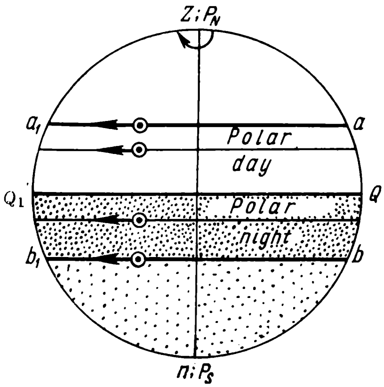

Fig. 30

2. In its diurnal motion, the sun follows a parallel of latitude, and its altitude is always equal to the declination.
3. The sun reaches highest altitude during the days of solstice, at which time $h_\odot = \delta_{\odot max} = 23°27'$.

SEC. 17. AN EXPLANATION OF THE APPARENT ANNUAL MOTION OF THE SUN

Like its diurnal motion, the apparent annual motion of the sun over the celestial sphere is only an apparent motion that reflects the motion of the observer together with the earth.

This view, which was expressed in the 16th century by the great Polish scholar Nicolaus Copernicus (1473-1543), differed radically from the then accepted **geocentric hypothesis** of Ptolemy which considered the sun's motion as its true motion about a stationary earth. The **heliocentric system** of Copernicus was later proved definitively and became the generally accepted theory. The basic geometric and

mechanic regularities of this theory were developed in the works of Kepler (1571-1630) and Newton (1643-1727). The **Kepler laws** are:

1. *The orbit of each planet (including the earth) is an ellipse with the sun at one focus.*

2. *The radius vector drawn from the sun to the planet sweeps out equal areas in equal times.*

This law takes into account the nonuniform motion of the earth about the sun: the earth moves faster on portions of the ellipse

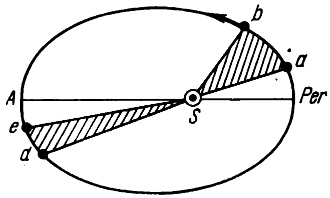

Fig. 31

closer to the sun (*ab* in Fig. 31) and slower on more distant portions (*ed*).

3. *The cubes of the mean distances of the planets from the sun are proportional to the squares of their times of revolution about the sun.*

$$\frac{S_1^2}{S_2^2} = \frac{a_1^3}{a_2^3} \tag{4.4}$$

where S_1 and S_2 are the times of revolution of two planets relative to the stars

a_1 and a_2 are semimajor axes of the ellipses described by these planets.

This law states that planets closer to the sun move faster than those greater distances away.

Kepler's laws are dynamically accounted for in *Newton's law of universal gravitation*, which states:

Every particle in the universe attracts every other particle with a force (F) that is directly proportional to the product of their masses, and inversely proportional to the square of the distance between their centres of mass, or

$$F = -k^2 \frac{mM}{r^2} \tag{4.5}$$

where m and M are the masses of two particles (infractions of solar mass),

r is the distance between them (in astronomical units) and

k is a coefficient (about $^1/_{58}$) equal to the force of attraction of two unit masses separated by a unit distance.

The Copernicus system and these laws account very accurately for peculiarities of the *apparent* motion of the sun.

Indeed, in Fig. 32 constructed on the plane of the solstitial colure let the centre of the celestial sphere of a very great radius coincide with the centre of the sun. The earth's motion will be counterclockwise in an elliptical curve with an eccentricity of about $^1/_{60}$, called the **earth's orbit.** The sun (S) is located at one focus of the orbit.

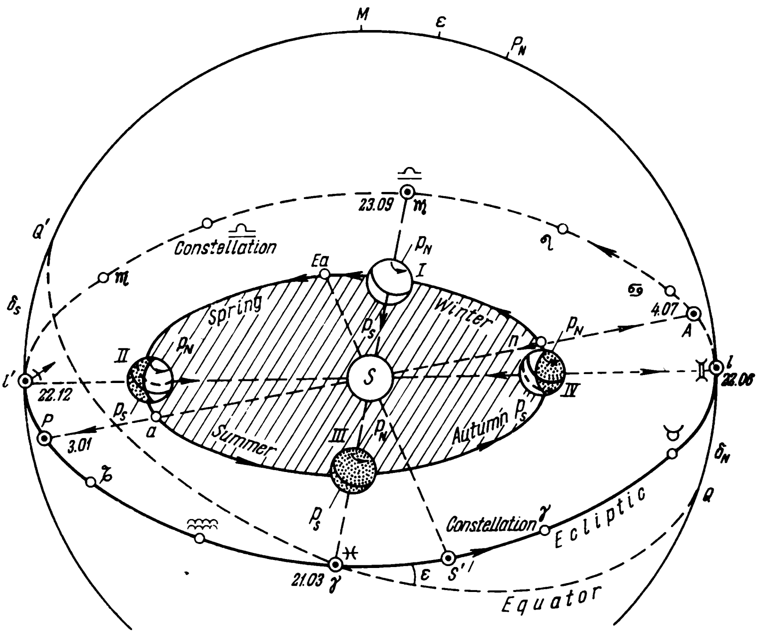

Fig. 32

Due to the motion of the earth (Ea) in its orbit, an observer on the earth will regard the sun as moving over the celestial sphere (point S') in the same direction as the earth. Thus, the *ecliptic is the projection of the earth's orbit on the celestial sphere* or, to be more precise, the *ecliptic is a great circle of the celestial sphere obtained by its intersection with the plane of the earth's orbit* (or *with a plane parallel to it*).

Let the earth arrive in position I at 21.03; then the sun will be seen from the earth in the direction of ♈ of the sphere (in the region of the constellation Pisces). In this position, a straight line connecting the centres of the sun and earth will intersect the earth's equator, that is, the sun will be in the zenith on the equator, and since this

line is perpendicular to the axis $P_N P_S$, the earth's surface will be illuminated from pole to pole, and in all latitudes day will be equal to night. In position I, spring will set in in the northern hemisphere, since the sun has moved into the northern half of the sphere.

In roughly three months (22.06), the earth will reach position II. From this position the sun is projected on point l of summer solstice (the region of the constellation Gemini).

A straight line connecting the centres of the sun and earth will intersect the tropic of Cancer, that is, the sun will be in the zenith at $\varphi = 23°27'$N; the angle between this line and the axis $P_N P_S = 66°33'$. Light is received, in the main, by the northern hemisphere,

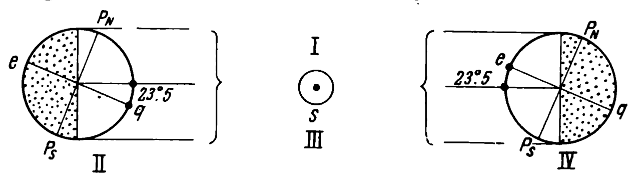

Fig. 33

where summer sets in. At $\varphi > 66°33'$N we have a polar day, at $\varphi > 66°33'$S, polar night.

The earth arrives in position III on 23.09. The sun is projected on the autumnal equinox ♎ (in the region of the constellation Virgo); all phenomena will be similar to position I, but in the northern hemisphere autumn will be setting in.

Finally, the earth will arrive in position IV on 22.12. The sun will appear at the point of winter solstice l' (in the region of the constellation Sagittarius). The straight line connecting the centres of the earth and sun will cross the Tropic of Capricorn ($\varphi = 23°27'$S) where the sun will be in the zenith; the southern hemisphere will be more illuminated and summer will set in; winter will begin in the northern hemisphere. As may be seen from Fig. 33, the sun's rays intersect the earth's axis (which retains a constant direction in space) at angles from 90° to $\pm 66°33'$ or with the equator from 0° to $\pm 23°27'$; this explains the variation of the sun's declination on the sphere from 0° to $\pm 23°27'$N and S.

Let us now see how the earth's velocity in orbit varies, and consequently how the longitude of the sun changes on the sphere. According to Kepler's second law, the earth's speed is greatest when closest to the sun (point p of the orbit). This point is the **perihelion** (Greek: *peri*—about, *helios*—sun)* of the orbit which the earth enters on

* If we consider not the earth's orbit but the apparent orbit of the sun moving (by convention) about the earth, then the nearest and farthest points of the orbit are called perigee and apogee. Then we say that the "sun is in perigee", the "longitude of apogee", etc.

January 2-4. At this time, the sun is seen from the earth at point p of the ecliptic, the longitude of which is about 282°. The diurnal variation (ΔL) of longitude of the sun at this point is greatest $\Delta L \approx \approx 61'.2$. The angular diameter of the sun ($2R_\odot$) at this point as seen from the earth should obviously be greatest, $2R_\odot = 32'31'' \approx \approx 32'.5$. At the opposite point (a) of the orbit, which is farthest from the sun, the earth will be moving at its lowest speed. This point is called the **aphelion** (from the Greek, *apo*—distant, *helios*—the sun) and is reached by the earth on July 3 to 5. At aphelion, the angular diameter of the sun as seen from the earth is smallest: $2R_\odot = = 31'27'' \approx 31'.5$. The point of the ecliptic A, on which the sun will be projected, has a longitude of 102°, while the diurnal variation of longitude of the sun at this point is smallest: $\Delta L = 57'.2$. The earth's orbital velocity near the equinoxes (positions I and III) is close to the mean velocity of 29.77 km/s, while the diurnal variation of longitude of the sun is close to the mean rate of $59'.2$.

The seasonal changes of the year that we have already mentioned are due to the fact that when the earth moves round the sun it alternately turns to the sun its northern and southern hemispheres (Fig. 33). The angle of incidence of the sun's rays on a given hemisphere increases (onset of spring and summer, II in Fig. 33) and decreases (onset of autumn and winter, IV in Fig. 33). In summer (for the northern hemisphere of the earth) the earth is farther from the sun, while in winter it is closer, but the amount of heat is more dependent on the angle of incidence of the rays on the earth's surface than on the slight difference in the sun's distance from the earth.

Since the earth's velocity is different on various portions of its orbit the time it takes the earth to cover these portions (see Fig. 32, I-II, II-III, III-IV, and IV-I) differs. For this reason, the seasons have different lengths and for the northern hemisphere of the earth they are: spring—92.9 days, summer—93.6 days, autumn—89.8 days, and winter—89.0 days.

Thus, the summer (warm) period for the northern hemisphere is at present equal to roughly 186 days and is longer than the winter (cold) period by 7 days.

The foregoing and the physical proofs that follow permit us to conclude that the *observed annual motion of the sun over the sphere is a reflection of the actual movement of the earth round the sun.*

SEC. 18. VARIATIONS IN THE EQUATORIAL COORDINATES OF THE SUN

It has been established that the sun's motion along the ecliptic is not uniform, as a result of which its longitude also varies in nonuniform fashion. Let us find out what variations there are in right

ascension and declination of the sun at various points on the ecliptic. This is very important for practical astronomy. We first obtain

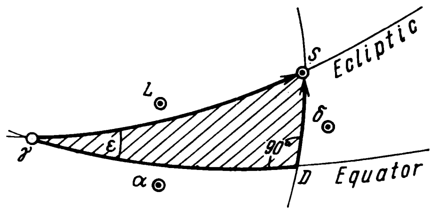

Fig. 34

the increments $\Delta\alpha_\odot$ and $\Delta\delta_\odot$ as a function of the increment of longitude. Using the cotangent and sine formulas (or Napier's mnemonic rules), let us express $\alpha_\odot$ and $\delta_\odot$ in terms of L and ε from the spherical right triangle ♈ SD (Fig. 34):

$$\cos\varepsilon = \tan\alpha\cdot\cot L, \text{ whence } \tan\alpha = \cos\varepsilon\cdot\tan L \tag{4.6}$$

and

$$\sin\delta = \sin\varepsilon\cdot\sin L \tag{4.7}$$

Differentiating (4.6) with respect to α and L, and (4.7) with respect to δ and L, we have

$$\frac{d\alpha}{\cos^2\alpha} = \cos\varepsilon\,\frac{dL}{\cos^2 L}$$

whence

$$d\alpha = \frac{\cos^2\alpha}{\cos^2 L}\cos\varepsilon\, dL \tag{4.8}$$

and

$$\cos\delta\, d\delta = \sin\varepsilon\cos L\, dL$$

whence

$$d\delta = \sin\varepsilon\,\frac{\cos L}{\cos\delta}\, dL \tag{4.9}$$

By the cosine formula of side ♈S, from triangle ♈SD we get

$$\cos L = \cos\alpha\cdot\cos\delta \tag{4.10}$$

Putting $\cos L$ into (4.8) and (4.9) and passing to finite increments (which is sufficiently exact for small values of ΔL), we finally get

$$\Delta\alpha = \frac{\cos\varepsilon}{\cos^2\delta}\cdot\Delta L \tag{4.11}$$

$$\Delta\delta = \sin\varepsilon\cdot\cos\alpha\cdot\Delta L \tag{4.12}$$

Let us investigate the diurnal variations $\Delta\alpha$ and $\Delta\delta$ near the points of equinoxes and solstices, making use of the above-given values of change of longitude.

Table 2

Point of ecliptic	$\delta_{\odot}$	$\alpha_{\odot}$	$\Delta L_{\odot}$ per day	$\Delta\alpha_{\odot}$ per day	$\Delta\delta_{\odot}$ per day
Spring equinox	0°	0°	59′	$59' \cos\varepsilon \approx 54'$	$59' \sin\varepsilon \approx 23'.5$
Summer solstice	$\delta_{max} = +\varepsilon$	90	57	$57 \sec\varepsilon \approx 62$	0
Autumnal equinox	0°	180	59	$59 \cos\varepsilon \approx 54$	$59 \sin\varepsilon \approx 23.5$
Winter solstice	$\delta_{max} = -\varepsilon$	270	61	$61 \sec\varepsilon \approx 66$	0

Using the data given in Table 2, it is possible to construct a curve of the values of $\delta_{\odot}$ and $\alpha_{\odot}$ during one year (Fig. 35).

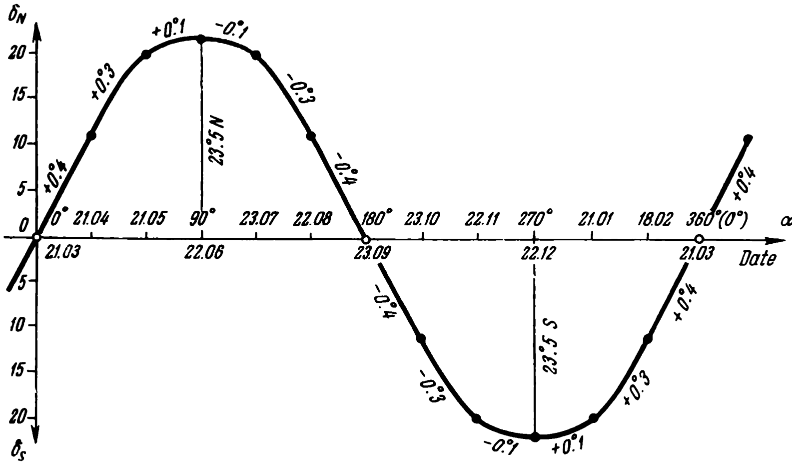

Fig. 35

The diurnal $\Delta\alpha_{\odot}$ reaches a maximum on about December 24 (66′.6), and a minimum about September 16 (53′.8). On the average for the year, the diurnal $\Delta\alpha_{\odot} = 59'$ or approximately 1°.

The diurnal variation of solar declination, however, fluctuates between 0° about the solstice days to $\pm 23'.5$ about the equinoxes. Using the formula (4.12) we can obtain the values of $\Delta\delta_{\odot}$ for the middle of the first, second and third months following the spring equinox, i.e., about April 5, May 5, and June 4. Taking $\sin\varepsilon = 0.40$, ΔL consecutively 59′, 58′, 57′ and $\alpha = 15°, 44°, 73°$, we

get

for the 15th day following equinox $\Delta\delta_{\odot}=23' \approx 0^{\circ}.4$ per day;
for the 45th day following equinox $\Delta\delta_{\odot}=17' \approx 0^{\circ}.3$ per day;
for the 75th day following equinox $\Delta\delta_{\odot}= 7' \approx 0^{\circ}.1$ per day.

Similar values of $\Delta\delta$ are obtained also for the 15th, 45th, and 75th day before the spring equinox, and also at these periods following the autumnal equinox. Consequently, declination varies approximately symmetrically relative to equinoxes and solstices (see Fig. 35). This circumstance may be utilized in approximate calculations of declination.

SEC. 19. APPROXIMATE SOLUTION OF PROBLEMS ASSOCIATED WITH THE SUN'S MOTION

I. APPROXIMATE CALCULATION OF $\alpha_{\odot}$ AND $\delta_{\odot}$ FOR A GIVEN DATE

As a basis for the approximate calculation of $\alpha_{\odot}$ and $\delta_{\odot}$ we take the absolute values of the coordinates given in Table 2 or on the curve in Fig. 35, and the diurnal variations of $\Delta\alpha$ and $\Delta\delta$, for the values of which, on the basis of the foregoing section, we take

$\Delta\alpha_{\odot}=1^{\circ}$ per day during the whole year;
$\Delta\delta_{\odot}=\pm 0^{\circ}.4$ per day for first month before and first month after equinox;
$\Delta\delta_{\odot}=\pm 0^{\circ}.3$ per day for second month before and second month after equinox;
$\Delta\delta_{\odot}=\pm 0^{\circ}.1$ per day for first month before and first month after solstices.

To calculate the right ascension and declination of the sun for a given date, multiply the value of diurnal variation $\Delta\delta_{\odot}$ or $\Delta\alpha_{\odot}$ by the number of days to the *nearest date* of equinox or solstice, take the result and add it to (or subtract it from) the value $\delta_{\odot}$ and $\alpha_{\odot}$ of that date.

Example 1. Compute $\delta_{\odot}$ and $\alpha_{\odot}$ for May 3.
(a) The nearest date will be 21.03; $\delta_{\odot}=0^{\circ}$, $\alpha_{\odot}=0^{\circ}$.
(b) Number of days to nearest date: 43.
(c) We calculate the variation $\Delta\delta$ and $\Delta\alpha$. The diurnal variation $\Delta\delta = +0^{\circ}.4$ and $0^{\circ}.3$.

$$\Delta\delta_1=0.4^{\circ}/\text{d}\cdot 30\text{d}=12^{\circ}$$
$$\underline{\Delta\delta_2=0.3^{\circ}/\text{d}\cdot 13\text{d}=3^{\circ}.9}$$
$$\Delta\delta_{\odot} \qquad =15^{\circ}.9, \qquad \Delta\alpha_{\odot}=1^{\circ}/\text{d}\cdot 43\text{d}=43^{\circ}.$$

(d) $\delta_{\odot}=0^{\circ}+15^{\circ}.9=15^{\circ}.9\text{N}$; $\qquad \alpha_{\odot}=0^{\circ}+43^{\circ}=43^{\circ}$.

Example 2. Compute $\delta_{\odot}$ and $\alpha_{\odot}$ for November 16.

(a) The nearest date will be 22.12; the number of days: 36 (back).

(b) $\Delta\delta_1 = 0.1°/d \cdot 30d = -3°.0$

$$\begin{array}{ll} \Delta\delta_2 = 0.3°/d \cdot 6d = -1°.8 & \\ \hline \Delta\delta_{\odot} \qquad = -4°.8, & \Delta\alpha_{\odot} = 1°/d\,(-36d) \\ & = -36°. \end{array}$$

(c) 16.11. $\delta_{\odot} = 18°.7S$, $\alpha_{\odot} = 234°$.

The relationship which we have established between $\delta_{\odot}$, $\alpha_{\odot}$, and the dates of the year permit of an approximate solution of a number of problems on solar motion.

II. DETERMINING THE DATE OF ONSET AND END OF POLAR DAY AND NIGHT AND THEIR DURATION IN A GIVEN LATITUDE

On the basis of the conditions of the rising and setting of an astronomical body (3.1), we find that: (1) the polar day will continue as long as $\delta_{\odot} \geqslant 90° - \varphi$ and the polar night of the same name as φ will continue as long as $\delta_{\odot} \geqslant 90° - \varphi$ and of opposite name to φ; (2) the onset and end of the polar day are determined by the equation $\delta_{\odot} = 90° - \varphi$ and are of the same name; (3) the onset and end of the polar night are determined by the equation $\delta_{\odot} = 90° - \varphi$ and are of opposite names.

Example 3. Determine the dates of onset and end of polar night and its duration at $\varphi = 72°.5N$.

(a) Condition for onset and end of night: $\delta_S = 90° - \varphi_N$, i.e., $\delta_{\odot} = 17°.5S$.

(b) We calculate two dates corresponding to $\delta_{\odot} = 17°.5S$. The nearest date will be 22.12$\delta_{\odot} = 23°.5S$. The diurnal variation is 0°.1 and 0°.3.

The difference $\Delta\delta = 23°.5 - 17°.5 = 6°$. For one month prior to and following 22.12: $0.1°/d \cdot 30d = 3°$; the remainder 3°: $0.3°/d = 10$ days, a total of 40 days.

Consequently, $\delta_{\odot} = 17°.5S$ will occur 40 days prior to and following 22.12; in other words, the polar night will commence 12 November and will end on 31 January; duration: 80 days.

III. DETERMINING THE LATITUDE AT WHICH THE POLAR DAY (NIGHT) WILL CONTINUE A SPECIFIED NUMBER OF DAYS

This problem is the reverse of the preceding one. We shall therefore make use of the same conditions: $\delta_{\odot} \geqslant 90° - \varphi$.

Example 4. At what northern latitude does the polar day last 48 days?

(a) The condition for the commencement and end of the day: $\delta_N = 90° - \varphi_N$.

(b) Due to a symmetrical variation in declination, the number of days to the nearest date is $48 : 2 = 24d$. The nearest date is 22.06.

(c) $\delta_{\odot} = 23°.5 - 24d \cdot 0.1°/d = 21°.1N$.

(d) $\varphi = 90° - 21°.1 = 68°.9N$.

Note. The condition used in these problems $\delta_{\odot} \geqslant 90° - \varphi$ is approximate because it disregards refraction, the apparent semidiameter of the sun, and the amount of dip of the horizon. With these factors taken into consideration, $\delta_{\odot} \geqslant [(90° - \varphi) \pm 0°.9]$ and the polar day will continue for a longer time, while the night will be shorter than in the foregoing problems.

IV. DETERMINING DATES ON WHICH THE SUN PASSES THROUGH THE ZENITH IN A GIVEN LATITUDE

The condition for passage through the zenith is: $\delta_{\odot} = \varphi$ and they are of the same name; the problem reduces to calculating the date from the solar declination.

Example 5. $\varphi = 10°S$. Determine the dates on which the sun will pass through the zenith at noon.

(a) $\delta_{\odot} = \varphi = 10°S$.

(b) The nearest dates are 23.09 and 21.03, $\Delta\delta_{\odot} = 0°.4$ per day.
(c) Number of days to nearest date: $10° : 0.4°/d = 25d$.
(d) The sun passes through the zenith on 18 October and 24 February.

V. DETERMINING THE MERIDIAN ALTITUDE OF THE SUN ON A GIVEN DATE IN A GIVEN LATITUDE

Applying formulas (3.8) and (3.9) for the sun, we get

$$H_{\odot} = 90° - \varphi_N \pm \delta_S^N$$

Example 6. Determine $H_{\odot}$ at Vladivostok ($\varphi = 43°.1N$) on 20 August.
(a) $\delta_{\odot} = 0° + 0.4°/d \cdot 30d + 0.3°/d \cdot 4d = 13°.2N$.
(b) $H_{\odot} = 46°.9 + 13°.2 = 60°.1S$.

VI. FOR A GIVEN DATE DETERMINE THE CONSTELLATIONS VISIBLE AT MIDNIGHT IN THE SOUTH, EAST AND WEST (IN φ_N)

Using the date, we calculate $\alpha_{\odot}$ approximately. Obviously, stars with $\alpha_{*} = \alpha_{\odot}$ will at midnight be near lower transit; with $\alpha_{*} = \alpha_{\odot} \pm 180°$, they will be near upper transit (in the south); with $\alpha_{*} = \alpha_{\odot} - 90°$, they will be east of the meridian; with $\alpha_{*} = \alpha_{\odot} + 90°$, to the west. The magnitudes of α_{*} are taken from Table 5 (pp. 126-127) or from a star map. The solution is checked by a drawing.

Example 7. On 10 December $\varphi = 50°N$. What constellations are visible in the south, to the east and to the west of the meridian at about midnight?

(1) $\alpha_{\odot} = 270° - 12d \cdot 1°/d = 258°$.

(2) For stars to the south, $\alpha_* = 258° - 180° \approx 78°$. From the table or from a map we find that Orion, Taurus and Auriga will be in the south.

(3) For stars in the east, $\alpha_* = 258° - 90° \approx 168°$. From the table or map we choose the constellation Leo.

(4) For stars in the west, $\alpha_* = 258° + 90° \approx 348°$.

In the west we will have the rather faint constellation Pegasus.

In similar fashion we determine the positions of constellations after sunset or before sunrise, but the relationships between α_* and $\alpha_\odot$ will be different.

The following examples should be solved without the aid of an almanac, by computing $\delta_\odot$ and $\alpha_\odot$ approximately for the given day.

Examples.

8. $\varphi = 0°$. What will the meridian altitude of the sun be at noon on 10.02 and what is the azimuth on that date at sunrise?

9. $\varphi = 0°$. What will $H_\odot$ be at noon on 16.07 and what will $A_\odot$ be on this day at sunset?

10. $\varphi = 16°N$. On what date will the meridian altitude of the sun at noon be equal to 85° to the north?

11. $\varphi = 5°N$. When will the sun pass through zenith at noon?

12. $\varphi = 20°S$. Will the sun pass through the zenith on 5.01 at noon, and if not, will it cross the prime vertical on that day?

13. $\varphi = 15°N$. From what day to what day will the sun not cross the prime vertical?

14. $\varphi = 70°N$. From what date to what date will there be polar day and polar night?

15. In what latitude on 24.10 will a nonsetting sun be first observed?

16. On what parallel will the polar night last 20 days?

17. $\varphi = 63°N$ on 25.04. What constellations and stars will be visible to the south, west and east at midnight?

18. $\varphi = 47°S$; 16.01. What stars are visible at midnight on the meridian of the observer (in the north)?

19. $\varphi = 40°N$; 19.07. What star transits at midnight and what is its meridian altitude?

C H A P T E R 5

PHENOMENA ASSOCIATED WITH THE REVOLUTION AND ROTATION OF THE EARTH

SEC. 20. ANNUAL PARALLAX OF STARS. THE DIURNAL PARALLAX

The annual motion of the earth along its orbit will inevitably give rise to a shift of the stars on the background of the sky if the stars are at certain finite distances from the earth. Indeed, from point T_1 of the earth's orbit (Fig. 36) a celestial body C is seen on the celestial sphere (of very great radius) at the point C'; from point T_2 it is seen on the sphere at C''. As the earth revolves in its orbit during the year, the star will describe on the sphere an ellipse very close to a circle. *The angular magnitude of the semimajor axis of the ellipse of shift equal to the largest angle at which the semiaxis of the earth's orbit is seen from the star is called* **the annual parallax π of the star.** The annual parallax of stars is proof of the earth's motion round the sun, because the parallactic ellipse is a projection of the earth's orbit on a celestial sphere of very great radius.

To prove the basic principle of the Copernican system (movement of the earth round the sun) many astronomers of the 17th and 18th centuries made attempts to detect a parallactic shift of stars on the celestial sphere. For a long time these attempts did not yield any results due to the smallness of the shifts and crude instruments.

Only at the start of the 19th century did Bessel and V. Struve succeed in detecting an annual parallax for the stars 61 Cygni (0″.3) and Vega (0″.25). On present data, the following stars have the largest annual parallaxes: Proxima Centauri, 0″.762, and α Centauri, 0″.756. Considerable parallaxes have been obtained for Sirius, 0″.4, Procyon, 0″.3, and others. At the present time, the parallaxes of over 4,000 stars have been measured.

From the triangle CST_1 (Fig. 36) we get

$$\frac{\sin x}{\sin T} = \frac{a}{D}$$

whence, due to the smallness of the angle x,

$$x'' = \frac{a}{D \operatorname{arc} 1''} \cdot \sin T \qquad (*)$$

But from a determination of the annual parallax π, we have (for $T = 90°$)

$$\sin \pi = \frac{a}{D}$$

or

$$\pi'' = \frac{a}{D \text{ arc } 1''} \tag{5.1}$$

Putting this expression into formula (*), we get

$$x'' = \pi'' \cdot \sin T \tag{5.2}$$

Consequently, due to annual parallax the stellar directions will deviate from SC towards the sun by a very small angle x.

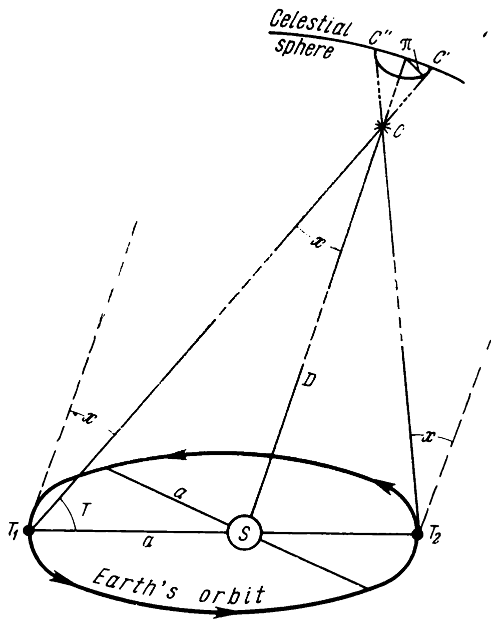

Fig. 36

Knowing the annual parallax π of a star, it is easy to determine the distance to it D from the formula

$$D = \frac{a}{\pi'' \cdot \text{arc } 1''} \tag{5.3}$$

where a is the mean distance from the earth to the sun, the so-called **astronomical unit** (AU) $= 149.5 \times 10^6$ km.

Stellar distances are so great that it is more convenient to use larger units: the **light year,** which is the distance traversed by light in one year (equal to 9.46×10^{12} km) or the **parsec**, the distance to a star whose parallax is 1″; a parsec equals 3.26 light years or 30.8×10^{12} km. The distance to α Centauri is 1.3 parsecs, to Sirius, 2.7 parsecs.

The annual parallax changes the coordinates α and δ of a star, but the changes are so slight that they may be disregarded in problems of nautical astronomy.

Besides the annual parallax we also have a similar phenomenon with a diurnal period called the **diurnal parallax** (or simply **parallax**). A diurnal parallax is observed in the case of celestial objects of the solar system at relatively small distances from the earth. As a result, from different points of the earth these bodies lie in different directions. During a 24-hour period, the coordinates of these bodies undergo certain changes. The diurnal parallax of the moon is particularly evident. This problem will be considered in more detail in Sec 74, Chapter 14. The annual parallax is sometimes called the heliocentric parallax, and the diurnal parallax is called the geocentric parallax.

SEC. 21. STELLAR ABERRATION

In the first part of the 18th century, the English astronomer Bradley attempted to determine the parallactic shift of stars and detected deviations in stellar directions towards the motion of the observer together with the earth. This phenomenon was explained by Bradley and was given the name **aberration** (from the Latin *aberratio* meaning deviation) of light.

Aberration is due to two causes: (a) the motion of the observer in space, and (b) the fact that the rate of propagation of light is finite and is commensurable with the velocity of motion of the observer. We distinguish **annual aberration** due to the motion of the earth in its orbit, and **diurnal aberration** due to the daily rotation of the earth. We shall confine ourselves to annual aberration, since diurnal aberration is too small (about 0″.3) to be considered in nautical astronomy.

Suppose that in an interval of time ΔT, an observer A (Fig. 37) with telescope AB moves with the earth in orbit from the point A to A_1 over a distance $AA_1 = v \cdot \Delta T$, where v is the orbital velocity of the earth, equal to 29.77 km/s, or about 30 km/s.

A ray of light during time ΔT will traverse a distance $AB = V \cdot \Delta T$, where V is the velocity of light, equal to 299,793 km/s, or about 3×10^5 km/s. As a result, the apparent direction AC' towards the star will deviate from the actual direction AC towards

the observer's motion by an angle y, which is called aberration. In other words, in order to see the star in the centre of the field of view, it is necessary to incline the telescope at A through an

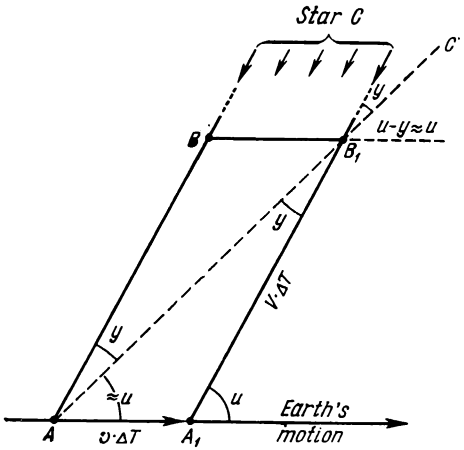

Fig. 37

angle y towards the light ray CA; then the light ray will move along the axis of the telescope and arrive at A_1 at the same time as the centre of the field of view of the telescope.

From the triangle AB_1A_1, in which we take the angle $u - y \approx u$ due to the smallness of y, we get

$$\frac{\sin y}{\sin u} = \frac{v \cdot \Delta T}{V \cdot \Delta T}$$

whence we have

$$y'' = \frac{v}{V \operatorname{arc} 1''} \cdot \sin u = k'' \cdot \sin u \tag{5.4}$$

where $k = \frac{v}{V \cdot arc\ 1''} = 20''.47 \approx 20''.5$, the aberration constant, which is equal to the angle y for $u = 90°$.

Due to aberration, in one year the image of a star will describe on the celestial sphere an ellipse with dimensions tens of times greater than the ellipse due to parallax, and the directions of its axes will be different. For this reason, the coordinates α and δ of stars will vary approximately up to $\pm 1'.0$ with an annual period, and these variations must be taken into account in practical astronomy. Stellar aberration is also one of the physical proofs of the earth's motion about the sun.

SEC. 22. THE ESSENTIALS OF PROCESSION AND NUTATION

If we compare the coordinates α and δ of certain stars that have been computed in different epochs and registered in catalogues, it will be evident that in addition to the periodic annual variations due to the causes discussed in the preceding sections we will find considerable increases in α and changes in δ of most stars.

If we examine the ecliptic coordinates, we will find that the latitudes of the stars do not change, while the longitudes of all stars

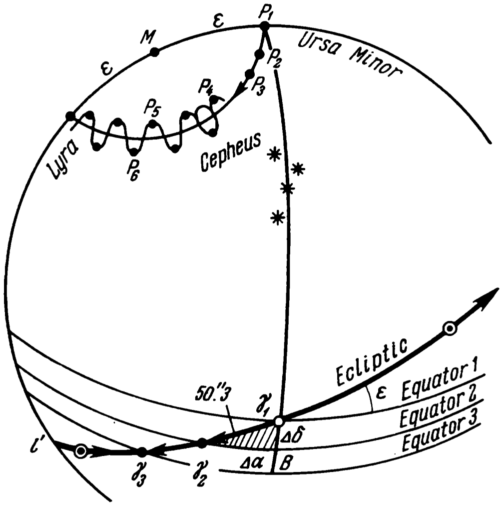

Fig. 38

increase by 50".3 a year. This phenomenon was discovered in the second century B. C. by the Greek scholar Hipparchus in the course of comparing his own latitudes and longitudes of a series of stars with their coordinates obtained 150 years before that time. An analysis of this phenomenon shows that the plane of the ecliptic remains fixed relative to the stars, while the celestial equator is constantly inclining without changing the angle $\varepsilon = 23°.27'$, as a result of which the point Υ moves along the ecliptic towards the sun by about 50".3 a year. For this reason, the sun arrives at Υ *before* (position Υ_2, Fig. 38) a revolution has been completed (position Υ_1) round the sphere by $\frac{50''.3}{59'.14 \times 60''} \times 24 \times 60 \times 60s = 20m\ 24s$ per year.* The equinox will occur that much earlier—**precession** (the Latin

* In the proportion 59'.14—24h, 50".3—x, where 59'.14 is the diurnal change in the sun's longitude.

praecessio means the act of preceding). Thus, the tropical year will be 20m24s shorter than one complete circuit of the sun round the sphere relative to the stars, which is called a **sidereal year** and is equal to 365.2564 mean days (365d 6h 9m 10s).

Due to precession, the celestial pole P will move round the pole of the ecliptic M in a small circle of radius 23° 27′, while the celestial axis will describe a conic surface and return to its original position in $\frac{360 \times 60 \times 60''}{50''.3/\text{year}} = 25{,}800$ years. Hence, precessional motion gives rise to movements of the celestial pole and the vernal

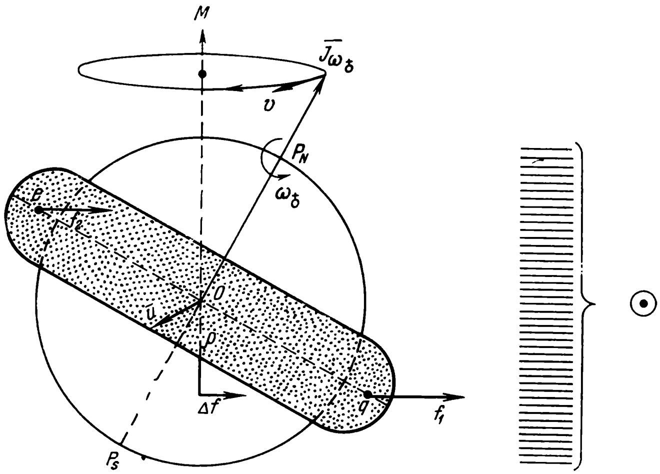

Fig. 39

equinox point among the constellations. If P_N is at present close to the star α Ursae Minoris (Polaris), then in 6,000 years the pole will be near α Cephei, and in 12,000 years, near α Lyrae (Vega). α Ursae Minoris will be closest to P_N in the year 2102 (about 28′). In Hipparchus' day, the vernal equinox point was located in the constellation Aries, while the autumnal equinox point was in the constellation Libra, whence their names. At the present time, these points have moved (see Fig. 32) into the constellations Pisces (the first point of Aries) and Virgo (the first point of Libra), which is nearly 30° towards the motion of the sun.

If we return from shifts on the celestial sphere to the motions of the earth, we find that the precessional motion of the celestial pole on the sphere is only a reflection of the actual precessional motion of the earth's axis in space.

Physically, the motion of the earth's axis is due to the gravitational action of the sun, moon and the planets on the rapidly rotating terrestrial ellipsoid. There would be no precession if the earth had the shape of a sphere of uniform mass. Let us suppose (Fig. 39) that the excess mass (that is, in addition to that of a sphere) of the geoid is concentrated in a ring round the equator with the centres of gravity of the closest and most distant parts of the ring at points q and e. Considering (for the sake of simplicity) only the effect of the attraction of the sun* on the mass of the ring and applying the law of universal gravitation, we get forces f_1 and f_2 ($f_1 > f_2$ due to the difference in distances from the sun; the drawing is made for summer solstice).

Applying to the centre of the earth the moment of difference of forces $f_1 - f_2 = \Delta f$, equal to $\Delta f \cdot \rho$, construct at point O the vector of this moment $\bar{u}$ perpendicular to the drawing.

The vector $\overline{I\omega_♁}$, which is the principal angular momentum of the earth (here, I is the moment of inertia, and $\omega_♁$ is the angular velocity of terrestrial rotation), will be under the action of an external force with a moment u and, by the laws of mechanics, will begin to move. On the basis of the Résal theorem, the instantaneous velocity v of the end of the vector of the principal angular momentum $\overline{I\omega_♁}$ will be equal and parallel to the vector of the moment of the external forces $\bar{u}$. As a result, the axis of the earth will slowly begin to move in space (precess) in the direction indicated by the arrow v.

Due to the gravitational action of the moon, this phenomenon becomes stronger and more complicated. Then if we add to the lunar-solar precession the small precession due to the planets, we get a **general precession,** whose constant p is equal (according to the findings of Newcomb) to

$$p = 50''.2564 + 0''.00022\,(t - 1900) \tag{5.5}$$

where t is the number of years reckoned from the epoch 1900.0.

Due to precession, the equatorial coordinates of all stars vary far more substantially than due to all other causes. From the triangle $\Upsilon_1\Upsilon_2B$ (Fig. 38) we see that for stars near the meridian of equinoxes $P_1\Upsilon_1$ the changes of α and δ per year will be

$$\Delta\alpha'' = 50''.3 \cdot \cos\varepsilon \approx 46''$$

$$\Delta\delta'' = 50''.3 \cdot \sin\varepsilon \approx 20''$$

These changes in stellar coordinates may be found in catalogues.

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* The effect of the moon is still greater because it is closer to the earth (of the 50″, 16″ is due to the sun and 34″ to the moon).

For stars located at other places on the sphere, the changes in α and δ due to precession may be computed from the formulas

$$\left.\begin{aligned}\Delta\alpha'' &= (m + n \cdot \sin\alpha \cdot \tan\delta) \cdot T \\ \Delta\delta'' &= n \cdot \cos\alpha \cdot T\end{aligned}\right\} \tag{5.6}$$

where T is the number of years reckoned from the epoch for which α and δ are given;

m and n are precessional constants in right ascension and declination equal to: $m = 46''.085 + 0''.00028\ t$ and $n = 20''.047 - 0''.000085\ t$, reckoning t from 1900.0.

Approximately, we can take

$$\left.\begin{aligned}\Delta\alpha'' &= (46'' + 20'' \cdot \sin\alpha \cdot \tan\delta) \cdot T \\ \Delta\delta'' &= 20'' \cdot \cos\alpha \cdot T\end{aligned}\right\} \tag{5.7}$$

Due to the precession (caused by the planets) the obliquity of the ecliptic does not remain constant either, but changes from 22° 59′ to 24° 36′. The mean obliquity ε is at present

$$\varepsilon = 23°27'8''\cdot 3 - 0''.468t \tag{5,8}$$

where t, as before, is the number of years after epoch 1900.0.

Precession is accompanied by slight oscillatory motions of the earth's axis called **nutations** (see Fig. 38, positions P_5 and P_6) with a maximum period of about 18.6 years and with axes of the deviation ellipse about 18″ and 14″. Nutations are due both to changes in the perturbing forces themselves (mainly due to the moon) and in their directions.

Nutations likewise alter the coordinates of stars, but the changes are smaller; in the case of "navigational" stars, $\Delta\alpha$ and $\Delta\delta$ may be of the order of 0′.3 per year.

SEC. 23. ON VARIATIONS IN THE EQUATORIAL COORDINATES OF STARS

The coordinates α and δ of stars do not remain constant, as we have seen, but are constantly changing due to the motion of the coordinate system itself (via the phenomena of precession and nutation), and due to factors operating for each star separately. These factors include aberration, the annual parallax and the "motion proper" of the stars. The coordinates of stars observed directly (after refraction is taken into account) are **apparent** coordinates.

If the coordinates of a star (α_0, δ_0) are known for a certain initial instant (epoch), then the apparent coordinates α_* and δ_* may be

computed for any instant from the formulas

$$\left.\begin{aligned} \alpha_{*} &= \alpha_0 + \Delta\alpha_{pr} + \Delta\alpha_{nut} + \Delta\alpha_{aber} \\ \delta_{*} &= \delta_0 + \Delta\delta_{pr} + \Delta\delta_{nut} + \Delta\delta_{aber} \end{aligned}\right\} \qquad (5.9)$$

where the increments $\Delta\alpha$ and $\Delta\delta$ are computed from special formulas that take into account the time elapsed since the initial epoch.

If in formulas (5.9) we disregard all but precession, we get so-called *mean coordinates that determine the mean positions of stars*; if in addition we take nutations into account, we get the *true coordinates of the stars*; if aberration is included, we get the *apparent coordinates* or the *apparent places of the stars* at the given instant.

The *precomputed apparent coordinates (places) of stars* for future years, months and dates are recorded in nautical almanacs. For this reason, when selecting stellar coordinates for a given day no corrections need be made in practical work.

C H A P T E R 6

THE APPARENT MOTIONS OF THE MOON AND PLANETS

SEC. 24. PROPER MOTION OF THE MOON AND ITS EXPLANATION

Observations of the apparent diurnal motion of the moon (symbol ☾) during even a few days reveal more markedly those peculiarities that distinguish solar movements, namely:

(1) an eastward movement of the moon among the constellations, as a result of which we have a daily lag in the time of transit (meridian passage) of the moon;

(2) a daily change in the meridian altitude of the moon;

(3) a daily movement of the points of moonrise and moonset along the horizon.

Considering these phenomena from the viewpoint of a fixed earth and a celestial sphere rotating about it, we can conclude that the moon (like the sun) has its own **apparent motion** over the sphere, but a much faster one.

If we determine $\alpha_{☾}$ and $\delta_{☾}$ from observations and use these coordinates to mark the path of proper motion of the moon on the celestial sphere, the result will be a large circle passing near the ecliptic and situated in the belt of zodiacal constellations. This large circle is called the **apparent lunar orbit.**

The apparent lunar orbit (Fig. 40) intersects the ecliptic at an angle $i \approx 5° 8'$ in two points called the **lunar nodes*:** the **ascending** node (☊), where the moon passes across the ecliptic from the southern hemisphere into the northern, and the **descending** node (☋), where the movement is just the reverse. The movement of the moon over the sphere is in the same direction as the annual motion of the sun, that is, it is direct, but the diurnal movement of the moon is much greater, about 13°.

This apparent lunar movement over the sphere is, as we know, due to the actual motion of the moon in its orbit about the earth. Thus, the moon is not a planet, as was thought in antiquity, but a satellite of the planet earth.

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* The nodal points were called "draconic points" in antiquity. Dragons were believed at these points to be waiting for the moon to swallow it during eclipses. The symbols for the nodes resemble dragons.

The *apparent lunar orbit*, in which the moon moves over the sphere in its proper motion, *is the projection of the actual lunar orbit on a celestial sphere* of very great radius with centre at the earth's centre.

Fig. 40 shows the lunar orbit—an ellipse, in one focus of which is the earth, the mean earth-moon distance being about $384.4 \times \times 10^3$ km. The points *Per* and *A* signify **perigee** and **apogee** (the

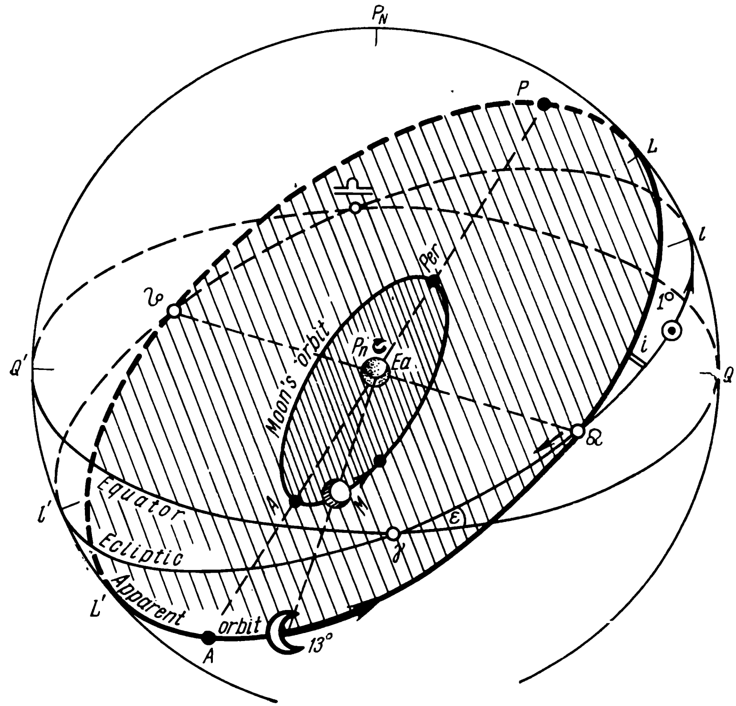


Fig. 40

closest point to the earth and the most distant point from the earth in the lunar orbit, respectively, at which points the velocity of the moon is greatest and least). At these points, the *angular radius of the moon* as seen from the earth will be greatest and least, respectively, just like the *diurnal parallax of the moon*. On the average, the angular radius of the moon $R☽ = 15' 32'' \approx 15'.5$, and the parallax $P☽ \approx 57'$. Due to the fact that the orbital velocity of the moon and the position of the orbit itself in space change rapidly, the proper motion of the moon on the sphere is exceedingly nonuniform.

Since the earth-moon system is not isolated from the action of solar gravitation and planetary gravitational effects, the lunar orbital motion does not correspond strictly to Kepler's laws, but is subjected to perturbations or inequalities, the most important of which are:

(1) regression of the line of nodes—the motion of the line ☊☋ counter to the motion proper of the moon by 19°.3 per year; as a result of this motion, the nodes execute a complete circuit round the sphere every 18.6 years;

(2) direct motion of the perigee-apogee line by 40°.7 per year;

(3) periodic oscillations of the angle of inclination i from 4° 59′ to 5° 17′ and of the eccentricity of the orbit from $^1/_{14}$ to $^1/_{23}$.

The complete theory of lunar motion is extremely complicated; for instance, the longitude of the moon is expressed by a series containing 655 terms, the latitude has 300 terms.

SEC. 25. PERIODS IN LUNAR MOTION

The entire apparent path over the sphere, that is, the full circle relative to the stars that the moon traces out during 27d 7h 43m $\approx$ $\approx$ 27d.32 is called the **sidereal** month or revolution. With respect to the sun, however, which moves 27° along the ecliptic during this same time, the moon executes a revolution in a larger interval of time called the **lunar** or **synodic** month, equal to an average of 29d 12h 44m $\approx$ 29d.53.

The lunar month serves as the basis for the calendar month.

We also distinguish periods of revolution of the moon relative to the node and perigee; these are, respectively, the draconitic month and the anomalistic month, but they are not used in nautical astronomy.

Twelve lunar months come out to about 354 days, which means the lunar year is shorter than the solar year by about 11 days, with the result that the same days of the lunar month come on different days and even different months of the calendar, year after year.

During one diurnal rotation of the sphere, the moon covers a distance (relative to the stars) of $\frac{360°}{27d.32} \approx 13°.2$ or 53m; relative to the sun, $\frac{360°}{29d.53} \approx 12°.2$ or 49m. For this reason, for the moon to execute a full diurnal circuit relative to the same terrestrial meridian, we need another 50 minutes or so in addition to the solar day, which means that the lunar day is longer than the solar day and is roughly equal to 24h 50m.

For this same reason, the moon's transit will lag relative to the transits of the stars and sun. Indeed, let us assume (Fig. 41) that

on a certain day the earth occupies a position T_1 in its orbit. At that time, the moon, sun and a star lie on one celestial meridian and, consequently, will transit simultaneously on the terrestrial meridian $P_N a$. One day later the earth will have reached position T_2 in its orbit and will have made one rotation on its axis. During this period of time, the moon will have moved (due to its proper motion) to the left by 13°.2, while the sun will have moved 1° to the left. The direction to star C remains practically constant. In this position, first to arrive at the terrestrial meridian $P_N a$ will be the star C (position a_1), which will be followed by the sun (a_2) and, finally, the moon (a_3). If, as indicated above, we compute the magnitudes of the arcs a_1a_2 and a_2a_3, we will find that the sun will transit approximately 4 minutes after the star, and the moon about 53 minutes after the star or 49 minutes after the sun.

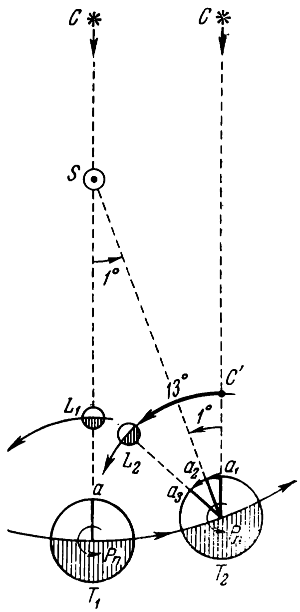


Fig. 41

Due to the rapid motion proper of the moon, its transit (meridian passage or culmination) on different terrestrial meridians occurs at different times. To the west of the given meridian, transit is late, to the east it is ahead of the time of transit for the given meridian. The amount of lag or gain will be

$$\frac{49\text{m}}{360°} \approx \frac{2\text{m}}{15°}$$

which is about 2 minutes for every 15° of longitude.

If we know the time of the moon's transit on some meridian (Greenwich, for instance), it has to be increased in western longitudes and reduced in eastern longitudes by about 2 minutes per 15° of longitude.

For example, if T_{tr} on Greenwich is 18h 4m, then the longitude $\lambda = 102°\text{E}$; $T_{tr} = 18\text{h }4\text{m} - 2\text{m}\,\frac{102°}{15°} \approx 17\text{h }50\text{m}$.

SEC. 26. PHASES AND AGE OF THE MOON. CONDITIONS OF SEEING

Like the planets and their satellites, the moon is a dark body that shines by the reflected light of the sun. From the earth we see only a part of the moon that is illuminated by the sun, and since

the position of the moon relative to the sun and earth varies, the moon also changes its appearance periodically producing a sequence of phases of the moon (phase change).

In astronomy, the regularly recurring appearances of a planet (or a satellite) as seen from another planet are known as phases. Fig. 42 shows the various orbital positions of the moon and its appearance as seen from the earth situated in the centre. In the various positions, the illuminated part of the moon is indicated by a solid diameter, while the portion visible from the earth is shown by a dotted

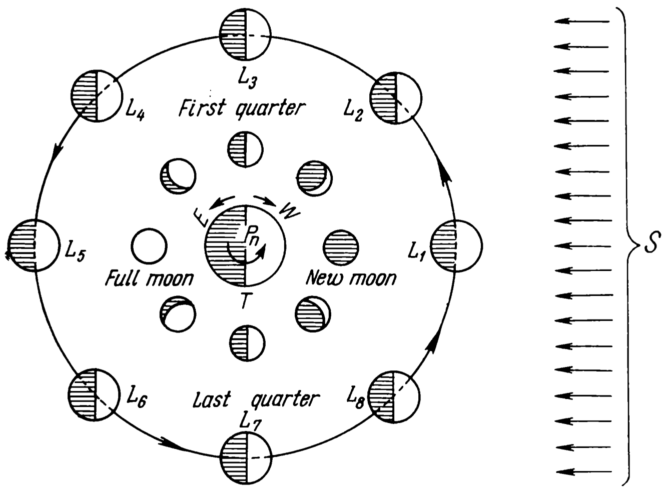


Fig. 42

line. In position L_1, the moon and sun are located on one and the same meridian, and the nonilluminated part of the moon faces the earth. This phase is called **new moon** and is taken to be the commencement of the phase sequence. The phases are also associated with the so-called **age of the moon,** *which is the time interval measured in days from new moon to the given position of the moon.* The age varies from 1d to 29d.5 or roughly 30d (0d). At new moon, the age is 0d (30d), but an age of 29 and 1 is also taken as new moon.

In positions L_2, L_3, the "young" moon is seen first as a thin crescent that gradually increases and that is convex to the west (towards the sun). The boundary of the illuminated portion (the terminator) has the form of an ellipse. The phase L_3 is called the **first quarter;** here the moon is in the form of a half-disc and is about 7.5 days old. In the position L_4, the moon is gibbous and convex westwards. In the position L_5—the full disc—the phase is called **full moon,** 14-15-16d old. In L_6 the moon is gibbous and convex eastwards.

The moon phase in position L_7 is called the **last** or **fourth quarter** where the moon is in the form of a half-disc convex to the east (towards the sun) and is 22.5 days old. The half-disc then turns into a crescent (L_8) and, finally, at L_1 we again have new moon.

The positions of the moon at new moon and full moon are also called **syzygies** or **spring**; the positions in the first and last quarters are called **quadratures** or **neap.** These terms are mainly used in studying the tides.

In position L_1, the moon, not seen from the earth, is on the same meridian as the sun and its upper transit occurs together with that

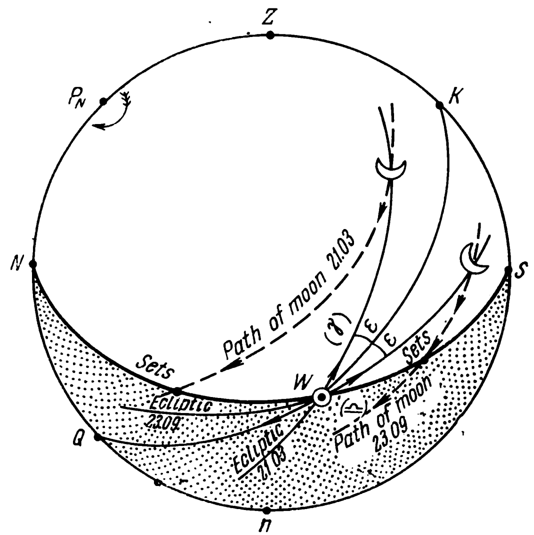


Fig. 43

of the sun, that is, at noon. Hence, on that day, the moon rises in the morning and sets in the evening, and the night is moonless. On each subsequent day the time of transit and the times of moonrise and moonset commence later and later, and when the moon is in the first quarter (L_3) after 7.5 days, its upper transit will occur at about 6 o'clock in the evening. Accordingly, moonrise will be close to noon, and moonset at about midnight.

On the day of full moon (L_5), the moon will lag behind the sun by 0.5 circle and will transit at midnight; in the last quarter (L_7) the moon will rise during the night, transit will occur in the morning hours (about 6 o'clock in the morning) and the moon will set in the daytime.

The seeing conditions are not the same in the evening and in the morning for the various times of the year. Fig. 43 shows the diurnal paths of a "young" moon on a spring evening and an autumn evening in northern latitude. Let us suppose that the moon is on the ecliptic (that is, we disregard the angle of inclination of the orbit i). Then, close to 21 March, at sunset, the point ♈ will coincide with W, and the ecliptic will be situated at an angle to the

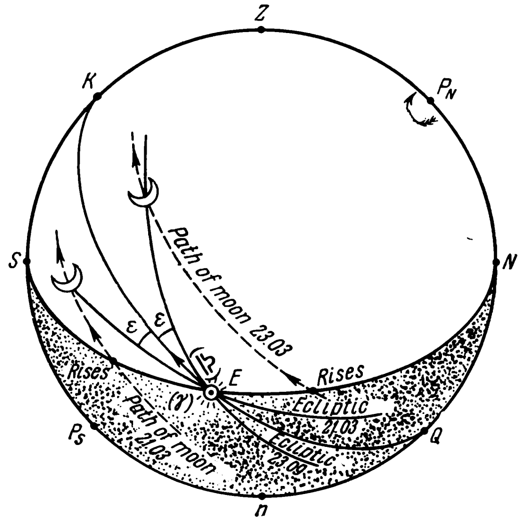


Fig. 44

horizon equal to $90° - \varphi + \varepsilon$. The apparent path of the young moon passes high above the horizon and seeing conditions will be best in the evening. In the autumn, about 23 September, point ♎ of the ecliptic will coincide with W and the obliquity of the ecliptic to the horizon will be $90° - \varphi - \varepsilon$. Then the apparent path of the moon passes just above the horizon, making seeing conditions of the moon the worst.

In the morning hours, these conditions are just the reverse. Fig. 44 shows the parallels of latitude of the old moon and its positions at sunrise on 23.09 and 21.03. Obviously, the morning seeing conditions will be better in autumn than in spring.

For joint observation of the moon and sun, the convenient phases are those of the first and last quarters and the days close to these phases, so that there will be from 4 to 6 favourable days every month for such observations.

SEC. 27. CHANGES IN LUNAR COORDINATES

Due to the proper motion of the moon, its right ascension and declination are changing constantly and rather rapidly. On the average, the magnitude of $\alpha_☾$ varies by 13°.2 per day; however, this change varies within rather broad limits.

The declination of the moon, during a circuit over the sphere, attains one maximum N value and, about 13.5 days later, one maximum S value. Due to the motion of the line of nodes, these maximum values will likewise vary from $\varepsilon + i$ to $\varepsilon - i$, depending on which node coincides with the point ♈: the ascending node (☊) or the descending node (☋). In the former case, $\delta_☾ = 23°27' + 5°8' = 28°35'$N or S, and in the latter, $\delta_☾ = 23°27' - 5°8' = 18°19'$N or S. Intermediate positions of the line of nodes yield intermediate values of $\delta_☾$. Now, due to fluctuations in the angle of inclination i, the foregoing maximal values vary still more (from 28°44′ to 18°10′) and in a more complicated fashion. Lunar coordinates are computed for several years in advance and are given in nautical astronomical almanacs.

SEC. 28. APPROXIMATE SOLUTION OF PROBLEMS ASSOCIATED WITH THE MOTION OF THE MOON

In certain practical cases, it is required to determine quickly the approximate position of the moon on the sphere and its illumination of the horizon. If the needed tables are lacking, *approximate* formulas may be used that follow from the peculiarities of lunar motions.

I. Formula for calculating the age of the moon ($B_☾$):

$$B_☾ = M + N + D \tag{6.1}$$

where M is an empirical number given in Table 3 and usually called the "Lunation number" (the numbers in the table are increased by 11 every year)

N is the number of the month of the year

D is the date.

If $B_☾$ comes out to more than 30 d, then 30 is rejected.

Table 3

Magnitudes of the *M* number

Year	1967	68	69	70	71	72	73	74	75
M **number**	16	27	8	19	0	11	22	3	14

II. Formula for computing the local time of upper transit of the moon, T_{tr}:

$$T_{tr} \approx 12\text{h} + B_{☾} \cdot 0\text{h}.8 \tag{6.2}$$

where 12h is the approximate local time of transit of the sun;
Oh. 8 (or 49m) is the diurnal lag in transit of the moon relative to the sun.

III. Formulas for rough computation of time of moonrise (T_r) and moonset (T_s);

$$T_r \approx T_{tr} - 6\text{h};\ T_s \approx T_{tr} + 6\text{h} \tag{6.3}$$

More precise results are obtained by plotting the lunar parallels of declination on a drawing of the sphere or on a star globe.

IV. Formula for computing the right ascension of the moon ($\alpha_{☾}$):

$$\alpha_{☾} \approx \alpha_{\odot} + B_{☾} \cdot 12^{\circ} \tag{6.4}$$

where $\alpha_{\odot}$ is an approximate value of the right ascension of the sun;
12° is the diurnal advance of the moon over the sun in motion proper.

V. An approximate (up to 5° to 8°) value of the declination of the moon $\delta_{☾}$ may be obtained from a drawing of the sphere from the plotted value of $\alpha_{☾}$ and the ecliptic. Disregarding the angle i, the position of the moon is taken at the point of intersection of the ecliptic and the meridian of the moon.

Example 1. 6 August 1968, $\varphi = 48°$N. Determine $B_{☾}$, T_{tr}, T_r, T_s, $\alpha_{☾}$, and the position of the moon at 23h.

(1) $B_{☾} = 27 + 8 + 6 = 11\text{d}$. The moon is gibbous and convex westwards.

(2) $T_{tr} = 12\text{h} + 11\text{d} \cdot 0.8\text{h/d} \approx 21\text{h}$.

(3) $\alpha_{☾} = 135° + 11\text{d} \cdot 12°/\text{d} = 267°$.

(4) $T_r = 21\text{h} - 6\text{h} = 15\text{h}$; $T_s = 21\text{h} + 6\text{h} = 3\text{h}$.

(5) Using Fig. 45, we find the position of the moon at 23h: $A \approx$ SW 30°; h = 10°.

The M number may be computed approximately from the peculiarities of lunar motion. As far back as in ancient Greece it was found that every 19 years the age of the moon recurs on the same days of the year. The reason, as indicated by the Greek astronomer Meton, is that 19 tropical years are equal to 235 lunar months to within 0.1 day. The period became known as the lunar cycle or the Metonic cycle, and the number of the year in this 19-year cycle is the "golden number" (R). Knowing the golden number and taking into account that the lunar year, equal to 12 synodic months, is shorter than the tropical year by 11 days, we may obtain the so-called epact (Ep), or the age of the moon at the commencement of any year of the Metonic cycle. The first year of our era (the number of which is 0) coincided with the first year of the Metonic cycle, and the

moon's age was zero. Therefore, dividing the number of the year +1 by 19, we get the golden number as the remainder, and from the latter we get the epact and the M number with an accuracy to within 1 to 2 days.

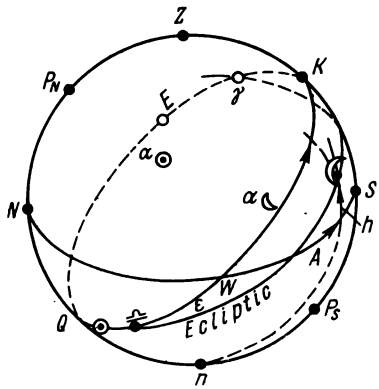


Fig. 45

Example 2. 1) $1958+1=1959$; 2) $1959:19=103+\frac{2}{19}$ and remainder 2; hence, $R=2$. 3). The epact $=(R-1)\cdot 11=11$ days. When Ep >30, the period 30 is dropped. 4) $M=$ epact $-3=8$, where 3 is an empirical coefficient. Thus, in 1958 $M=8$ (more precisely, 7).

SEC. 29. THE FUNDAMENTALS OF TIDES

Observations of the ocean level show that in the general case it attains two peaks a day (high water or high tide) and two minima (low water or low tide). These phenomena are of a wave nature and lag daily about 50 minutes, thus indicating their relationship with the movements of the moon. Celestial mechanics gives a dynamic explanation of the tides. The flood tide is due to the attraction of particles of water in the oceans by the moon and sun, and since the tidal force is inversely proportional to the cube of the distance, the attraction of the moon is 2.2-fold that of the sun. The tidal wave forms on the terrestrial meridian nearest the moon, and also on the meridian opposite it (Fig. 46), and follows the moon. However, the tidal maximum does not occur at the time of upper or lower transit of the moon, but later by an amount known as the **lunitidal interval.** This tidal lag is due to the friction of the particles of water, the shape of the sea bottom and coastline, and to other local causes.

The size of the tidal wave depends on the position of the moon (L) relative to the sun (S) and the earth (T). The largest tides occur during new and full moon (Fig. 46) and are called spring tides, the

smallest occur at quadrature and are known as neap tides. Tides are characterized by the time and height of high and low water. To determine the times of flood and ebb, use is made not of the variable lunitidal interval, but of a constant interval called the **High**

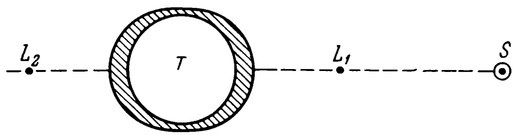


Fig. 46

Water Full and Change, which is the *arithmetic mean of the lunitidal intervals in the mean equinoctial syzygy.*

Tidal data are given in special tables of tides and other manuals. The tides are very important in navigation and are therefore studied in oceanography and in pilotage (tidal aids and how to work with them).

SEC. 30. PLANETARY MOTIONS PROPER

In ancient times it was already known that not all stars are fixed. Some of them—which the ancient astronomers called **planets—**

Data on Planets

Planets (ordered as to distance from sun)		Astronomical symbol	Equatorial diameter (km)	Angular dimensions (greatest and least)	Eccentricity	Inclination to plane of earth orbit
Inferior	Mercury	☿	4,840	4″.7-12″.9	0.2056	7°0′.2
	Venus	♀	12,400	9.9-65.2	0.0068	3 23.6
	Earth	♁	12,756	—	0.0167	0
Superior	Mars	♂	6,780	3.5-25.5	0.0934	1°51′.0
	Asteroids }	—	770 and less	—	Mean ≈ 1/7	Mean 9°.5
	Jupiter	♃	143,640	30.5-50.1	0.0484	1°18′.4
	Saturn	♄	120,500	14.7-20.7	0.0557	2 29 .4
	Uranus	♅	53,400	3.4-4.3	0.0472	0 46 .4
	Neptune	♆	49,600	2.2-2.4	0.0086	1 46 .5
	Pluto	♇	⩽13,000	0.19-0.24	0.2470	17 6 .5

wander over the celestial sphere like the moon, though much more slowly and along more intricate pathways.

To account for such motions and to be able to precompute the positions of the planets, the ancient astronomers who adhered to the geocentric system of Ptolemy had to think up complex systems of circular planetary motions round the earth. However, even the most intricate of such motions yielded inaccurate results. It was only the heliocentric system of Copernicus plus the laws of Kepler that reduced planetary motions to a simple and integrated system. Planets are celestial bodies revolving about the sun in specific orbits; hence the earth is a planet too. The planetary orbits, like that of the earth, describe ellipses, at one focus of which is the sun. Table 4 contains some essential facts about the planets and their orbits.

From the table it will be seen that the planetary orbits have different eccentricities and inclinations to the earth's orbit, though these inclinations are very small, which means that all the planets revolve about the sun in approximately the same plane and in the same direction. Consequently, the projections of all planetary orbits (except that of Pluto) on the celestial sphere lie close to the ecliptic in the belt of zodiacal constellations.

Table 4

Data on Orbits and Planet Motion

Mean distance to sun		Sidereal period of revolution	Orbital velocity km/s	Mean angular orbital velocity per day	Mean distance from earth, 10^6 km			
mln km	AU				inferior conjunction	superior conjunction	opposition	conjunction
57.9	0.387	87d23h15m.7	47.8	4°5′.5	91.6	207.4	—	—
108.1	0.723	224 16 49.1	35.0	1 36.1	41.4	257.6	—	—
149.5	1.000	365 6 9.2	29.8	0°59′.1				
227.8	1.524	686d2h33m0.8	24.1	0°31′.4	—	—	78.3	377.3
Mean 418	Mean 2.8	1.76 to 13.7 y	—	—	—	—	—	—
777.8	5.203	11.826 y	13.1	0 5 .0	—	—	628.3	927.3
1,426.1	9.539	29.457 y	9.6	0 2 .0	—	—	1,276.6	1,575.6
2,869.1	19.191	84.015 y	6.8	0 0 .7	—	—	2,719.6	3,018.6
4,495.6	30.071	164.782 y	5.4	0 0 .4	—	—	4,346.1	4,645.1
5,929	39.656	249.73 y	4.7	0 0 .2	—	—	4,315	7,542

The planets whose orbital semiaxes are less than that of the earth's orbit (Fig. 47) are called **inferior** planets (Mercury and Venus). All the other planets whose orbits lie beyond the earth's orbit are called superior planets.

The angular and linear velocities of the inferior planets are greater than the velocity of the earth, those of the superior planets are less, as should follow from Kepler's third law. The inferior planets can occupy the following positions relative to the earth and sun (Fig. 47): **inferior conjunction**—the planet at a lies between the sun and earth; **superior conjunction**—the planet lies beyond the sun at point b; **elongation**—western at point d and eastern at c, which is the greatest angular recession of the planet from the sun. The greatest elongation of Venus is 48°, of Mercury, 28°.

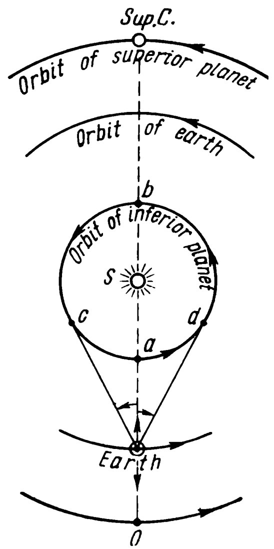


Fig. 47

The superior planets may occupy the following positions (Fig. 47): b at conjunction when the planet is on the other side of the sun, O at opposition, when the earth lies between the sun and the planet. From the foregoing it is obvious that the inferior planets can, in their apparent motion, recede from the sun on the sphere only to strictly definite angular distances (elongations), while the superior planets may recede to any distances (up to $\pm 180°$).

If from observations we obtain α and δ of a planet and plot its apparent path on the celestial sphere or a map, we will get a curve close to the large circle of the ecliptic, but one that has an intricate loop-like appearance. Fig. 48 shows the apparent path of the inferior planet Mercury between 1.01 and 1.07.56. Arrows indicate the direction of motion. Up to point C_1 the planet moves with the sun, *direct* motion; C_1 is a so-called *stationary* point of the planet, which means that its coordinates do not change for a certain time. From C_1 to C_2 the planet moves towards the sun describing retrograde motion. C_2 is another stationary point; from here the planet is in rather fast direct motion up to C_3, where a change of direction takes place again, and so forth. Fig. 49 shows the apparent path of the superior planet Mars between May and December 1956. At first we have direct motion; C_1 is a stationary point, followed by retrograde motion up to C_2, and, finally, again direct motion.*

* Between C_1 and C_2 there occurred the "great opposition" of Mars of 1956 when the earth-Mars distance decreased to 56.5 million kilometres.

We have already mentioned that planetary movements over the celestial sphere are rather simply accounted for by the orbital motion of the planets in one direction but with different velocities.

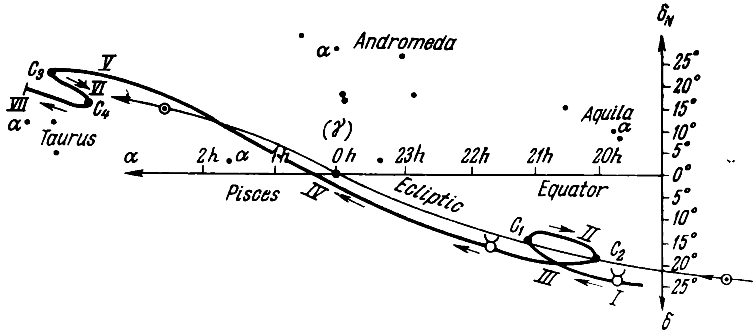


Fig. 48

Indeed, let Fig. 50 be a projection of the celestial sphere on the plane of the ecliptic and the projections of the orbit T of the earth and the orbit of the inferior planet Venus H. When the earth is at

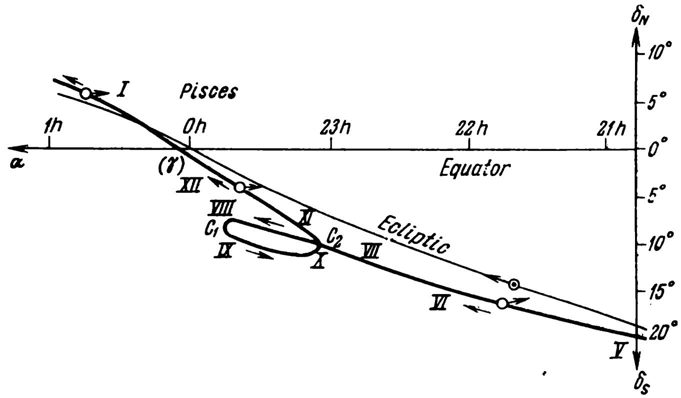


Fig. 49

T_1, the planet H_1 is in inferior conjunction with the sun and is seen on the sphere at point *1* (arrows indicate earth-planet directions). When the earth moves to T_2, the planet (whose orbital velocity is greater than that of the earth) will cover a greater distance and will come to H_2, which is projected on the celestial sphere at point *2*, and its motion between *1* and *2* and *3* will be retrograde motion. In position near T_3 the planet is seen from the earth along the tangent to its orbit as receding from the earth in an unchanging direc-

tion; all positions of the planet near H_3 are projected on the sphere at point *3*, which is a stationary point of the planet. As the earth moves from T_3 to T_4 and T_5, the motion of the projection of the planet from point *3* to *4* and *5* will be direct and more rapid than the motion of the sun. At T_5 the planet H_5 is in superior conjunction with the sun, and about the position H_3 it will be in western elongation. During the motion of an inferior planet, its illuminated

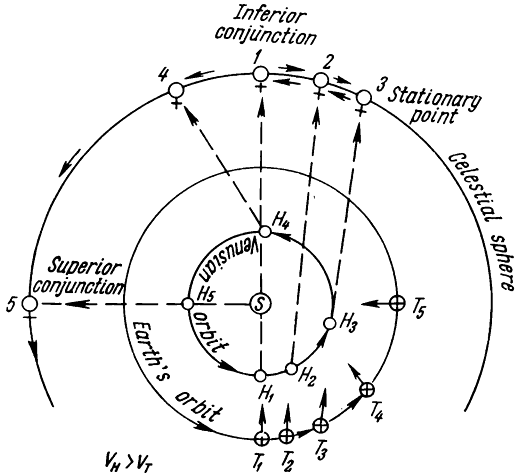


Fig. 50

portion is turned towards the earth, and then away from the earth, which means that, like the moon, the planet is seen in various phases.

The motion of a superior planet B is explained in Fig. 51. The orbital velocity of the earth T is greater than that of the planet B, with the result that the projection of the planet on the celestial sphere first executes direct motion (*1-3*), then is stationary (*3*, *4*, and *6*, *7*), then is in retrograde motion (*4-6*); and at T_5 planet B_5 is in opposition. As will be seen from the figure, no change of phases is observed in the superior planets.

Due to the apparent motion proper of the planets, the equatorial coordinates α and δ are changing constantly and nonuniformly. For the inferior planets the diurnal change of α may be greater than for the sun. For instance, for Venus the greatest $\Delta\alpha$ will be of the order of 1°24′ per day or 3′.5 per hour. Now the superior planets have smaller $\Delta\alpha$ than the sun. The α of a planet increases in direct motion and diminishes in retrograde motion. The maximal declinations of the brightest planets do not go beyond the limits of 27°N or S due to the fact that they move close to the ecliptic.

Only the four brightest planets are utilized in nautical observations: Venus, Mars, Jupiter and Saturn. The brightnesses and visibility conditions of these so-called navigational planets vary depending on the distance from the earth and their position on the celestial sphere.

The inferior planet Venus is lost in the sun's rays and is not visible from the earth at superior and inferior conjunction (Fig. 50). In position H_3 (western elongation) Venus is seen *in the morning* before sunrise; in eastern elongation it is seen *in the evening* after sunset. Venus is brightest (about Mag. 4.2)* in phase 0.25, when a quarter of the disc is visible, since in this position it is considerably closer to the earth than in the full-disc phase.

The brightest planets, Venus and Jupiter, are seen even in broad daylight, but only in the telescope of a sextant. At such a time, the location of a position may be found from simultaneous observations of, say, the sun and Venus.

The superior planets (Mars, Jupiter, Saturn) are invisible only near superior conjunction when they are lost in the sun's rays. These planets vary considerably in brightness. Mars ordinarily has a brightness of about Mag. 1, while during a great opposition the brightness increases to Mag. —2.5. The brightness of Jupiter fluctuates between Mag. —2.5 to —1.5.

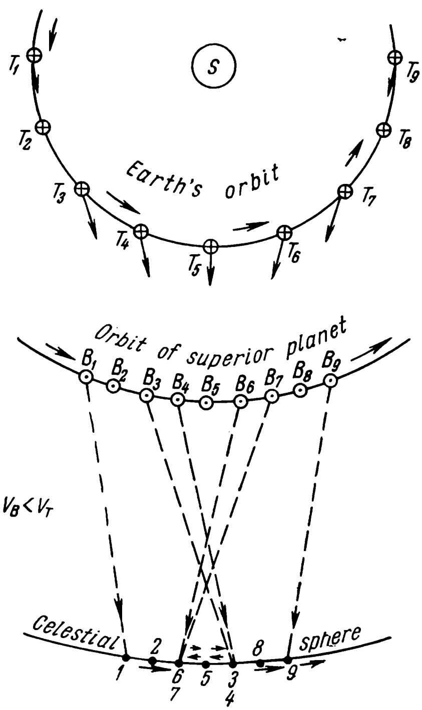


Fig. 51

It is comparatively easy to recognize the navigational planets: Venus is always close to the sun and is therefore seen only as a *bright white evening* or *morning star*. Mars has a *reddish-orange hue*, Jupiter is *yellowish*, Saturn is *white*. The planets do not twinkle as do even the brightest stars. Almanacs indicate the visibility conditions of the planets for every month of the given year.

* About brightness see Sec. 31.

CHAPTER 7

THE STELLAR SKY

SEC. 31. ON THE CLASSIFICATION OF STARS

Already in remote antiquity, the stars in the sky had been divided into groups, called **constellations,** each with its specific configuration. This was done for purposes of orientation in the sky and frequently had a religious basis. That classification of stars according to "mutual positions on the sphere" has, with slight modifications, come down to the present day.

The names of the constellations were taken mainly from ancient myths and legends (for example, Perseus, Andromeda, and others) and only in the southern part of the sky do we encounter modern names (such as Compass) that were given in the nineteenth century. The boundaries of the constellations were drawn to fit the objects which they represented. It was only in 1928 that the boundaries of constellations were internationally agreed upon and followed the arcs of celestial meridians and parallels of declination. The old boundary lines were of course taken into account. The accepted practice is to consider all stars (including those not visible) within the limits of a constellation as belonging to that constellation.

One must bear in mind that constellations are artificial groupings of stars located at various distances (sometimes very great distances) from each other and are simply projected onto the celestial sphere in a contiguous group.

At the present time there are 88 constellations in all; of this number, less than 50 are used for purposes of navigation.

The constellations and certain of the brighter individual stars have been given names. These names have come down to us mainly from ancient Greek and Arabian times (10-12th centuries) and are likewise connected with myths and religious beliefs.

To illustrate, the star beta (β) of the constellation Perseus was called "El-gul" (Demon Star) by the Arabs because it periodically varied in brightness from 2 to 4 stellar magnitudes,* which fact—

* The star β Persei is a double star; variations in brightness are due to eclipses of the brighter star by the faint companion. Such stars are now called eclipsing binaries or Algol-type stars.

on the background of the other "invariable" stars—was believed to be the doings of the devil. The Europeans modified this name to Algol, and that is what it is today.

Each bright star in the sky is designated by a letter of the Greek alphabet (α, β...) according to the place it occupies in the constellation; the smaller stars of the given constellation are simply numbered. Stars with names are also given these designations. For instance, α Ursae Majoris is called Dubhe, α Bootis is Arcturus, and so forth.

Lists of large numbers of stars in their constellations are given in star catalogues with coordinates α and δ; shorter lists are given in almanacs.

The *apparent brightness* of a star is characterized by its **stellar magnitude,** denoted by Mag. The brightest stars in the sky (there are about 20 in all) are of the first magnitude and brighter, the faintest stars visible to the unaided eye (about 5,000 in all) are of the sixth magnitude.

In the middle of last century, this rather arbitrary division of stars as to brightness was given a physical justification. It was established that stars of first magnitude appear to the eye brighter than those of second magnitude to the same extent that second-magnitude stars are brighter than third-magnitude stars, and so forth. In other words, the stellar brightnesses (I_1, I_2...) of different magnitudes form a geometric progression. To obtain the value of the common ratio of the progression q, it was agreed that first-magnitude stars are to be considered 100 times brighter than sixth-magnitude stars, or $q^5 = \frac{I_1}{I_6} = 100$, whence $q = \sqrt[5]{100} = 2.512$.

Thus, the relationship between the brightnesses (or brilliance) I_1 and I_2 of two stars and their stellar magnitudes m_1 and m_2 is expressed by the formula

$$\frac{I_1}{I_2} = 2.512^{(m_2-m_1)} \tag{7.1}$$

or

$$\log \frac{I_1}{I_2} = 0.400\,(m_2 - m_1) \tag{7.2}$$

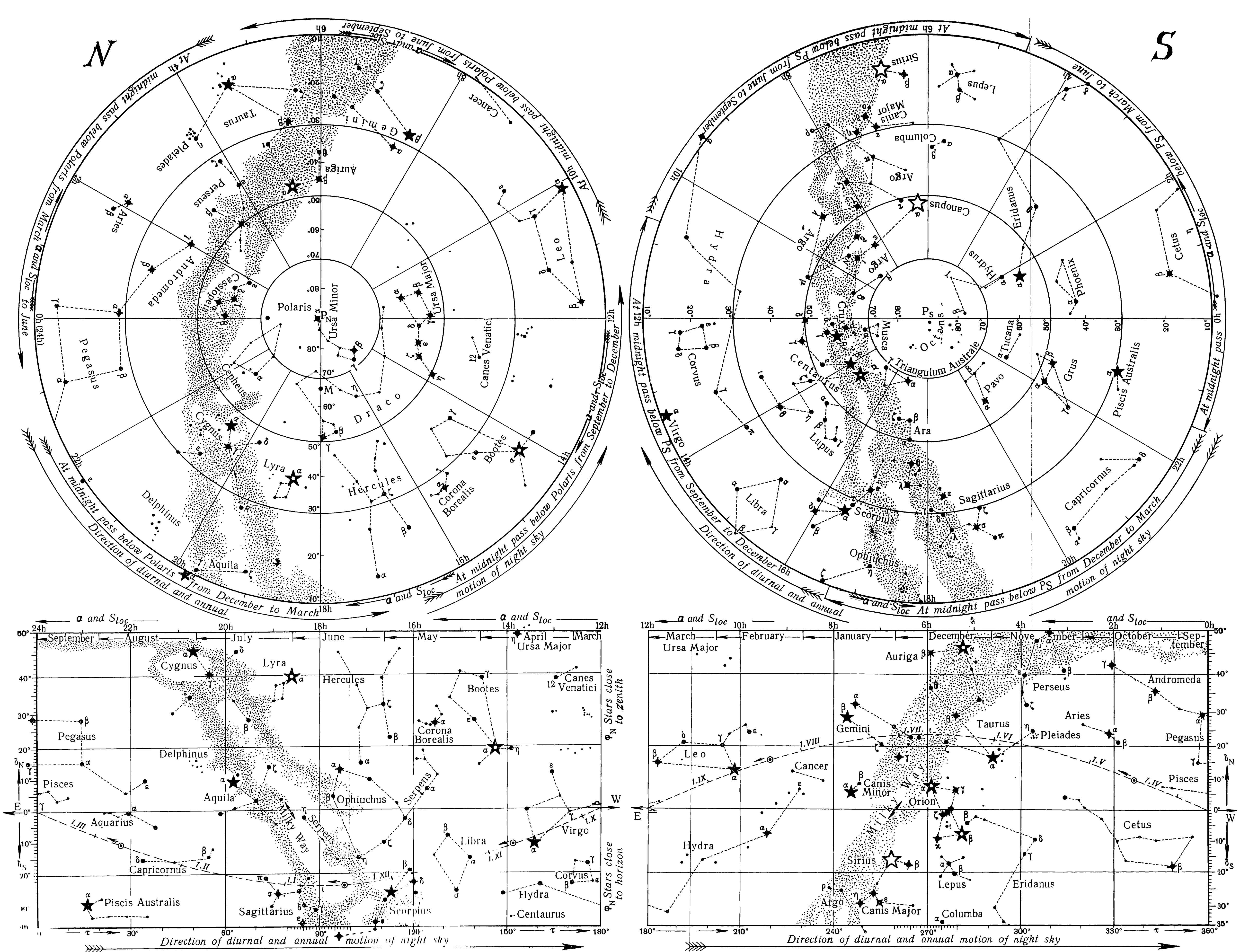
Formula (7.1) shows that for an increase in stellar magnitude by one unit the brightness of the star must increase approximately 2.5 times.

On this basis, all stars (celestial bodies) may be arranged in a series as to brightness (and, hence, magnitude); stellar magnitudes may come out fractional or even negative.

Table 5

Principal Navigational Constellations and Stars

No.	Constellation name	Star	Proper name	Magnitude	Approx. α_*	τ_* 1960.5	Annual variations	δ_* 1960.5	Annual variations
1	Andromeda	α	Alpheratz	2m.2	00h 06m	358°24′.8	—0′.78	N 28°52′.2	+0′.33
2	Argo	α	Canopus	—0. 9	6 23	264 14 .4	—0 .33	S 52 40 .5	+0 .03
3	Gemini	α	Castor	2. 0	7 32	246 59 .3	—0 .96	N 31 58 .5	—0 .13
		β	Pollux	1. 2	7 43	244 16 .9	—0 .92	N 28 07 .3	—0 .15
4	Ursa Major	α	Dubhe	2. 0	11 01	194 41 .0	—0 .92	N 61 58 .1	—0 .32
		ε	Alioth	1. 7	12 52	166 55 .8	—0 .66	N 56 10 .8	—0 .33
		η	Benetnasch	1. 9	13 46	153 30 .3	—0 .59	N 49 30 .9	—0 .30
5	Canis Major	α	Sirius	—1. 6	6 43	259 09 .3	—0 .66	S 16 39 .8	+0 .08
6	Auriga	α	Capella	0. 2	5 14	281 34 .0	—1 .11	N 45 57 .4	+0 .06
7	Bootes	α	Arcturus	0. 2	14 14	146 32 .1	—0 .68	N 19 23 .5	—0 .31
8	Hydra	α	Alphard	2. 2	9 26	218 35 .6	—0 .74	S 08 29 .3	+0 .26
9	Virgo	α	Spica	1. 2	13 23	159 13 .3	—0 .79	S 10 57 .3	+0 .31
10	Ophiuchus	α	Rasalhague	2. 1	17 33	96 43 .2	—0 .70	N 12 35 .4	—0 .04
11	Cassiopeia	α	Schedar	{ 2. 1* / 2. 6	00 38	350 26 .1	—0 .85	N 56 18 .9	+0 .33
12	Cetus	β	Diphda	2. 2	00 42	349 36 .0	—0 .75	S 18 12 .1	—0 .33
13	Cygnus	α	Deneb	1. 3	20 40	49 58 .3	—0 .51	N 45 08 .3	+0 .21
14	Leo	α	Regulus	1. 3	10 06	208 26 .2	—0 .80	N 12 09 .7	—0 .29
15	Lyra	α	Vega	0. 1	18 36	81 05 .6	—0 .51	N 38 44 .9	+0 .06



N
S
At midnight pass below Polaris from December to March
Direction of diurnal and annual motion of night sky
α and S_{loc}
At midnight pass below Polaris from September to December
At midnight pass below PS from June to September
At midnight pass below PS from March to June
At midnight pass below PS from September to December
At midnight pass below PS from December to March
Polaris
Ursa Minor
Ursa Major
Draco
Cassiopeia
Cepheus
Cygnus
Lyra
Hercules
Corona Borealis
Bootes
Canes Venatici
Leo
Cancer
Gemini
Auriga
Taurus
Pleiades
Perseus
Aries
Andromeda
Pegasus
Delphinus
Aquila
Sirius
Canis Major
Lepus
Columba
Argo
Canopus
Eridanus
Hydrus
Cetus
Phoenix
Tucana
Octans
Musca
Crux
Centaurus
Triangulum Australe
Pavo
Grus
Piscis Australis
Hydra
Corvus
Virgo
Lupus
Ara
Libra
Scorpius
Sagittarius
Capricornus
Ophiuchus
September
August
July
June
May
April
March
February
January
December
November
October
Canis Minor
Orion
Pisces
Aquarius
Serpens
Milky Way
Stars close to zenith
Stars close to horizon
Direction of diurnal and annual motion of night sky

Fig. 52

16	Ursa Minor	α	Polaris	2. 1	1 56	330 58 .5	—10 .9	N 89 04 .4	+0 .29
17	Canis Minor	α	Procyon	0. 5	7 37	245 41 .9	—0 .78	N 5 19 .6	—0 .15
18	Aquila	α	Altair	0. 9	19 49	62 46 .9	—0 .73	N 8 45 .9	+0 .16
19	Orion	α	Betelgeuse	{ 0. 1* 1. 2	5 53	271 45 .0	—0 .81	N 7 23 .9	+0 .01
		β	Rigel	0. 3	5 13	281 50 .8	—0 .72	S 8 14 .9	—0 .07
		γ	Bellatrix	1. 7	5 23	279 15 .2	—0 .80	N 6 18 .8	+0 .05
20	Pegasus	α	Markab	2. 6	23 03	14 18 .0	—0 .75	N 14 59 .5	+0 .32
21	Perseus	α	Mirfak	1. 9	3 21	309 37 .9	—1 .07	N 49 43 .0	+0 .21
22	Corona Borealis	α	Alphecca	2. 3	15 33	126 44 .6	—0 .64	N 26 51 .1	—0 .20
23	Scorpius	α	Antares	1. 2	16 27	113 15 .0	—0 .92	S 26 20 .7	+0 .13
24	Sagittarius	σ	Nunki	2. 1	18 53	76 47 .5	—0 .93	S 26 20 .7	—0 .08
25	Taurus	α	Aldebaran	1. 1	4 34	291 35 .6	—0 .86	N 16 25 .7	+0 .12
26	Centaurus	α	Rigil Kentaurus	0. 3	14 37	140 46 .0	—1 .02	S 60 40 .5	+0 .25
27	Eridanus	α	Achernar	0. 6	1 36	335 56 .7	—0 .56	S 57 26 .0	—0 .30
28	Crux	α	Acrux	{ 1. 6* 2. 1	12 24	173 53 .9	—0 .84	S 62 53 .1	+0 .33
29	Piscis Austrinus	α	Fomalhaut	1. 3	22 55	16 07 .9	—0 .83	S 29 49 .7	—0 .32
30	Triangulum Australe	α	Atria	1m. 9	16h44m	108°52′.2	—1′.59	S 68°57′.5	+0′.11

* Variable.

The Pole Star (Polaris), of magnitude 2.15, is taken as the basis of the scale of stellar magnitudes.

On this scale we get the following magnitudes for a number of stars: Capella 0.2; the brightest star in the sky, Sirius, —1.6; Venus (mean) —3.8; the moon (at full-moon) is —12.5; and the sun is —26.8, etc.

Star brightnesses are determined by a variety of methods: visual, photographic, photoelectric and radiometric. For a single star, these techniques yield several different stellar magnitudes, but the differences are not significant for nautical observations.

In nautical astronomy, the term "navigational star" is used. These stars are the brightest in various parts of the sky and are convenient for nautical observations (as a rule, they are brighter than second magnitude); in all there are about 60 such stars.

Table 5 lists the main navigational constellations and stars, their magnitudes, approximate values of α_* and exact values of τ_* and δ_* for the epoch 1960.5 with their variations for one year.

Star maps (Fig. 52, insert) are made in rectangular projections: the polar zones in stereographic projection, the equatorial zones in Mercator projection. The sky is that seen by an observer looking from inside the celestial sphere, that is, at the actual sky. The maps show the brightest (mostly navigational) constellations, and also the constellations of the zodiacal belt and constellations used for orientation; for which reason stars of magnitude less than 3.5 are included. Stellar magnitudes are indicated by the following designations:

☆ — brightness greater than Mag. — 0.5

✡ — from —0.5 to +0.5

★ — from +0.5 to +1.5

✦ — from +1.5 to +2.5

● — from +2.5 to +3.5

· — less than +3.5.

At the top of the equatorial-zone map is a scale of months. The stars given under a date on the scale are seen at midnight near their *upper transit*. On maps of the polar zones, the stars are shown as seen for their *lower transit*. Stars situated on the opposite meridian will be seen near their upper transit. These maps may be used for identifying stars. To do this, hold the map over your head, position the zenith on the parallel equal to φ and turn the map to the given T_{loc}.

SEC. 32. STAR AND CONSTELLATION IDENTIFICATION

To identify stars, note the following:
(1) the constellation in which it is located;
(2) the configuration of stars;
(3) the brightness and the colour of the stars.

It is convenient to identify constellations and stars relative to easily recognizable and prominent constellations which one should know very well and be able to find at any time in the night. For this purpose, let us take the constellations Ursa Major and Orion.

The principal orientation constellation of the *northern sky* is *Ursa Major*, which has the shape of a dipper with a long, slightly

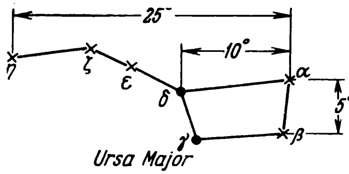


Fig. 53

bent handle (Fig. 53). Fig. 14 shows its location in the sky in autumn for different hours. It is always very easy to locate this constellation.

All seven principal stars* are white in colour and have approximately the same magnitude, about 2 and are suitable for nautical observations. Nearly all the basic constellations of the northern sky may be located via their positions *relative to Ursa Major* by dividing the sky into sectors: upwards from the dipper, towards the bottom, towards the handle, and so forth. To find individual stars located in these sectors, let us connect the stars of Ursa Major with straight lines (these are the arcs of large circles on the sphere) and extend them to indicated distances. To do this, first remember all the stars of the constellation Ursa Major (abbreviated UMa).

I. Let us examine the constellations located upwards from the dipper of Ursa Major.

(a) Connecting the stars α and β UMa with a straight line (Fig. 54) and extending it towards the star α to a distance five times that between α and β (the α—β distance is roughly 5°), we see the *Pole Star* (*Polaris*—α *Ursae Minoris*) —a white and rather faint star which

* The constellation of Ursa Major occupies an area of 1,280 square degrees and numbers 125 stars brighter than Mag. 6.0. The angular distances between the principal stars of Ursa Major are shown in Fig. 53.

is, however, quite prominent among the surrounding fainter stars. The constellation Ursa Minor also has the form of a dipper, but it is smaller and is made up of very faint stars. The Pole Star is located at the tip of the handle of this dipper. As a check, it is well to bear in mind that the azimuth of the Pole Star is close to N, and its altitude is approximately equal to the latitude of the ship.

(b) If we connect δ UMa and the Pole Star (Polaris) with a straight line and extend it the same distance beyond Polaris, we will find ourselves in the constellation *Cassiopeia*, the five principal stars of which are of magnitude between 2 and 3 and are white in colour, forming a figure that resembles a stretched W.

This constellation is used for orientation in the adjacent region of the sky. It will be noted that Polaris is located at a distance of about 1° from the celestial pole in the direction of Cassiopeia.

These constellations, together with the fainter Cepheus and Draco, are *circumpolar* constellations and are visible in the latitudes of the Soviet Union.

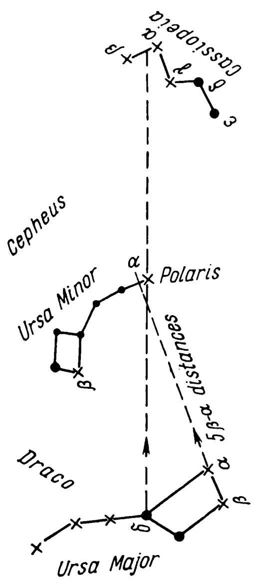


Fig. 54

(c) Extend the two lines β-α UMa and δ UMa-Polaris beyond the constellation Cassiopeia to a distance about equal to the Polaris-

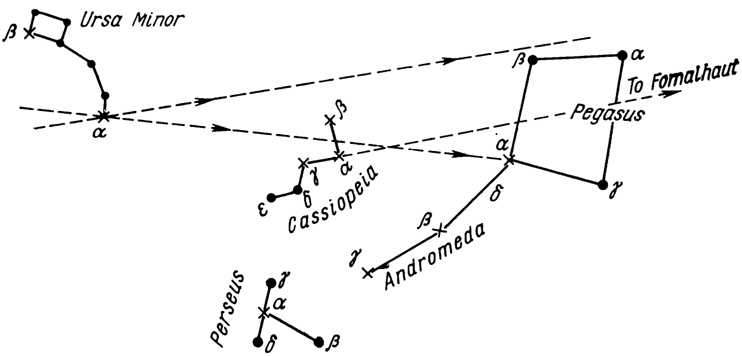


Fig. 55

Cassiopeia distance (Fig. 55). This is the constellation *Pegasus* in the form of a large square made up of fainter stars than in Cassio-

peia. One of the stars of the square (closest to Cassiopeia) belongs to the constellation Andromeda, which encloses Cassiopeia in a chain of three white stars. One end of this chain takes us to constellation *Perseus*, which likewise has the shape of a short chain directed towards Cassiopeia. The star β Persei is Algol mentioned above. The stars of these constellations are rarely used for nautical observations.

(d) There is a bright star *Fomalhaut* (α *Piscis Australis*) in the direction of the straight line γ-α Cassiopeiae, through the middle of Pegasus, two Cassiopeia-Pegasus distances to the south.

II. Let us examine a sector of the northern sky above the handle of the dipper of Ursa Major (Fig. 56). This area includes three constellations with bright stars.

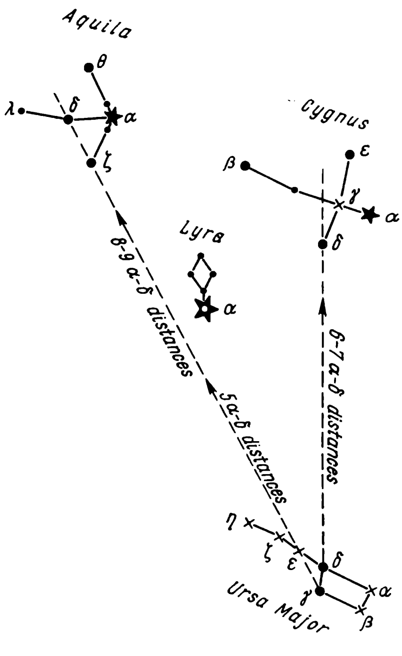


Fig. 56

(a) Connecting the stars γ and δ UMa (Fig. 56) and continuing 6 to 7 distances between the stars δ and α UMa ("the openings of its dipper"), which we take as unity (equal to 10° in the sky), we find the constellation *Cygnus* the Swan (also has the form of a cross or airplane) with the bright white star *Deneb* (which means "tail") to the rear. The Swan is "flying" along the Milky Way on which the star is located.

(b) Connecting the stars γ and ε UMa with a straight line and extending it five α-δ distances, we find nearby a bright white star, Vega (α Lyrae) of magnitude 0.1. This is the brightest star of the northern sky. Near it is a diamond made up of faint stars in the constellation *Lyra*.

(c) If we continue this line much farther—8 or 9 times the α-δ distance—we will see the constellation *Aquila*, somewhat reminiscent of a jet-plane flying towards the Swan (Cygnus). The brightest star of this constellation is white Altair situated slightly in front. Altair, Vega and Deneb are constantly used for nautical observations in the summer time*.

* These three stars are located at the vertices of a triangle called the "summer triangle".

III. Towards the handle of the dipper of Ursa Major and under the handle are four bright constellations (Fig. 57).

(a) Connecting the stars ε and η UMa with a straight line and extending it 3 to 4 α-δ distances, we find the constellation *Corona Borealis* with the rather bright star of *Alphecca* (or *Gemma*) in the middle of a chain of small stars that resemble a crown.

(b) Connecting the stars ζ and η UMa with a straight line and extending it a long distance (8 to 9 α-δ UMa distances) we will see

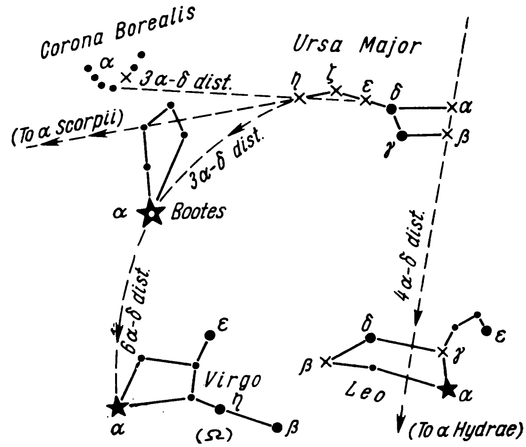


Fig. 57

a large cluster of bright stars called the constellation *Scorpio* with a bright reddish star, Antares (α Scorpii). The name Antares means "similar to Mars" and is due to its bright reddish colour.

(c) Continuing the arc formed by the stars of the "handle of the dipper" ε-ζ-η UMa out to 3α-δ distances, we find a bright light-orange star of magnitude 0.2. This is Arcturus (α *Boötis*), one of the brightest stars that is constantly used in nautical observations.

(d) Continuing this arc ε-ζ-η still farther beyond Arcturus to the same distance, we find the constellation *Virgo* with the rather bright white star Spica (α Virginis)*.

IV. "Under the bottom of the dipper" of Ursa Major there are two constellations: Leo and Hydra.

(a) Connecting the stars α and β UMa with a straight line and extending it in a direction opposite that of Polaris (Fig. 57) out to 4 or 5 α-δ distances, we find the constellation *Leo* in the form

* Between the stars η and β of this constellation is the autumnal equionx point.

of a resting lion (perhaps more like a flat-iron). In the lower corner is a bright yellowish star, Regulus (α Leonis) and Denebola (which means "tail of the lion") at the other end of the constellation.

(b) Continuing this line another 2α-δ distances, we see a rather faint but prominent star in this star-poor region: Alphard (α Hydrae), which means "lonely".

V. Let us examine the constellations and stars located in areas opposite the "handle of the dipper", that is, towards its edge and

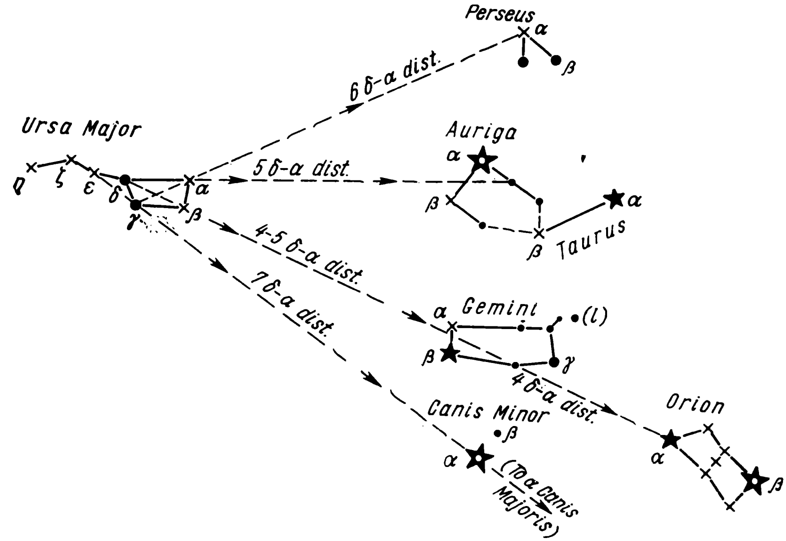


Fig. 58

lower corner. In these directions, there are no bright constellations and stars near Ursa Major, but at 5 to 7 α-δ distances is a portion of the sky rich in bright stars. Here are located the constellations Auriga, Gemini, Taurus, Canis Minor and Canis Major, and the most beautiful of all constellations—Orion.

(a) Connecting the stars δ-α UMa (Fig. 58) and continuing the line away from the "handle of the dipper" by 4 to 5 α-δ distances, we see the constellation *Auriga* with a very bright (Mag. 0.2) yellow star Capella (α Aurigae). If we include the star β Tauri, this constellation has the shape of a pentagon of bright stars.

(b) Connecting the stars δ and β UMa with a straight line and continuing it to 4 or 5 α-δ distances in the same direction, we shall see two rather bright stars of the constellation *Gemini*: Castor (α Geminorum, Mag. 2.0) and Pollux (β Geminorum, Mag. 1.2). This constellation consists of two parallel chains of faint stars. At present, the summer solstice point (l) is located at the end of the northernmost one.

(c) Continuing the straight line δ-β for another 4 α-δ distances, we find a large cluster of bright stars: the constellation of *Orion*, which is described below.

(d) Connecting the stars ε and γ UMa with a straight line and extending it to 6 or 7 α-δ distances, we see a bright (Mag. 0.5) white star Procyon (α Canis Minoris). Canis Minoris is considerably fainter. Another 4 α-δ distances bring us to the constellation Canis Major, which is described below.

(e) Connecting the stars γ and α UMa and continuing the line to 5 or 6 α-δ distances, we see the star Mirfak (α Persei).

If one remembers the above-described locations of the constellations relative to Ursa Major, a glance at the position of the latter constellation in the sky will make it possible to determine the portion of the sky in which the desired constellation is located. In addition, the position of Ursa Major relative to Polaris gives the approximate sidereal time (see Appendix VI).

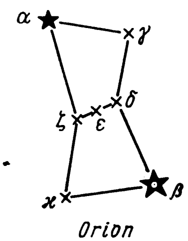


Fig. 59

In low and medium latitudes, *Orion* (Fig. 59) is most convenient for identifying constellations and stars in that portion of the sky. Orion has the characteristic shape of a trapezium with a belt of three stars round the middle. The ancients personified the mythical hero Orion in this constellation; the three stars were his belt. The brightest star of this constellation is Rigel (β Orionis, Mag. 0.3), which is white; the second brightest one, Betelgeuse (α Orionis, Mag. 0.9), is of a reddish hue. Nearly all the stars of this constellation may be used for purposes of observation.

(1) Connecting the stars α and γ Orionis with a straight line (Fig. 60) and continuing it towards the star α for 3 to 4 α-γ distances (in Orion), we will find the familiar constellation of Canis Minor with the bright star Procyon.

(2) Connecting the stars δ and α Orionis and continuing the line 4 α-γ distances, we find the stars α and β Geminorum.

(3) Connecting the stars δ and γ Orionis and continuing the line towards γ for 5 α-γ distances, we find the constellation Auriga and the star Capella.

(4) Connecting the stars of Orion's belt, ζ, ε, δ, with an arc (Fig. 60) and continuing it towards δ for 3 α-γ distances, we find a bright reddish star called Aldebaran (α Tauri, Mag. 1.1). Continuing this arc another α-γ distance, we get to the stellar cluster *Pleiades*.

(5) Extending the arc of Orion's belt in the opposite direction 3 α-γ distances, we see the brightest star in the heavens, Sirius (α Canis Majoris, Mag. —1.6), which in ancient Egypt was worship-

ped as the sacred star Sothis. The other stars of this constellation are not so bright and are not usually used in observations.

Let us examine some ways of identifying the chief constellations and stars of the southern sky that are not seen in the latitudes of the Soviet Union. In identifying them, we shall proceed from Orion and the Southern Cross (Crux).

(1) If we connect the stars ζ and ϰ Orionis with a straight line and continue it 5 or 6 ζ-ϰ distances in the southern direction

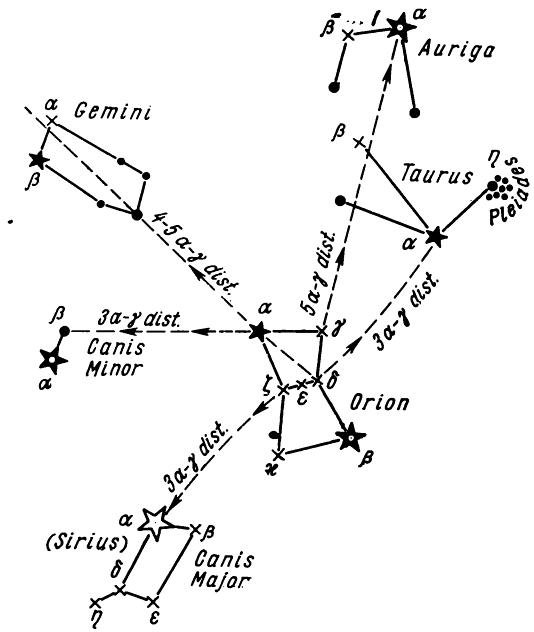


Fig. 60

(Fig. 61), we will come to the constellation *Argo*, which has the second brightest star (after Sirius) in the sky: Canopus (α Argus, Mag. —0.9). Argo, or as it is sometimes called, *Argo Navis*, is now divided into four smaller constellations: *Carina*, *Puppis*, *Pyxis*, and *Vela*. Canopus is α Carinae. However, globes and maps also give the generic name Argo Navis or Argo. Some of the stars of this constellation (Mag. 1.7 to 3) may be used in observations. They should be identified on a globe or map.

(2) Connecting the stars ε and β Orionis with a straight line and continuing it to 7 or 8 ζ-ϰ (Orionis) distances southwards (Fig. 61), we will see the bright star Achernar (α Eridani, Mag. 0.6). The remaining stars of this constellation are fainter.

Among the constellations of the other part of the southern sky, *The Southern Cross* (Crux) may be used as a starting point (Fig. 62).

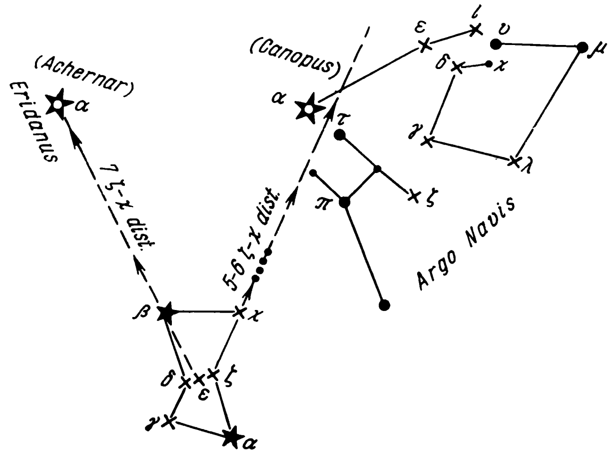


Fig. 61

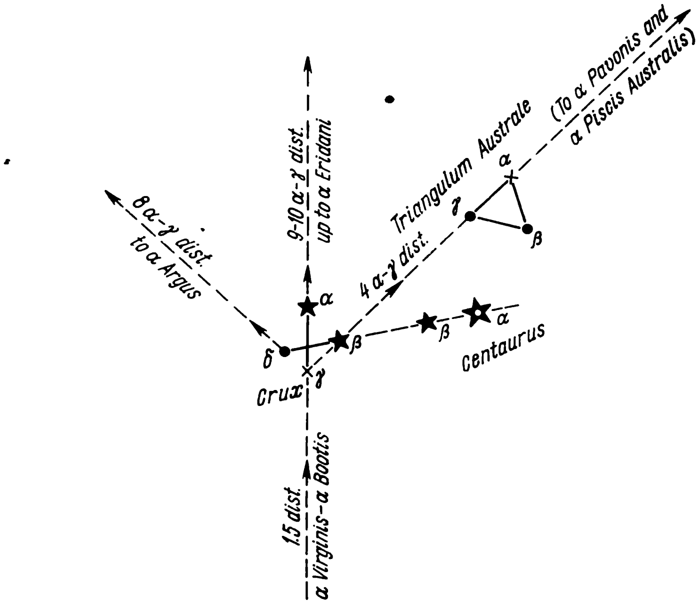


Fig. 62

It has the form of a small cross made up of four rather bright stars (Mag. 1.5 to 3). In low latitudes, it may be located by connecting

the stars Arcturus and Spica and continuing the line southwards to 1.5 the distance between them.

(3) Two δ-β (Crucis) distances towards the star β are two bright stars: β and α Centauri, Mag. 0.9-0.3 (see Fig. 62).

(4) Connecting the stars γ and β Crucis with a straight line and continuing it 4 α-γ distances, we will see the constellation *Triangulum Australis*, the alpha star of which is suitable for observations.

(5)Continuing this line to 8 α-γ distances, we will see α *Pavonis* (Mag. 2.1). Another 11 α-γ distances brings us to two stars of the same magnitude (Mag. 2.2): α and β *Gruis*.

Continuing in the same direction another 2 or 3 α-γ distances (Crucis), we will see the bright star α *Piscis Australis*.

Fig. 62 also shows the directions of the familiar stars α Argus and α Eridani.

CHAPTER 8

MEASUREMENT OF TIME

SEC. 33. FUNDAMENTALS OF MEASURING TIME

One of the problems of spherical astronomy is *measuring time*; namely, establishing the principles for determining time, the units for measuring time, and the system of time keeping. Measuring time is associated with certain difficulties that follow from its properties, one of which is its *irreversibility*, which means the property of changing in only one direction (forward). The irreversibility of time makes it difficult to reproduce its units of measurement. The unit for measuring time must be one that varies periodically and is of the same duration. In addition, like any other unit of measurement, it must be easy to determine from observations and convenient for everyday use.

Since ancient times, the basic unit for measuring time is the period of one revolution of the celestial sphere on its axis (one day), which is a reflection of the actual rotation of the earth on its axis.

This period is to a high degree of accuracy* a constant quantity and is readily obtained from observations of the alternation of day and night or the motion of the sun.

The rotational period of the celestial sphere should be reckoned from some plane (fixed for the given observer) to some specific point of the sphere (a sort of "index') that participates in diurnal motion. The observer's meridian was taken as the initial plane; for the moving point (index) it was agreed to take the first point of Aries, the apparent sun or the mean sun, which is a fictitious point on the equator moving with a certain mean velocity.

Depending on the choice of point, we get different units of time measurement: *sidereal* and *solar*.

The period of rotation of the sphere (or earth) relative to these points will obviously differ in accord with the different rates of motion over the sphere; hence, there is a difference in the duration of these units.

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* In practical atronomy we disregard the slight secular decrease in the period of the earth's rotation and its minute accidental fluctuations.

Since the rotation of the celestial sphere is uniform, the duration of a complete circuit or of any part of it may be evaluated by the *angle of rotation* of the sphere or the corresponding *arc* of the equator (the hour angle). Thus, the quantitative measurement of time reduces to measuring a totally different physical quantity, the arc (or angle) of rotation of the celestial sphere. We must be clear on the point that *time is not an arc*, it is simply numerically equated to the arcs of the hour angles for the sake of convenience in measuring.

Consequently, for measuring time one may use the hour angles of the first point of Aries, the apparent sun or the mean sun (Fig. 63).

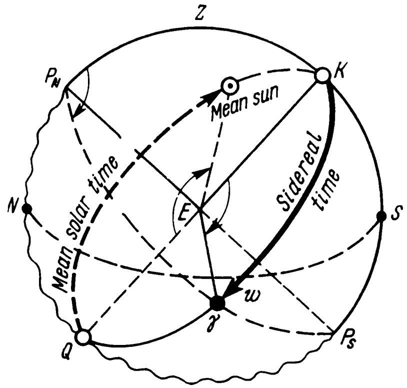


Fig. 63

Depending on the reference origin and the units of measurement, we distinguish two basic systems for reckoning time: *sidereal* and *mean*. Time reckoned in the first system is usually called sidereal time, that in the latter system, mean solar time or civil time. There are several times in each system, depending on the meridian from which the time is reckoned: local, Greenwich, or some other one.

From the foregoing it follows that in these systems time is reckoned in fractions of a revolution of the celestial sphere measured as the hour angles of definite points. But hour angles (as arcs of the equator) may be measured either *in degrees* or *in hours of arc*, as was mentioned in Sec. 2. Indeed, during one day (24h) in any system, a certain point will complete one revolution of 360° relative to the initial plane, hence, 24h = 360°; 1h = 15°, etc. To measure time, the hour angles will be expressed in time to within 0s.5 and in degrees to within 0′.1.

Practical astronomical problems involve both arcs and angles expressed as degrees and hours, so it is important to be able to convert one into the other. For this purpose, in addition to the earlier

mentioned special tables (39a, б, в, in MT-63 and others), make use of the following rules:

(a) **To convert an angle or arc from arc units into time units:**

(1) Divide degrees by 15 to obtain whole hours;

(2) Multiply remainder by 4 and add the quotient obtained in the division of minutes (′) by 15; the sum gives minutes of time;

(3) Multiply remainder, obtained in the division of minutes (′) by 15, by 4, disregarding the fractional part. The result yields seconds of time.

(b) **To convert time into arc:**

(1) Multiply hours by 15 and add quotient, obtained in division of minutes of time, by 4 to get degrees of arc;

(2) Multiply remainder (of division of minutes of time by 4) by 15 and add quotient obtained in division of seconds of time by 4 (to within tenths) to get minutes of arc with tenths.

Example 1. $65°43'.5 = \left(\frac{60°}{15}\right) \text{h} + (5° \times 4) \text{ m} + \left(\frac{30'}{15}\right) \text{m} + + (13'.5 \times 4) \text{ s} = 4\text{h } 22\text{m } 54\text{s}.$

Example 2. $5\text{h } 18\text{m } 15\text{s} = (5\text{h} \times 15)° + \left(\frac{16\text{m}}{4}\right)° + (2\text{m} \times 15)' + + \left(\frac{15\text{s}}{4}\right)' = 79°33'8.$

The earth's rotational period is not the only basis for establishing a standard of time. At the present time, the oscillation periods of *atomic* and *molecular* systems are employed for measuring frequencies and time. Their features are extreme constancy and very slight dependence on external conditions. Atomic systems oscillate with a very high frequency, thus making them difficult for practical use. And so the oscillations of molecular systems are used. These are the oscillations of several connected atoms (for instance, of the molecules of ammonia). Although the new time standard is quite independent of the earth's rotation, it is still based on fractions of the revolution of the celestial sphere and yields hours, minutes and seconds. The advantages of the new standard are: high constancy of the period of oscillation and the possibility of reproducing and storing the "units of time", irrespective of astronomical observations of the motion of the sphere. At the present time in celestial mechanics, particularly in the theory of lunar motion, there is a marked inconvenience due to a slight nonuniformity in the earth's rotation, which means a lack of constancy of the fundamental unit. In this connection, a new concept has been introduced: *ephemeris time*, an absolutely uniform time of Newtonian mechanics that appears in all gravitational theories of the motion of bodies of the solar system. Ephemeris time may be taken as analogous

to mean time (based on the motion of an ephemeris sun); that is, we equate it to the hour angle of an ephemeris sun. To convert from this time to Greenwich time, use the following equation: ephemeris time = Greenwich time + ΔT, where ΔT is +34s.0 for 1960.

To reproduce the foregoing time units and systems we have special mechanisms with uniform movements, clocks. *Regulators* ensure uniform rate. One of the first regulators was the *pendulum*, whose period of oscillation under given physical conditions is a constant quantity. This regulator is utilized in nearly all timepieces and chronometers now in practical use. At present we also have such regulators as the *oscillations* of *quartz* crystals (the quartz clock) and *molecular oscillations* (the atomic clock).

SEC. 34. SIDEREAL UNITS. SIDEREAL TIME

If we take the observer's meridian as the reckoning point in rotation of the celestial sphere, a complete circuit (one rotation) of some fixed point of the celestial sphere relative to this meridian may be called a sidereal day. The difficulty lies in the fact that there are no such fixed points on the celestial sphere convenient for reckoning purposes. For that reason, the vernal equinox point or first point of Aries (Υ) was chosen to mark the rotation of the celestial sphere. An added convenience is that this point serves as a reference point in the second equatorial system of coordinates. *The sidereal day is an interval of time between two successive transits (of the same name) of the vernal equinox point on the same meridian. For the origin of a sidereal day we take the instant of upper transit of* Υ on the given meridian (point K in Fig. 63). From this basic unit we obtain the smaller units: the sidereal hour (1h), equal to $^1/_{24}$ of the sidereal day, the sidereal minute (1m), equal to $^1/_{60}$ of one hour, and the sidereal second (1s), equal to $^1/_{60}$ of a minute.

The sidereal day is not equal to a complete rotation of earth (or the celestial sphere) on its axis, but is somewhat shorter due to the motion of the initial point (Υ) along the equator. We know that the point Υ moves 50″.3 along the ecliptic every year due to precession of the earth's axis in space. During one sidereal day, the point Υ will move along the equator $\frac{50''.3 \cos\varepsilon}{366.\,2422} = 0''.126 =$ $=$ 0s.0084 in the direction of the diurnal rotation of the celestial sphere. The sidereal day is just this much shorter than one complete revolution of the celestial sphere. However, this circumstance is not important in establishing units.

Now let us see how time is reckoned with these sidereal units. Systems of time in which the starting point is the instant of upper

transit of Υ and the intervals are expressed in sidereal units may be generally called systems of sidereal time. Thus, *sidereal time* (S) *is an interval of time from the instant of upper transit of the first point of Aries to a given instant expressed in sidereal units.* Due to the uniform rotation of the celestial sphere, the angles of its rotation relative to a meridian and, hence, the hour angles of Aries (Υ) vary uniformly from 0° (0h) to 360° (24h).

On this basis, the magnitude of the hour angle of Aries (Υ) may serve as a numerical estimate of the intervals of time that have elapsed from the beginning of the sidereal day, in other words, *sidereal time is numerically equal to* the west hour angle of the vernal equinox point (t^{Υ}), i.e.,

$$S = t^{\Upsilon} \tag{8.1}$$

Sidereal time may be expressed in time units (hours) and arc units (degrees), for example:

$$S = 14\text{h}\ 53\text{m}\ 18\text{s} \ \text{ or } \ t^{\Upsilon} = S = 223°19'.5$$

Large intervals of time are not measured in sidereal days and therefore *sidereal time has no dates.*

Sidereal time may be read on special sidereal clocks or chronometers adjusted so that the hands move exactly 24h 00m 00s every sidereal day.

SEC. 35. THE BASIC FORMULA OF TIME

Between the sidereal time S, the hour angle t of any celestial body and the right ascension α of that body we can establish a simple relationship which was obtained above (Sec. 2) and which also follows from Fig. 64 constructed on the plane of the celestial equator. The wavy line depicts the lower branch of the meridian of the observer. Indeed,

$$\text{arc } K\Upsilon = \text{arc } KD + \text{arc } D\Upsilon$$

or

$$S = t + \alpha \tag{8.2}$$

which means that the *sidereal time at any instant is equal to the west hour angle of the celestial body at that instant plus its right ascension**.

If the sum $t + \alpha$ exceeds 24h or 360° (Fig. 65), then to obtain S we drop 24h (360°), which is permissible because sidereal time has no date.

* We again stress that all equations in which time intervals are equated with magnitudes of arc have only numerical meaning.

The basic formula of time holds for any celestial body and any point of the sphere; that is, we can write

$$S = t_* + \alpha_* = t_\odot + \alpha_\odot = t_☾ + \alpha_☾ = t_♃ + \alpha_♃ \text{ and so forth.}$$

From (8.2) we can determine the hour angle of the body

$$t = S - \alpha \tag{8.3}$$

Applying this formula to any star, we get

$$t_* = S - \alpha_*$$

Add 360° to the right side and put $360° - \alpha_* = \tau_*$, then

$$t_* = S + \tau_* \tag{8.4}$$

where τ_* is the "sidereal hour angle", which is given in the Nautical Almanac for navigational stars.

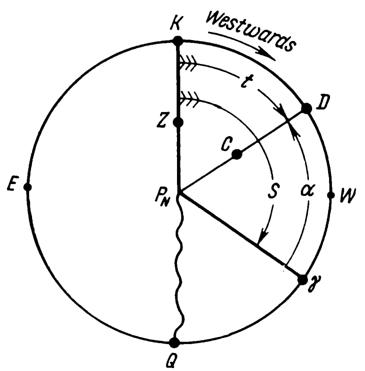


Fig. 64

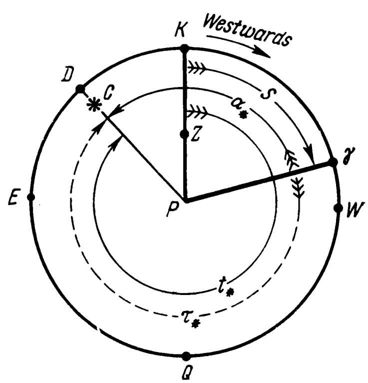


Fig. 65

Formulas (8.3) and (8.4) are used to compute the hour angles of celestial bodies in the most important problems of nautical astronomy.

At the instant of upper transit of a body, its hour angle is 0° (360°); at the instant of lower transit, 180°. The basic formula (8.2) then takes the form:

for upper transit: $S = \alpha_{body}$

for lower transit: $S = 180° + \alpha_{body}$

Thus, at the instant of upper (or lower) transit (meridian passage), it is possible to determine the sidereal time if we know the right ascension of the body; conversely, if we know the sidereal time

(for example, from a sidereal chronometer), it is possible to determine the right ascension of the body. This principle is utilized in practical astronomy.

From the foregoing it will be evident how convenient Aries (Υ) is as the basic point in reckoning sidereal time as well: by using Aries (Υ) we can relate the time numerically to the equatorial coordinates t and α of celestial bodies.

Examples. 3. *Given*: $\alpha_{*}=4\text{h}\,29\text{m}\,32\text{s}$; $t_{*}=51°48'\,\text{W}$. Determine $S=t^{\Upsilon}$ analytically and by drawing.

Solution:

$$\begin{array}{r|l} +\begin{matrix}\alpha_{*}\\ t_{*}\end{matrix} & \begin{matrix}4\text{h}29\text{m}32\text{s}\\ 3\ 27\ \ 12\end{matrix} \\ \hline S & 7\text{h}\,56\text{m}\,44\text{s (see Fig. 64)} \end{array}$$

4. $S=5\text{h}\,2\text{m}\,17\text{s}$; $\tau_{*}=244°18'.5$. Determine t_{*} and α_{*} (in the drawing).

Solution:

$$\begin{array}{r|l} S=t^{\Upsilon} & 75°34'.2 \\ \tau_{*} & 244\ 18\ .5 \\ \hline t_{*} & 319°52'.7\text{W}=40°7'.3\text{E (Fig. 65)} \end{array}$$

5. Given: $\alpha_{*}=15\text{h}\,17\text{m}\,48\text{s}$. Aries ($\Upsilon$) in upper transit. Determine t_{*} and S.

6. Given: $\alpha_{*}=18\text{h}\ 23\text{m}\ 27\text{s}$. Aries ($\Upsilon$) in lower transit. Determine t_{*} and S.

7. Given: $\alpha_{*}=21\text{h}\ 12\text{m}\ 38\text{s}$. Star in upper transit. Determine S.

8. Given: $t_{*}=107°18'\text{E}$; $\tau_{*}=304°25'$. Determine S in hours of time.

9. Given: $\varphi=50°\text{N}$; $h_{*}=35°$, $A=SW\ 50°$; $\alpha_{*}=5\text{h}\ 53\text{m}$. Determine S from a drawing of the sphere.

10. 12 November 1959. $S_{loc}=20\text{h}\ 30\text{m}$. Determine approximately the hour angles of the sun and moon at this instant.

11. On 9 October 1959, a star with $\tau_{*}=247°$ was observed at the instant of upper transit of the moon. Determine approximately S_{loc} and the hour angles of the star and the sun at this instant.

SEC. 36. APPARENT SOLAR DAY

Since ancient times, man has divided his time into the light and dark portions of the day, that is, following the diurnal motion of the sun. As is easy to establish, during the year the sidereal day begins at different times of the solar day.

Fig. 66 shows a portion of the celestial sphere with equator and the ecliptic. Sidereal time is determined by the hour angle of Aries (t^{Υ}), solar time is determined by the hour angle of the centre of the sun ($t^{\odot}$).

On 21.03 the sun passes the point Υ; at this time both the sun (C_0) and Aries (Υ) will transit simultaneously and the sidereal day will begin at apparent noon, which is the instant of the upper transit of the sun. One day later, the sun, due to its proper motion towards the diurnal motion, will arrive at C_1 and Aries (Υ) will transit before the sun, which means that the sidereal day will commence prior to noon; a month later, on 21.04, the sun will arrive at C_2, and so forth, until three months from then (on 22.06) the

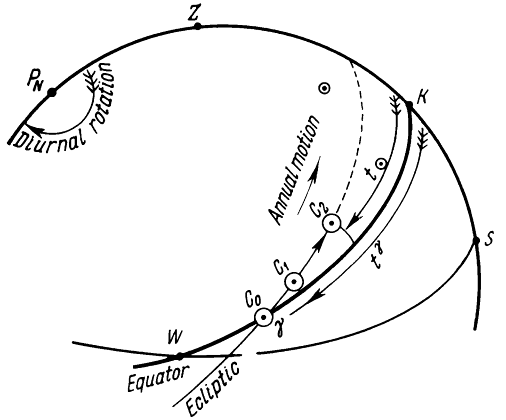


Fig. 66

sun moves 90° from Υ. Then the sidereal day will begin in the morning. In half a year, on 23.09, the commencement of the sidereal day will come at midnight; in 9 months, in the evening, and in a year, again at noon.

Quite obviously, it is inconvenient to arrange one's life and work by the sidereal day and sidereal time. For this reason, the sidereal time systems are used in astronomy. In ordinary life we reckon time by the sun.

If we take the true sun as the initial point for reckoning time (as has been done for many centuries), we get another unit of time measurement: the **solar day.**

The apparent solar day is an interval of time between two successive transits (of the same name) of the centre of the visible disc of the sun on one and the same meridian.

From Fig. 66 it will be seen that the solar day will differ from the sidereal day by the magnitude of the projection of arc $\Upsilon\ C_1$ on the equator, which is the diurnal movement of the sun over the equator, or by the magnitude of the diurnal variation of the right ascension of the sun ($\Delta\alpha_\odot$).

Earlier (Sec. 18) we found that $\Delta\alpha_{\odot}$ is a variable quantity (varying from 53′.8 to 66′.6 per day), while the motion of the sun is counter to the diurnal motion of the celestial sphere, from which we conclude that:

(1) the apparent day is longer than the sidereal day by the quantity $\Delta\alpha_{\odot}$,

(2) the duration of the apparent day is variable, because the diurnal variation of the right ascension of the sun changes. The difference between the longest and the shortest solar days in the year is 66′.6 — 53′.8 = 12′.8 or 51s, or almost one minute.

It is obviously inconvenient both theoretically and practically to take a variable as a unit for reckoning time. For this reason, the apparent day and apparent time is not now used in measuring time. Apparent time is used only as the hour angle of the true sun ($t_{\odot}$).

SEC. 37. MEAN SOLAR DAY. MEAN SOLAR TIME

The mean duration of a day for a year is known as the **mean solar day,** or simply the mean day. The number of mean days per year is equal to the number of apparent solar days, but the duration of a mean day is a constant quantity. The mean day serves as the basis for establishing the mean solar units of time.

In order to introduce the mean day as a unit of time, it is necessary that the motion of the sun should not differ appreciably from time reckoning in mean days; that is, the transit of the sun should be close to noon by the clock, etc. Besides, time obtained in mean solar units should readily be obtainable and verifiable from astronomical observations. To fulfill these conditions, it is best to introduce a certain *fictitious point of the celestial sphere* called the **mean sun** ($\oplus$) that replaces the true sun in questions of measuring time. The motion of this point, unlike that of the real sun, must be strictly uniform, which means that the variation of its right ascension should be a constant quantity and be equal to the mean value of $\Delta\alpha_{\odot}$. As mentioned earlier, during one tropical year equal to 365.2422 mean solar days, the right ascension of the sun changes by 24h (360°), and therefore the mean diurnal value of $\Delta\alpha_{\odot}$ or the diurnal variation of $\alpha_{\oplus}$ will be

$$\Delta\alpha_{\oplus} = \frac{24\text{h}}{365\text{d}.2422} = 3\text{m } 56\text{s}.56 \text{ sidereal units per day}$$

The mean solar day is *longer* than the sidereal day by this quantity.

The proper motion of the mean sun is in the same direction as the true sun, but is along the equator (Fig. 67, point C_2). To reproduce the mean day and not depart far from the true sun C, let us introduce first the point C_1 in motion along the ecliptic with the mean

annual velocity of the true sun and passing through points P and A of the ecliptic (perihelion and aphelion of the earth's orbit, see Sec. 17). However, the diurnal motion of C_1 will be nonuniform due to the obliquity of the ecliptic. Now let us impose the condition that the longitude (l) of C_1 should always be equal to the right ascension ($\alpha_{\oplus}$) of the mean sun.

The rate of motion of the mean sun over the equator will then be equal to the mean rate of motion of the true sun along the ecliptic. By virtue of these conditions, the mean sun will change its right

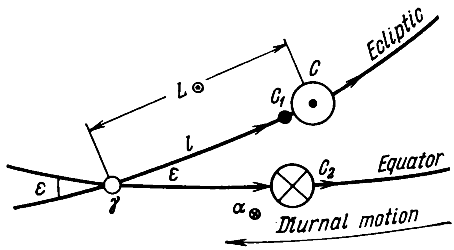


Fig. 67

ascension by 360° during one tropical year, that is, simultaneously with the true sun, and their meridians will coincide four times a year, while at intermediate dates these meridians will be close to one another. The diurnal variation of the right ascension of the mean sun comes out equal to the above-obtained mean value, or $\Delta\alpha_{\oplus}$ = 3m 56s. 56 per mean day.

On the basis of the foregoing, the *mean day is the interval of time between two successive* (*of the same name*) *transits of the mean sun on one and the same meridian*, that is, the time during which the mean sun in its diurnal motion describes an arc of 360° on the equator.

For the beginning of the mean day we take the instant of lower transit of the mean sun on the given meridian, which is the so-called mean midnight. The mean day is divided (like the sidereal day) into 24 *mean hours*, each hour into 60 *mean minutes*, and each minute into 60 *mean seconds* and thence into decimal fractions.

The mean day, hour, minute and second are the generally accepted units of measuring time in ordinary life, science and technology and in practical astronomy. These units are designated in the same way as the sidereal units: h, m, s, ordinarily without indicating "mean", whereas the sidereal units are always designated "sidereal".

Now let us see how time is measured with mean units. We consider the mean solar systems of reckoning time.

*The mean or civil time** (T) of a given meridian is *that interval of time which elapses from the instant of lower transit of the mean sun to the given instant* and is expressed in mean units. The mean time is always accompanied by a calendar date: the time T is written as follows: T=17h 28m 30s 20.06. Since the hour angle of the mean sun varies with perfect regularity during the diurnal rotation of the celestial sphere, the magnitude of the hour angle $t_{\oplus}$ may serve as

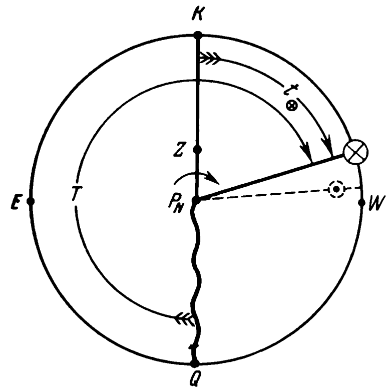


Fig. 68

a *numerical evaluation* of intervals of mean time. Like arcs of the equator, hour angles are expressed in degrees or hours; however, for mean time the accepted practice is to express them only in hours of time.

Due to the fact that the commencement of reckoning of hour angles of the mean sun (noon meridian) does not coincide with that of the reckoning of mean time (midnight meridian), a relationship is established between them that is clearly illustrated in Fig. 68

$$T = t_{\oplus} \pm 12\text{h} \tag{8.5}$$

which means that mean time is numerically equal to the west hour angle of the mean sun ±12h; in other words, it is equal to the hour angle reckoned from the midnight. The plus and minus signs are to be used so as to obtain T between 0h and 24h.

Since mean time is measured in strictly uniform fashion (due to the uniform motion of the mean sun), it is possible to create a clock that will read mean time. The situation is handled similarly to the sidereal chronometer, but the pendulum is adjusted to mark 86,400

* From 1925 onwards, mean time has been equal to civil time. Previous to 1925, mean time was reckoned from the instant of upper transit of the mean sun, and civil time, from its lower transit.

seconds *per mean day.* If the hands of this so-called "mean" chronometer read 0h0m0s at the instant of lower transit of the mean sun, it will read mean time at any instant.

SEC. 38. EQUATION OF TIME

In nautical astronomy it is often required to convert from mean time T to the hour angle of the true sun $t_{\odot}$. For problems of this nature it is necessary to know the so-called **equation of time** (η) by which is meant the *difference between mean time and apparent time*, reckoning them from one and the same branch of the meridian.

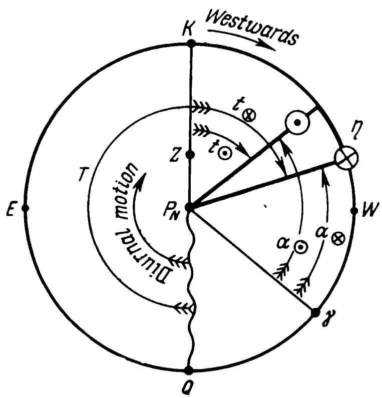


Fig. 69

This difference is numerically equal to the arc of the equator between the meridians of the mean sun and true sun (Fig. 69):

$$\eta = t_{\oplus} - t_{\odot} = \alpha_{\odot} - \alpha_{\oplus} \tag{8.6}$$

or the *equation of time is numerically equal to the difference of the hour angles of the mean and true suns*, or the difference of the right ascensions of the true and mean suns*.

The sign "+" is given if the meridian of the mean sun in diurnal motion is ahead of the meridian of the true sun (Fig. 69), and "—" if behind.

The magnitude and sign of the equation of time is obviously dependent on the conditions that the motion of the mean sun was

* In the nautical literature, the equation of time is given in the meaning of mean minus apparent time, whereas in other manuals and tables (Astronomical Almanac, for example) we have the inverse difference: apparent minus mean time, with only the sign of η changed.

subjected to. To obtain point C_1 (Fig. 67), the motion of the sun on the ecliptic was reduced to uniform motion. This part of the equation of time is called the equation of the centre and is expressed by the formula

$$y_1 = -2e \cdot \sin (L - \omega)$$

or approximately up to 0m.1

$$y_1 = +7\text{m}.7 \sin (78° + L)$$

where L is the longitude of the sun

e is the eccentricity of the earth's orbit ($e \approx 0.0167$)

ω is the longitude of perihelion of the earth's orbit; for 1960, $\omega = 102°15'$.

y_1 is zero near 3 January and 5 July. We have to transfer this quantity to the equator; the second part of the equation of time is thus called reduction to equator and is found from the formula

$$y_2 = -\sin^2 \frac{\varepsilon}{2} \cdot \sin 2L$$

or approximately

$$y_2 = -9\text{m}.5 \cdot \sin 2L$$

where ε is the obliquity of the ecliptic.

The quantity y_2 is zero on days of equinox and solstice.

The overall quantity $\eta = y_1 + y_2$ is zero four times a year (Fig. 70), namely: 15.04, 14.06, 1.09, 25.12 and has four extremal

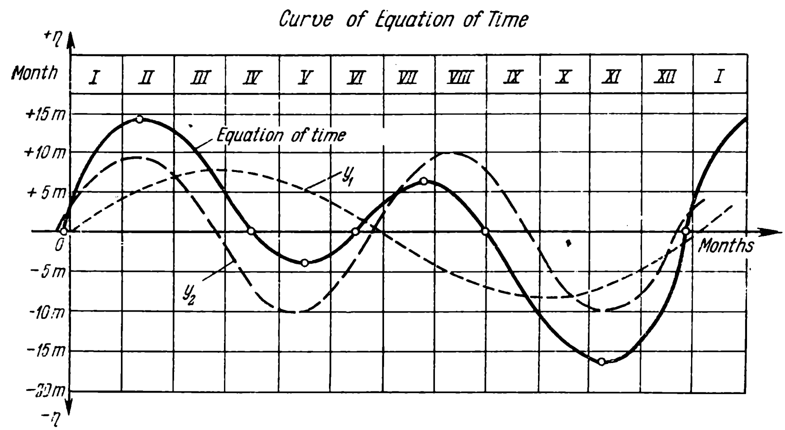


Fig. 70

values: 11.02 (+14m.4); 26.07 (+6m.4); 15.05 (—3m.8) and 3.11 (—16m.4). Fig. 70 shows the curve of the equation of time (solid line); the dotted line with annual period is the curve of the equation of the centre (y_1); the dashed line with half-year period is the curve of reduction to the equator (y_2).

Knowing the equation of time, we can solve two important practical problems:

(1) Determine the hour angle of the true sun $t_{\odot}$ from the mean time T.

We have the formulas

$$\eta = t_{\oplus} - t_{\odot} \text{ and } T = t_{\oplus} \pm 12\text{h}$$

Solving them for $t_{\odot}$, we get

$$t_{\odot} = T \pm 12\text{h} - \eta \tag{8.7}$$

(2) Determine the mean time T of the upper (or lower) transit of the sun.

Solving equation (8.7) for T, we find

$$T = t_{\odot} \mp 12\text{h} + \eta \tag{8.8}$$

But at the instant of upper transit, $t_{\odot} = 0$, hence

$$T = 12\text{h} + \eta \tag{8.9}$$

From this formula it is obvious that when $\eta = 0$, the true sun transits together with the mean sun, that is, at noon by mean time. When η is positive, the true sun transits after noon, that is later than the mean sun, and when it is negative, before noon, which is ahead of the mean sun. The largest difference in transit times will be of the order of 30m (from 11h44m to 12h14m). For lower transit, it will obviously be

$$T = 24\text{h} + \eta$$

The use of mean time for practical everyday affairs would not cause any inconvenience since the departure of the true sun amounts to only about $^1/_4$ hour.

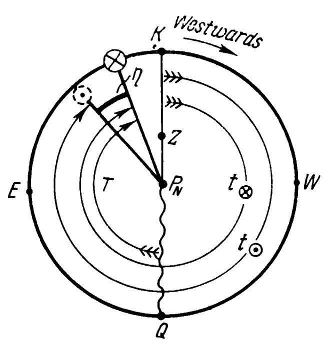
Examples. 12. *Given*: 2 March (2.03) at $T_{loc} = 10\text{h}\,55\text{m}\,13\text{s}$; $\eta = +12\text{m}\,28\text{s}$. Find $t_{loc}^{\odot}$ and show it on a drawing.

(a) From formula (8.7) we find:

$+\ T$	10h 55m 13s
12	12
$-\ t_{\oplus}$	22h 55m 13s
η	12 28
$t_{\odot}$	22h 42m 45s

$= 340°41'.2\text{W} = 19°18'.8\text{E}$.

(b) Make a drawing (Fig. 71):



13. *Given*: 19.01; $t_{\odot}=208°$; $\eta=+10m.5$. Find T and show it in the figure.
14. *Given*: 2.12; $T_{loc}=13h\,42m\,6s$; $\eta=-10m\,57s$. Find $t^{\odot}_{loc}$ in degrees of arc.
15. *Given*: 28.10; $\eta=-16m.0$. Find T for the upper transit of the sun.
16. *Given*: 16.02; η from the graph. Find T for the upper and lower transits of the sun.

SEC. 39. RELATIONSHIP BETWEEN SIDEREAL AND MEAN UNITS OF TIME MEASUREMENT

We have seen that the mean day and mean units of duration are "bigger" than the sidereal ones. We can establish the exact relationship between them in the following manner. The tropical year contains 365.2422 mean days, while the number of sidereal days in

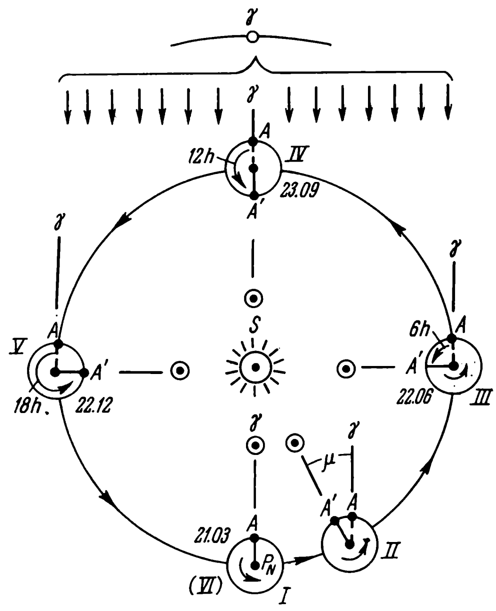


Fig. 72

the tropical year is one day more. Indeed, (Fig. 72), if in position *I*, Aries (♈) and the sun transit on the meridian A of the earth at the same time, then one day later (position *II*), the first point to arrive on meridian A will be Aries, then (after an interval of time μ) the sun; thus, relative to ♈, the earth executes one revolution $+\mu$.

Three months later (position *III*) we obtain ♈ $\approx$ 91 revolutions plus 6h relative to ♈; in six months (position *IV*) we have $\approx 2 \cdot 91$ revolutions + 12h, and so forth. Finally, one tropical year later

(position *VI*), the earth will have completed 24h = 1 sidereal day of revolutions relative to Aries more than relative to the sun, or 366.2422 sidereal days. Consequently,

1 tropical year = 365.2422 mean days = 366.2422 sidereal days (*)

(1) **Converting from mean to sidereal units of time.** Solving equation (*) for mean days, we get

$$1 \text{ mean day} = \frac{365.2422+1}{365.2422} \text{ sidereal days} = 1\text{d}\left(1+\frac{1}{365.2422}\right)$$

$$\text{sidereal days} = 1\text{d}\,(1+\mu) \text{ sidereal days} \qquad (8.10)$$

$$\text{where } \mu = \frac{1\text{d}}{365\text{d}.2422} = \frac{24\times 60\times 60\text{s}}{365.2422\times 24\text{h}} = \frac{3\text{m } 56\text{s}.56}{24 \text{ mean hours}} = 0.00273791.$$

Obviously, μ equals the change in $\alpha_{\oplus}$ in one mean day.

Multiplying both sides of (8.10) by a units, we get

$$a_{\text{mean days}} = a\,(1+\mu) \text{ sidereal days}$$

or, in smaller uniform units, we have

$$a_{\text{mean units}} = a\,(1+\mu) \text{ sidereal units of time} \qquad (8.11)$$

or

$$a_{\text{mean units}} = (a+\mu a) \text{ sidereal units}$$

(2) **Converting from sidereal to mean units of time.** Solving (*) for sidereal days, we get

$$1 \text{ sidereal day} = \frac{365.2422}{366.2422} \text{ mean days} = \frac{366.2422-1}{366.2422} \text{ mean days} =$$

$$= 1\text{d}\,(1-\mu') \text{ mean days} \qquad (8.12)$$

$$\text{where } \mu' = \frac{1\text{d}}{366\text{d}.2422} = \frac{3\text{m } 55\text{s}.91}{24 \text{ sidereal hours}} = 0.00273043,$$

i.e., μ' is equal to the change in $\alpha_{\oplus}$ during 1 sidereal day.

Multiplying both sides of (8.12) by b units, we have

$$b \text{ sidereal units} = b\,(1-\mu') \text{ mean units of time} \qquad (8.13)$$

Formulas (8.11) and (8.13) are used to convert intervals of time from mean units into sidereal and vice versa. For this purpose, there are special tables of the quantities μa and $\mu' b$; for example, Table 45a and 45б, MT-43.*

The relationship between the mean and sidereal units may be vividly shown by means of chronometers. If we take two chronometers, one adjusted to mean time, the other to sidereal time, and start them at the same time, exactly one day later (24h), the sidereal

* There are no such tables in MT-63.

chronometer will be *fast* by 3m 56s.56 sidereal units, and in exactly 24h (by the sidereal chronometer) the mean chronometer will be *slow* by 3m 55s.91 mean units.

Hence, in 24 mean hours the hour angle of Aries Υ, which is numerically equal to the sidereal time, will change by 24h3m56s.56= = 360°59′.14, or 15°2′.46 in one mean hour. Practically speaking, the hour angles of stars also change by this same amount in one mean hour.

SEC. 40. CONVERTING FROM MEAN TIME TO SIDEREAL TIME AND VICE VERSA

Since the sidereal time is numerically equal to the hour angle of the first point of Aries (Υ), and the mean time is equal to the hour angle of the mean sun ±12h, the relationship between them is determined by the quantity $\alpha_{\oplus}$ (Fig. 73 or 67). This likewise follows from the basic formula of time written for the mean sun:

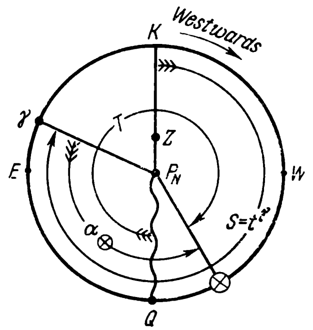


Fig. 73

$$S = t^{\Upsilon} = t_{\oplus} + \alpha_{\oplus} \qquad (8.14)$$

But $T = t_{\oplus} \pm 12h$. Substituting, we get

$$S = T \pm 12h + \alpha_{\oplus} \qquad (8.15)$$

Formula (8.15) enables us to calculate S from T and the date (and vice versa) both approximately and exactly. Approximate compu tations are performed by neglecting the advance of sidereal time over mean time and with an approximate value of $\alpha_{\oplus}$.

(1) Approximate calculation of sidereal time S at a given instant of mean time T.

In approximate calculations, it is more convenient to start from the basic formula of time and work as follows:

(a) from the given T calculate $t_{\oplus}$ using (8.5);

(b) from the date calculate the right ascension of the mean sun ($\alpha_{\oplus}$);

(c) calculate S from (8.14).

The quantity $\alpha_{\oplus}$ is calculated approximately in a manner similar to $\alpha_{\odot}$ (see Sec. 19), but in time units, that is, taking the diurnal $\Delta\alpha_{\oplus}$ = 1° = 4m or 2h per month and the following values of $\alpha_{\oplus}$*

21.03 $\alpha_{\oplus} = 0h$	23.09 $\alpha^{\oplus} = 12h$
22.06 $\alpha_{\oplus} = 6h$	22.12 $\alpha_{\oplus} = 18h$

* If the equation of time is taken into account, $\alpha_{\oplus}$ = 0° not on 21.03 but on 23.03, and $\alpha_{\oplus} = 180°$ on 22.09. A more precise diurnal change is $\Delta\alpha_{\oplus} = 4m$ 0s — 4s.

Example 17. *Given*: 20 August, $T = 21\text{h}\ 45\text{m}$. Determine S.
(a) $t_{\oplus} = 21\text{h}\ 45\text{m} - 12\text{h} = 9\text{h}\ 45\text{m}$;
(b) $\alpha_{\oplus} = 12\text{h} - 34\text{d} \times 4\text{m/day} = 9\text{h}\ 44\text{m}$;
(c) $S = t_{\oplus} + \alpha_{\oplus} = 19\text{h}\ 29\text{m}$ (Fig. 73).
A more exact calculation would have yielded 19h 35m.

(2) Approximate calculation of mean time T at a given instant of sidereal time S.

Work in the following order:
(a) from the date compute $\alpha_{\oplus}$;
(b) compute the hour angle of the mean sun from the formula

$$t_{\oplus} = S - \alpha_{\oplus};$$

(c) compute the mean time from (8.5).

Example 18. *Given*: 26 November, $S = 8\text{h}\ 47\text{m}$. Determine T.
(a) $\alpha_{\oplus} = 18 - 26\text{d} \times 4\text{m/d} = 16\text{h}\ 16\text{m}$.

(b)	$-\,S$	8h 47m
	$\alpha_{\oplus}$	16 16
(c)	$-\,t_{\oplus}$	16h 31m
	12h	12
	T	4h 31m 26.11

Approximate computations of S may be used, for instance, in identifying stars by means of a star globe, for computing the instant of transit and in other problems.

An approximate computation of T from S may be used for rough determinations of time on the basis of star positions (see Appendix VI).

(3) Exact calculation of sidereal time S on the basis of the mean time T on the same meridian.

For an exact calculation of S, put S in (8.15) as follows

$$S = T + (12\text{h} + \alpha_{\oplus})$$

Denote by S_0 the quantity $12\text{h} + \alpha_{\oplus}$ computed for $T = 0\text{h}$ of the given day. Then the exact value of S for the given time T will be

$$S = S_0 + T + \mu T \tag{8.16}$$

where $S_0 = 12\text{h} + \alpha_{\oplus}$ is the sidereal time at mean midnight on the given meridian
$T + \mu T$ is the conversion of the time interval T from mean units to sidereal units
μT is the lead of sidereal time over mean time and is taken from special tables.

and

$$T_1 - T_{gr} = \lambda_E^1; \qquad T_2 - T_{gr} = -\lambda_W^2$$

or in the general form

$$\left.\begin{aligned} S_{loc} &= S_{gr} \pm \lambda_W^E \\ T_{loc} &= T_{gr} \pm \lambda_W^E \end{aligned}\right\} \tag{8.18}$$

Converting from local time on one meridian to another we get (considering meridians A and B local, 1 and 2)

$$\left.\begin{aligned} T_{loc}^1 &= T_{loc}^2 + \Delta\lambda_E \\ T_{loc}^2 &= T_{loc}^1 - \Delta\lambda_W \end{aligned}\right\} \tag{8.19}$$

and also

$$\left.\begin{aligned} S_{loc}^1 &= S_{loc}^2 + \Delta\lambda_E \\ S_{loc}^2 &= S_{loc}^1 - \Delta\lambda_W \end{aligned}\right\} \tag{8.20}$$

From (8.19) and (8.18) it follows that

(1) the difference in the times of one system (S or T) reckoned on different meridians is numerically equal to the *difference in the longitudes* of the given meridians;

(2) the difference in the times of one system (S or T) reckoned at Greenwich and some point of the earth's surface at the same instant is numerically equal to the *longitude of the locality* of this point;

(3) to obtain local time from known Greenwich time, add the longitude of the point if the latter is located east of Greenwich, and subtract, if west of Greenwich. To avoid mistakes in regard to sign to be affixed to the longitude, remember the rule that the *time is greater to the east*; longitude west Greenwich time best, longitude east, Greenwich time least;

(4) for all observers on a single meridian of the earth, the local times of one system are the same, irrespective of the latitude of the observer.

Due to the fact that calendar reckoning of days is by mean days, a *date must be affixed to the mean time* (*whether local or Greenwich*).

When converting mean time from one meridian to another, it may happen that the sum $T + \Delta\lambda_E$ is greater than 24h. Since the mean time T indicates the number of mean hours, minutes and seconds that have elapsed from midnight of the given date, if the sum $(T + \Delta\lambda_E) > 24$h, we have to subtract 24 hours from it (one day); the result will represent mean time T of the next calendar day.

The opposite may happen, where the λ_W or $\Delta\lambda_W$ being subtracted is numerically greater than the given mean time. Then first add

24 hours to the given time and reduce the date by unity; then subtract λ_W or $\Delta\lambda_W$, and the mean time obtained will be that of the preceding calendar date. It is obvious that a case like this can happen only when the difference in longitudes is westwards.

Since sidereal time has no calendar date, this rule does not apply to sidereal time. But we can add (or subtract) 24 hours when dealing with sidereal time without invalidating the solution of a problem that requires such an operation.

The above indicated relationships between times on different meridians can of course be extended to the hour angles of any celestial bodies, that is,

$$\left.\begin{aligned} t_2 &= t_1 - \Delta\lambda_W \\ t_1 &= t_2 + \Delta\lambda_E \end{aligned}\right\} \tag{8.21}$$

and

$$t_{loc} = t_{gr} \pm \lambda_W^E \tag{8.22}$$

These formulas follow from Fig. 74 if the meridian of a body is drawn (this has not been done so as not to complicate the figure). The hour angle of a celestial body has no date of course.

From what has been said, it will be seen that if at some instant we know both the local time for some point and the Greenwich time (mean or sidereal, it makes no difference) or the hour angles of the body, then it is easy to compute the longitude of this point as the difference in times of hour angles, that is,

$$\lambda = T_{loc} - T_{gr} = S_{loc} - S_{gr} = t_{loc} - t_{gr}.$$

Examples:

22. *Given*: In longitude $\lambda_1 = 40°27'$W the local mean time $T'_{loc} = 21$h 45m 10s 12.05. Determine at this same instant time T''_{loc} at $\lambda_2 = 63°30'$E.

T'_{loc}	21h 45m 10s	12.05	Difference in longitudes of points
$\Delta\lambda_E$	6 55 48		
T''_{loc}	28h 40m 58s	12.05	$\Delta\lambda_E = 103°57' = 6$h 55m 48s
	= 4h 40m 58s	13.05	

This same problem may be done in terms of Greenwich, that is, by converting the time to T_{gr} by means of longitudes.

T'_{loc}	21h 45m 10s	12.05
λ_W	2 41 48	
T_{gr}	0h 26m 58s	13.05
λ_E^2	4 14 00	
T''_{loc}	4h 40m 58s	13.05

23. *Given*: $T_{gr}=2^h\,18^m\,54^s\,7.08$. Determine T_{loc} in longitude $\lambda=76°34'$ W.

T_{gr}	2h 18m 54s	7.08
$-\lambda_W$	5 6 16	
T_{loc}	21h 12m 38s	6.08

24. *Given*: $T_{gr}=19^h\,17^m\,43^s\,25.07$. Determine T_{loc} in longitude $\lambda=82°18'$ E.

T_{gr}	19h 17m 43s	25.07
$+\lambda_E$	5 29 12	
T_{loc}	0h 46m 55s	26.07

25. *Given*: $t^{*}_{gr}=293°27'$; $\lambda=34°21'$ E. Determine t^{*}_{loc}.

t^{*}_{gr}	293°27′
$+\lambda_E$	34 21
t^{*}_{loc}	327°48′ W = 32°12′ E

SEC. 42. ZONE TIME, LEGAL TIME, SHIP TIME

Since it is naturally impossible to reckon mean time from the meridians of each small point on the earth, a system of time reckoning was historically established over limited regions in which the time was the same: ordinarily the local solar (apparent) time of the nearest castle or town, and later, of the capital of a region or country (Paris time, St. Petersburg time).

This system was sufficiently convenient for a given stage in the development of production and society, but gradually inconveniences built up with the development of the telegraph, sea transport and railways. Indeed, the times of different points and regions differ by fractions of hours, minutes and seconds, and to compare times one has to know the exact differences of the longitudes of these places. As an observer moved from place to place, his clock had to be changed by the amount of the longitudinal difference.

In sea voyages, for the ship's clock to indicate apparent local time of the meridian on which the ship stood, it was necessary *to move* the hands forward all the time the ship was sailing eastwards, and back when sailing westwards (in earlier days the practice was to make the changes once a day, at apparent noon).

Towards the end of the 19th century, it became necessary to unify timekeeping on a world scale, and the astronomical congress of 1884 decided in favour of one of the systems of **zone time**.*

* Proposed in 1879 by engineer Fleming of Canada.

In the zone time system, the entire surface of the earth is divided by meridians into 24 so-called time zones of 15° (one hour) of longitude. The zone between meridians 7°.5W and 7°.5E with central meridian at Greenwich was considered the initial (or zero) meridian for numbering the zones from 1 to 12 eastwards and westwards. The twelfth zone in the eastern half is considered east, and in the western, west, like the zero zone. The longitudes of the central meridians

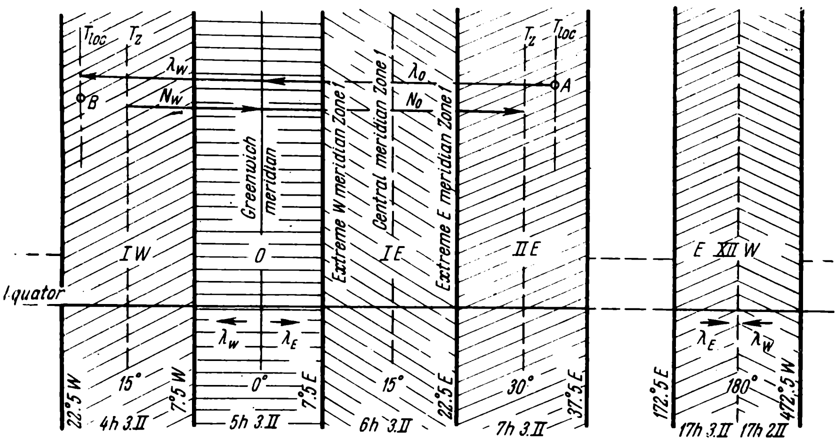


Fig. 75

of the zones are multiples of 15°, that is, 15°, 30°, 45°, . . ., 150°, 165°, 180°, and the numbers of the zones are equal to the longitudes of these meridians expressed in hours. The extreme meridians of a zone are $\pm 7°.5$ of longitude from the middle meridian (Fig. 75). To determine the number of a zone (zone description) that involves a point with a given longitude, divide its longitude by 15°. The quotient is the zone description (number) if the remainder is less than 7°.5; if the remainder is more than 7°.5, add unity and the result will be the zone description.

Throughout the territory of a given time zone, the *mean time is the same*, equal to the time of the middle meridian of the zone. This system of timekeeping is called the system of zone times, and time reckoned in this system is *zone time*, T_z.

Since the central meridians of adjacent zones are 15° apart and the mean sun covers 15° in one hour, the zone times of adjacent zones will differ by exactly one hour, and from other zone times by an *integral number of hours. The zone time of any zone differs from the time of the zero zone (Greenwich time) by the number of hours equal to the zone description.* From the rule for building zones, it follows that

local times of points within a zone should not differ from the zone time by more than ± 30m ($+7^\circ.5$), however the actual boundaries of zones are made to take into account various administrative and geographical factors and frequently do not coincide with the theoretical values. For this reason, local times within a zone may differ from the zone time by more than 30 minutes, but this is insignificant as far as everyday affairs go. The actual boundaries of the zones and other details of timekeeping in various countries are shown on *zone-time charts.* Ordinarily, western zones have the plus sign, eastern zones, the minus sign (that is, the sign for conversion to Greenwich time, T_{gr}).

During movements within a zone, clocks are not moved ahead or back because the entire zone keeps the time of the central meridian; *but when crossing zones, set the clock ahead one hour when moving eastwards and back one hour when moving westward.*

Problems in converting from zone time to local mean time and vice versa are best solved by the method "in terms of Greenwich", which is based on the fact that Greenwich mean time is *at the same time the local time* of the Greenwich meridian ($\lambda = 0^\circ$) and *zone time* of the zero zone. For that reason, the local time T_{loc} of, say, point A (Fig. 75) is converted by longitude to the Greenwich meridian, and then via the zone description is converted to the central meridian of the zone, that is, to zone time, T_z. When converting from zone time to local time, for point B (Fig. 75) for instance, the order is reversed. This conversion is done on the basis of the following formulas:

I. *Given* T_{loc}, find T_z:

$$\left.\begin{aligned} T_{gr} &= T_{loc} \mp \lambda_W^E \\ T_z &= T_{gr} \pm ZD_W^E \end{aligned}\right\} \qquad (8.23)$$

II. *Given* T_z, find T_{loc}:

$$\left.\begin{aligned} T_{gr} &= T_z \mp ZD_W^E \\ T_{loc} &= T_{gr} \pm \lambda_W^E \end{aligned}\right\} \qquad (8.24)$$

where λ is the longitude of the locality

ZD is the zone description where the observer is located.

The zone time of the first zone eastwards is sometimes called Central European time.

On the territory of the U.S.S.R., a decree of the Council of People's Commissars of 16 June 1930 established *zone time increased by one hour*, that is, as reckoned from the central meridian of the next zone eastwards.

Time reckoned in this system of timekeeping is called **legal time,** T_{leg}, and is connected with the zone relationship as follows

$$T_{leg} = T_z + 1\text{h} \tag{8.25}$$

Legal time was introduced in order to shift the working hours and daily affairs of life to the lighter morning hours of the day.

In a number of countries, clocks are put one hour fast, sometimes two hours fast of zone time, but only for the summer months. Zone time is returned to in autumn. This is called *summer time* and is introduced by special orders. In Great Britain, summer time (B.S.T.) is introduced from the middle of April to the beginning of October (Sunday from $T_{gr} = 2\text{h}$). In the United States, this time is called daylight saving time and is introduced by special order. In countries of the southern hemisphere, summer time is introduced from October to March.

Legal time of the second eastern zone, that is, the local mean time of the central meridian of the third zone, is called Moscow time (T_{Mos}) and is widely used on the territory of the U.S.S.R. Moscow time is used for making air, railway, sea and other timetables. *Moscow time is ahead of Greenwich time by three hours*

$$T_{Mos} = T_{gr} + 3\text{h} \tag{8.26}$$

Ship's clocks are set, as a rule, to some zone time, that of the zone in which the ship is located, or an adjacent zone. In small voyages, clocks are often not set back or ahead when zone boundaries are crossed. The result is that *ship time* T_{sh} *is the zone time of that time zone which the ship's clock reads.* This zone description should be written in the ship's log and marked when ship time is designated, for instance, $T_{-10} = 7\text{h}$ or $T_{sh} = 7\text{h}$ (ZD = 10E). In the waters of the U.S.S.R. ship's clocks are ordinarily set according to legal time, at open sea and in foreign waters, according to zone time, occasionally in accord with the time of the given country. When lying in a foreign port, the time of the port is taken. Ship time is usually reckoned with an accuracy to within one minute, in some cases to within 15 seconds or less, since most modern ship's clocks have second hands.

When crossing zone boundaries, all timepieces on the ship (with the exception of chronometers and clocks in the radioroom) are set ahead one hour for eastward sailing and back one hour for westward sailing. It is most convenient to set the timepieces *ahead* in the night watch (0-4h) and *back* in the day watches (8-12h, 12-16h). In actual practice, timepieces are sometimes set each 20m in the three successive watches so as to even the duration of working time. However, this upsets the principle of zone time.

Conversion from ship time to Greenwich time and local time is one of the most common problems and is done by the following formulas

$$T_{gr} = T_{sh} \mp ZD_W^E \tag{8.27}$$

$$T_{loc} = T_{sh} \mp ZD_W^E \pm \lambda_W^E \tag{8.28}$$

$$T_{sh} = T_{loc} \mp \lambda_W^E \pm ZD_W^E \tag{8.29}$$

where ZD is the zone description of the time zone to which the ship's clocks are set; for legal time, this is the zone description of the next zone eastwards of the given zone.

When using the formulas of this section, remember that local times are always corrected by *longitude*, while zone, legal and ship times are corrected by the *zone description*.

Examples:

26. *Given*: T_{loc} = 9h 21m 45s 4.08; $\lambda = 34°21'$E. Determine T_z and show it on a drawing.

$-\ T_{loc}$	9h 21m 45s 4.08	See Fig. 75 for observer *A*.
λ_E	2 17 24	
$+\ T_{gr}$	7h 4m 21s	
ZD_E	2	
T_{-2}	9h 4m 21s 4.08	

27. *Given*: T_{+10} = 17h 15m 28s 10.07; $\lambda = 145°23'$W. Find T_{loc}.

$+\ T_{+10}$	17h 15m 28s 10.07
ZD_W	10
$-\ T_{gr}$	3h 15m 28s 11.07
λ_W	9 41 32
T_{loc}	17h 33m 56s 10.07

28. *Given*: T_{loc} = 8h 53m 18s 5.04; $\lambda = 131°28'$E. Determine T_{leg} and show it on a drawing.
29. *Given*: T_{Mos} = 11h 21m 47s 14.05; $\lambda = 37°36'$E. Determine T_{loc}.
30. *Given*: T_{Mos} = 20h 33m 15s 10.06. Determine T_z at $\lambda = 131°54'$E.
31. *Given*: T_{sh} = 17h 35m 2.09; $\lambda = 136°29'$W. Find T_{gr}.
32. *Given*: T_{sh} = 6h 18m 30s (legal) 8.07; $\lambda = 144°25'$E. Find T_{loc}.
33. *Given*: 26.11 $t_{gr}^{*} = 130°$; $\tau_{*} = 64°$; $\alpha_{\oplus}$ = 16h 15m; $\lambda = 76°45'$W. Find T_{gr} and T_{loc}. Show them on a drawing.
34. On 22.08.69 at T_{gr} = 18h 20m; $\lambda = 41°$E, a star was observed with $\tau_{*} = 146°$. Find t_{loc}^{*} and the longitude in which the moon transits at that instant.
35. 5.08.59 T_{leg} = 20h 25m; $\lambda = 141°28'$E. Find S_{loc}.

SEC. 43. INTERNATIONAL DATE LINE

Due to the fact that in zone timekeeping, the time increases one hour with each zone, moving eastwards, and decreases one hour with each zone when moving westwards, there must be a single central meridian where the time difference is 24 hours, or one mean day. Suppose (Fig. 76) that on the Greenwich meridian (in the zero zone) it is 5h 3.II at some instant. The time will decrease westwards;

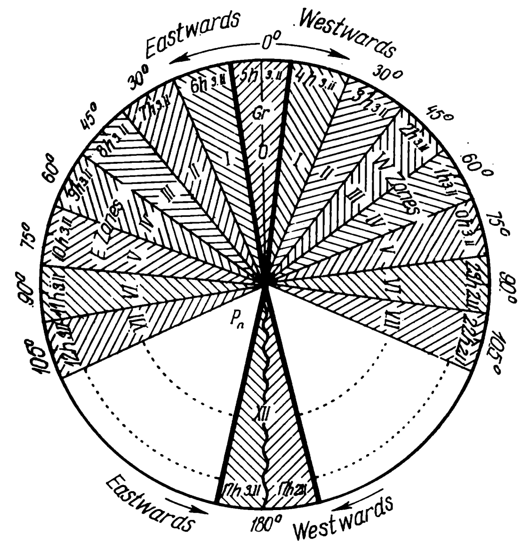


Fig. 76

in zone VI it will be 23h 2.II, in zone XII it will be 17h 2.II. Eastwards, on the contrary, the time increases; in zone VI it will be 11h 3.II; in zone XII it will be 17h, but 3.II. In zone XII the zone time is the same, 17h, but on one (east) side of the 180° meridian it will be 3.II, while on the other (west) side, it will be 2.II. Thus, the meridian $\lambda = 180°$ is the dividing line for changing dates. Along this meridian (with certain deviations made for the convenience of the population) lies the so-called **international date line.** The position of the date line at precisely this place is convenient because it lies entirely over water, and does not divide any islands or inhabited centres.

At the same instant, there are two dates on the two sides of this line. Therefore, when crossing this line, *one has to change dates.* Change of dates on ship is usually done not at the instant of crossing the line, *but at midnight after the crossing.*

Thus: (a) *When a ship is sailing eastwards (courses in the eastern quadrants of the compass), at midnight following a crossing of the date line, the date is repeated, that is, not changed.*

(b) *When a ship is sailing westwards (courses in the western quadrants of the compass), at midnight following a crossing of the date line, the date is changed two units at once; in other words, one date is dropped.*

To avoid errors in determining T_{gr} and the date in astronomical calculations when crossing the date line in the interval up to midnight, one should continue to reckon longitude above 180° E or W and correct the time in this reckoning up to midnight.

An entry is made in the ship's logbook about crossing of the date line and change of dates.

Examples:

36. 15.09.58, on course 140° true about $T_{sh} = 16$h (ZD = 12E), crossed date line. Change the dates.

Solution. Without changing time reckoning, continued up to 24h. At 0h, the date was 15.09.58 (instead of 16.09).

37. 21.05.58, course 294° true at $T_{sh} = 10$h 15m (ZD = 12W), crossed date line. Change the dates.

Solution. Without changing time reckoning, continued up to 24h. At 0h, the date was 23.05.58 (instead of 22.05).

38. 7.08.58, on course east at $T'_{sh} = 14$h 40m; first observations of sun made in $\lambda' = 179°47'$E. Second observations of sun made at $T''_{sh} = 17$h 30m; $\lambda = 179°16'$W. Find T_{gr}.

Solution. When obtaining T_{gr} in the latter case (before midnight) consider $\lambda = 180°44'$E, then

T'_{sh}	14h 40m		T''_{sh}	17h 30m
$-$ ZD_E	12		$-$ ZD_E	12
T_{gr}	2h 40m 7.08		T_{gr}	5h 30m 7.08

SEC. 44. CALENDAR

For reckoning large intervals of time, there have historically developed several systems based on the periods of revolution of the sun and moon. *These systems of reckoning large numbers of mean days for everyday human affairs are known as calendars.* The generally accepted system of reckoning dates involves such units as the week, month, and year. The month is based on the orbital period of the moon round the earth, the lunar month being equal to 29.53 mean days. The year is based on the earth's orbital period of revolution about the sun, the tropical year being equal to 365.2422 mean days. The week is an artificial unit.

As we see, both periods (the lunar month and the tropical year) contain a fractional number of mean days, while calendar units always involve periods made up of an *integral number of days.*

In the correlation of these artificial periods with the natural periods, three basic systems of reckoning have emerged: lunar, lunisolar, and solar calendars based on the movements of the moon, the moon and sun, and the sun alone. We shall consider only solar calendars, which are used by most countries of the world today.

The elementary arithmetical theory of the solar calendar is as follows: to find the number x of tropical years which includes a *whole number of* y mean days to a specified degree of accuracy.

To solve the problem, we form an indeterminate equation

$$0.2422x = y \tag{8.30}$$

We represent this equation in the form of a continued fraction

$$\frac{y}{x} = \frac{2422}{10,000} = \cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\dots}}}} \tag{8.31}$$

Suitable fractions with different degrees of accuracy are

$$\frac{y}{x} = \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{31}{128} \quad \text{and so forth}$$

Thus, in addition to 365 days we should consider (in order of increasing accuracy): one day in four tropical years; 7 days in 29 tropical years; 8 days in 33 years; 31 days in 128 years, etc. In the first case, the correction to the calendar will in four years be —0.0312 day (calendar with excess), in the second, +0.0238 (calendar with deficit), in the third, —0.0074, in the fourth, +0.0016. Thus, a solar calendar will be based on an alternation of 365 and 366 days in the periods indicated above.

The history of the solar calendar goes back into remote antiquity. In ancient Egypt, the year was first 360 days, then 365 days. Due to the considerable difference between this year and the tropical year, the new year was shifted backwards (it "wandered" among the seasons). Such a "wandering year" in the calendar was of course inconvenient.

In use in ancient Rome was a complicated and inaccurate lunisolar calendar that was frequently upset by the priests; in 46 B.C., this calendar was replaced by a solar calendar by order of Julius Caesar with the help of the Egyptian astronomer Sosigenes. This calendar, called the **Julian calendar,** utilized the first approximation of the above fraction, that is, three years were considered 365 days each, and the fourth was a *leap* year of 366 days. The correction relative to the standard (tropical year) is, as already stated, +0.0312 day every four years, one day in 128 years or 3.12 days in 400 years. During these time intervals, the calendar will lag behind the tropical

year. The Julian calendar was gradually accepted by all countries, and in the year 325 A.D. was recognized by the Orthodox church. In some countries, the Julian calendar is still in use today (termed "old style"). However, the error of one day every 128 years amounted to 10 days by 1582, and the vernal equinox was shifted to 11 March in the calendar, thus introducing confusion in religious holidays, particularly Easter. For this reason, Pope Gregory XIII, using a project of the Italian scholar Lilio, introduced a new, reformed calendar called the **Gregorian calendar** (or "new style"). In addition to correcting the error of 10 days that had accumulated by then, this calendar fixed the reckoning of leap years by a method rather less precise and more artificial than the third approximation given above.

In the U.S.S.R., the new style was introduced only after the October Revolution, in 1918. A decree of the Council of People's Commissars of the R.S.F.S.R. introduced the new style and made February 1 into February 14, because by that time the discrepancy between old and new styles had reached 13 days.

The calendar used in most countries today is the Gregorian (new style) calendar and is constructed as follows:

(a) Years are reckoned in **civil** (or **calendar**) years, which always contain a whole number of days but are of variable length, 365 days forming "common" years and 366 days forming "leap" years.

(b) Leap years are those *whose number is divisible by four*: the other years are common years. Exceptions are years that are multiples of 100, the first two figures of which are not divisible by 4; these years are also considered common. For example: 1600, 1956, 1960, 2000 are leap years; 1700, 1800, 1900, etc., are common years. For this reason, the error in the old style reached 13 days (in addition to the 10 that had accumulated by the year 1700). In the Gregorian calendar, the error will reach 1 day in 3300 years, which is of no practical significance.

An essential defect of the present calendar is the arbitrary distribution of days in the months (28, 29, 30, and 31 days) that has come down to us from ancient Roman times, the fractional number of weeks per year, and others. The United Nations Organization has given consideration to a number of projects, one of which was recommended—that of a new 12-month calendar in which the year would be divided into four quarters made up of 91 days each, each quarter being divided into three months, the first of which has 31 days, the others, 30 days. Every year and every quarter would begin on the same day of the week—Sunday. The number of days in the four quarters would come to exactly 364. The one remaining day would be inserted after December 30 every year to mark an international new-year holiday. In leap years, which would be determined by the customary rule, one day would be added after June 30 as well.

The initial reckoning date of civil years is called an **era**. In antiquity, the years were reckoned on the basis of reigning dynasties, from the year of the reign of some sovereign, from the "beginning of the world", from the first Olympian games, and so forth. One of the oldest eras is the Nabonassar Era that began in 747 B.C. In the year 284 of the Diocletian Era (532 A.D.), the clergy decided to reckon the years not from the date of the rule of the tormentor of the christians Diocletian, but from the mythical "birth of Christ", which was supposed to have taken place 532 years prior to that date (the number 532 was apparently taken on the basis of the Easter days that repeated those days of the month). This era became known as the new era (Anno Domini, A.D.) and was gradually taken up by most countries (in Russia, prior to 1700, years were reckoned from the mythical "creation of the world"). In the new era (Christian Era), the years are reckoned from year 1 as A.D. and back as B.C. (examples: 532 A.D., 46 B.C.).

In astronomy and chronology, large intervals of time are reckoned in Julian days or days of the so-called Julian period, by which is meant a continuous count of days and their fractions from a certain arbitrary date — 1 January 4713 B.C. The Julian day begins at $T_{gr} = 12^h$ and has no date. Its sequential number is given in astronomical almanacs. For example, noon of 10 November, 1958 (i.e., $10^d.5$) will correspond to the Julian day 2436518.0.

CHAPTER 9

NAUTICAL ASTRONOMICAL ALMANAC (MAE)

SEC. 45. ALMANACS

Due to the fact that the equatorial coordinates of all celestial bodies are constantly changing, it is necessary, when solving the astronomical triangle, to obtain the coordinates δ, t, α *at precisely the instant of observation.* Thus, for practical use, these coordinates must be known beforehand, for any instant of time in the future. For this purpose, the coordinates δ, t, α of celestial bodies and other data are computed by special computing centres (*precomputed* for definite instants of time a year or more in the future). Lists of such precomputations of coordinates at equidistant intervals of times are known as the **ephemerides of celestial bodies** and are published in special **astronomical almanacs.**

The practical problems of astronomy require a variety of coordinate accuracies, thus necessitating specialized almanacs. The following are published in the U.S.S.R:

(1) "Astronomical Almanac of the U.S.S.R." (abbreviated AE) is designed for observatories and field astronomical-geodetic studies. Coordinates are tabulated to within $\pm 0''.01$.

(2) "Nautical Astronomical Almanac" (MAE) designed for sea navigation. Equatorial coordinates are tabulated to within $\pm 0'.1$.

(3) "Air Astronomical Almanac" (AAE) designed for aviation. Equatorial coordinates are tabulated to within $\pm 1'$.

These almanacs are computed and compiled at the Institute of Theoretical Astronomy (U.S.S.R. Academy of Sciences) in Leningrad for several years ahead. Astronomical calendars are also published in the U.S.S.R. for astronomy amateurs.

The most important nautical almanacs published in other countries include:

1. "The American Nautical Almanac" and "The Abridged Nautical Almanac" published jointly by the United States and Great Britain (abbreviated N.A.).

(2) "Nautisches Jahrbuch" published in Germany.

(3) "Ephemerides Nautiques" published in France.

In addition to official almanacs, there are a number of publications put out by private companies, one of which in England is the popular "Brown's Nautical Almanac", and another, "Reed's Nautical Almanac", which contains a good deal of reference material in addition to astronomical data.

The contents of all nautical almanacs are about the same as regards the basic data, but they differ in arrangement. As a rule, all almanacs include declinations and Greenwich hour angles of the brightest bodies of the solar system, Aries, and δ and τ of the navigational stars.

Quite naturally, these and other quantities cannot be supplied continuously and are given in almanacs at time intervals of one hour, one day, or one month, depending on their variability and desired accuracy. Within an interval, an ephemeris is considered as varying uniformly and is interpolated in proportion to the time. The interpolation tables that serve this purpose are constructed on the principle of linear interpolation (based on the differential variations of the coordinates); here, there are slight differences in the almanacs of the various countries. However, there has recently been a tendency to standardize both the content and the auxiliary tables of nautical almanacs in all countries.

SEC. 46. THE STRUCTURE OF MAE TABLES FOR OBTAINING HOUR ANGLES AND DECLINATIONS OF CELESTIAL BODIES

The arguments for entering an almanac are: date, T_{gr} and name of body.* For these data, the MAE gives δ and t_{gr} of the Sun, Venus, Mars, Jupiter, Saturn and the moon and the sidereal Greenwich time (t_{gr}^{Υ}) at one-hour intervals of T_{gr}. These and certain other data are given in daily tables that occupy the major portion of the MAE. The star tables that follow this section include δ_{*} and τ_{*} for 159 navigational stars. In addition, an insert contains these data for 50 bright stars. The daily tables and ephemerides of stars make up the bulk of the MAE. The other data: tables for determining the azimuth and latitude from Polaris, visibility of the planets, etc., are of secondary importance and are found at the beginning and end of the MAE.

In the 1961 MAE, interpolation of all basic data is done on the basis of a single general interpolation table constructed so that corrections are obtained by extraction alone without any other operations (on the "minute principle"). For this purpose, the corrections for hour angles of Aries, the sun, planets and the moon are computed and given in the tables for every minute and second of the hour;

* In the MAE "Greenwhich mean time" is replaced by the term "Greenwich civil time".

the minutes are indicated at the top, the seconds on the left and right of the tables. The middle column of these tables gives the correction values (for interpolation of declinations and the values of quasi-differences $\overline{\Delta}$) obtained for tabular differences from 0'.0 to 18'.0 for the mid-point of the given minute. These tables are placed at the end of the MAE and occupy 30 pages.

Bear in mind that all data in the MAE refer to the Greenwich meridian, so that after extraction they have to be converted to the local meridian. Let us examine some basic problems.

I. OBTAINING SIDEREAL TIME (t^{Υ}_{gr} AND t^{Υ}_{loc})

Applying the basic formula of time (8.2) to the mean sun for the Greenwich meridian, we get

$$t^{\Upsilon}_{gr} = t^{\oplus}_{gr} + \alpha_{\oplus}$$

but

$$t^{\oplus}_{gr} = T_{gr} \pm 12\text{h}$$

whence

$$t^{\Upsilon}_{gr} = T_{gr} \pm 12\text{h} + \alpha_{\oplus} \tag{9.1}$$

In the MAE, this formula is used to precompute t^{Υ}_{T} for whole hours of T_{gr} (the subscript T indicates tabular values of t and δ). The increment Δt^{Υ}_{gr} in the interval between whole hours is computed on the basis of formula (8.11) of the preceding chapter. The formula for obtaining t^{Υ}_{gr} at a given instant of T_{gr} will be analogous to (8.16) and is of the form

$$t^{\Upsilon}_{gr} = (T_{gr} \pm 12\text{h} + \alpha_{\oplus})^{\circ} + \left(\Delta T_{gr} + \frac{\Delta\alpha_{\oplus}}{60\text{m}} \Delta T^{\text{min}}_{gr}\right)^{\circ} \tag{9.2}$$

where $\Delta\alpha_{\oplus}$ is the change in right ascension of the mean sun during one hour and is equal to 2'.46; the quantity $\frac{\Delta\alpha_{\oplus}}{60} \cdot \Delta T$ is equal to $\mu \cdot \Delta T$, the lead of sidereal time during the interval ΔT

ΔT_{gr} is the excess of Greenwich civil time over a whole hour.

The symbol "°" signifies that the quantities in brackets are expressed in degrees. The quantities in the first brackets are the Greenwich sidereal time in whole hours of T_{gr} and are given in the daily tables; the quantity in the second brackets is the increment Δt^{Υ}, more precisely the conversion of the interval ΔT from mean into sidereal units, and is given in the first column of the basic interpolation table (called "Increments and Corrections" in N.A.).

Formula (9.2) may be abridged to the form

$$t_{gr}^{\Upsilon} = t_T^{\Upsilon} + \Delta t^{\Upsilon} \tag{9.3}$$

Finally, the local sidereal time is obtained from formula

$$t_{loc}^{\Upsilon} = t_T^{\Upsilon} + \Delta t^{\Upsilon} \pm \lambda_W^E \tag{9.4}$$

Example 1. 12 September 1968 at $T_{gr}=21\text{h}\,37\text{m}\,42\text{s}$; $\lambda_c=38°40'\text{E}$. Find $t_{loc}^{\Upsilon}=S_{loc}$.

From the daily tables on 12.09, $T_{gr}=21\text{h}$; from the interpolation tables at $\Delta T_{gr}=37\text{m}\ 42\text{s}$.

t_T^{Υ}	306°37'.3
Δt^{Υ}	9 27 .0
$+\ t_{gr}^{\Upsilon}$	316°04'.3
λ_E	38 40 .0
t_{loc}^{Υ}	354°44'.3

II. OBTAINING LOCAL HOUR ANGLES AND DECLINATIONS OF STARS

From the basic formula of time (8.2) we get for the local meridian

$$t_{loc}^{*} = t_{loc}^{\Upsilon} - \alpha_{*}$$

or

$$t_{loc}^{*} = t_{loc}^{\Upsilon} + \tau_{*} \tag{9.5}$$

where $\tau_{*} = 360° - \alpha_{*}$.

From (9.5) it is seen that to obtain the local hour angle of a star it is necessary first to get the hour angle of Aries (t_{loc}^{Υ}), which is found from (9.4) in the order given above.

The ephemerides of stars τ_{*} and δ_{*} are given in the section of MAE entitled "Apparent Positions of Stars for the Year...", in the N.A. entitled "stars". On the left-hand page of this section are the values of τ_{*} with respect to the arguments: name of star and month of given year. The stars are given according to position in constellation, for example, α Andromedae, and so on in order of increasing α_{*}, which is given in the extreme left column to within one minute.

The tables of τ_{*} are constructed so that degrees are given once (in the first column), and minutes and their tenths are given for the first of every month. Interpolation of τ_{*} for a given date is done mentally, between columns; incidentally, it may often be disregarded. On the right-hand page in the same order are the declinations of stars according to the same numbers in the star list as on the left-hand page, and also according to the proper name of the star if it

has one. Stellar magnitudes, which indicate brightness (see Sec. 31, Chapter 7), are given in the extreme right-hand column. At the beginning of this section are lists of Latin and Russian names of constellations and stars; Latin names are needed in studying the star globe.

To simplify extracting stellar coordinates, the MAE has an insert with the values of τ and δ of the basic 50 navigational stars and Polaris at 10-day intervals. Unlike the basic tables, the insert gives the values of τ and δ of each star next to one another, the degrees at the top of the column and the minutes and tenths of minutes in the column according to the date. From this table, the coordinates of stars (with the exception of Polaris) are always obtainable to within $0'.1$ without interpolation.

Thus, the hour angle of a star is obtained by extractions from two sections: the daily tables and tables for stars (or the insert). In the general form, the formula for t^{*}_{loc} will be

$$t^{*}_{loc} = t^{\Upsilon}_{T} + \Delta t^{\Upsilon} \pm \lambda^{E}_{loc} + \tau_{*} \tag{9.6}$$

Example 2. 13 September 1968 at $T_{gr} = 23\text{h}\,57\text{m}\,24\text{s}$; $\lambda_c = 77°37'\text{E}$. Find t^{*}_{loc} and δ_{*} of α Leonis (Regulus).

From the daily tables for 13.09, $T_{gr} = 23\text{h}$.

	t^{Υ}_{T}	$337°41'.3$	
From the interpolation table for $\Delta T_{gr} = 57\text{m}\,24\text{s}$	Δt^{Υ}	$14\ 23\ .4$	
	$+\ t^{\Upsilon}_{gr}$	$352°04'.7$	
	λ_{E}	$77\ 37\ .0$	
	$-\ t^{\Upsilon}_{loc}$	$429°41'.7$	
	$360°$	$69\ 41\ .7$	
From the "Star" table No. 67	τ_{*}	$208\ 25\ .5$	$\delta_{*} = 12°9'.4\text{N}$
	t^{*}_{loc}	$278°07'.2\text{W}$	
		$= 81°52'.8\text{E}$	

III. OBTAINING THE HOUR ANGLES AND DECLINATIONS OF THE SUN, PLANETS AND THE MOON

To obtain the hour angle of the centre of the sun in Greenwich time T_{gr}, use the basic formula of time, written for $\oplus$ and $\odot$ and the meridian of Greenwich; thus

$$S_{gr} = t^{\oplus}_{gr} + \alpha_{\oplus}$$

and

$$S_{gr} = t^{\odot}_{gr} + \alpha_{\odot}$$

whence

$$t_{gr}^{\odot} = t_{gr}^{\oplus} + \alpha_{\oplus} - \alpha_{\odot}$$

or noting that $T_{gr} = t_{gr}^{\oplus} \pm 12\text{h}$, we get

$$t_{gr}^{\odot} = T_{gr} \pm 12\text{h} + \alpha_{\oplus} - \alpha_{\odot} \tag{9.7}$$

where $\alpha_{\odot} - \alpha_{\oplus} = \eta$ is the equation of time at the given instant.

Formula (9.7) is used to precompute the values of the hour angles of the sun ($t_T^{\odot}$) for whole hours of T_{gr} which are given in the daily tables of the MAE. For interpolating the hour angle to give intermediate values of T_{gr}, the MAE uses (for all celestial bodies with the exception of stars) an identical technique for the introduction of two (always positive) corrections: a constant correction that depends solely on ΔT_{gr} and a small correction (variable throughout the year) that depends on the value α of the body for the given day and hour. The latter correction is obtained by interpolation of the so-called **quasi-difference** $\overline{\Delta}$, which is given in the MAE at the bottom of the column of hour angles (for the moon it is next to $t_{gr}^{☾}$). For the sun the formula for obtaining $t_{gr}^{\odot}$ at a given instant of T_{gr} will be of the form

$$t_{gr}^{\odot} = t_T^{\odot} + \left(\Delta T_{gr} + \frac{\Delta\alpha_{\oplus} - \Delta\alpha_{\odot}^{max}}{60\text{m}} \cdot \Delta T_{gr}^{\min}\right)^{\circ} + \left(\frac{\Delta\alpha_{\odot}^{max} - \Delta\alpha_{\odot}}{60\text{m}} \cdot \Delta T_{gr}^{\min}\right)^{\circ} \tag{9.8}$$

where ΔT_{gr} is the excess of T_{gr} over a whole hour

$\Delta\alpha_{\oplus} - \Delta\alpha_{\odot}^{max} = -0'.3$ is a constant correction equal to the maximum change of the equation of time in the year (for one hour $\Delta\alpha_{\oplus} = 2'.46$, and $\Delta\alpha_{\odot}^{max} = 2'.78$)

$\Delta\alpha_{\odot}^{max} - \Delta\alpha_{\odot} = \overline{\Delta}$ is the quasi-difference for the sun, a quantity that varies from $0'.0$ to $0'.5$ (since the smallest $\Delta\alpha_{\odot} = 2'.24$ per hour) and is always positive.

Hence, *in the MAE the quasi-difference* $\overline{\Delta}$ *is an artificial difference between the hourly changes of right ascensions: the maximum possible change and the change for the given hour* ($\Delta\alpha_{body}$).

The quasi-difference may also be represented as the difference in the variation of the magnitude of the hour angle (Δt) for a given hour and the minimal variation (Δt_{min}) for a given celestial body, say the sun $\overline{\Delta} = \Delta t - 14°59'.7$.

The second term in (9.8) is the basic correction $\Delta_1 t$ of the hour angle of the sun, equal to $14°59'.7$ per hour, and is extracted via

ΔT_{gr} from the second column of the basic interpolation table (the page is found from the minutes, the line from the seconds).

The third term, $\Delta_2 t$, of (9.8) is taken from the middle column of these tables on the basis of $\overline{\Delta}$ and a minute of ΔT_{gr}.

In the general form, $t_{gr}^{\odot}$ is obtained from the formula

$$t_{gr}^{\odot} = t_T^{\odot} + \Delta_1 t + \Delta_2 t$$

The local hour angle of the sun is obtained by adding longitude to this formula

$$t_{loc}^{\odot} = t_T^{\odot} + \Delta_1 t + \Delta_2 t \pm \lambda_W^E \tag{9.9}$$

The declination of the sun is also taken out of the daily tables and is interpolated from the middle column of the basic tables on the basis of the following formula of linear interpolation

$$\Delta\delta_{\odot} = \frac{\Delta}{60\text{m}} \cdot \Delta T_{gr}^{\text{min}} \tag{9.10}$$

where Δ is the hourly change in declination of the sun given with its sign at the bottom of the column of declinations.

Example 3. 14 September 1968 at $T_{gr} = 15\text{h}\ 22\text{m}\ 54\text{s}$; $\lambda_c = 41°58'\text{W}$. Find $t_{loc}^{\odot}$ and $\delta_{\odot}$.

From daily tables for 14.09, $T_{gr} = 15\text{h}$	$t_T^{\odot}$	$46°06'.6\ (\overset{+}{0'.5})$	δ_T	$3°21'.1(\overline{0'.9})$
From interpolation table	$\Delta_1 t$	$3\ 13\ .4$	$\Delta\delta$	-0.2
	$\Delta_2 t$	$0\ .1$	$\delta_{\odot}$	$3°20'.9\text{N}$
at $\Delta T_{gr} = 22\text{m}\ 54\text{s}$, via $\overline{\Delta}$ and ΔT_{gr}	$t_{gr}^{\odot}$	$49°20'.1$		
	$-\ \lambda_W$	$41\ 58\ .0$		
	$t_{loc}^{\odot}$	$7°22'.1\text{W}$		

Similarly, from (9.8) and (9.9) we obtain the hour angles of the planets and the moon, but the values of the constant corrections and the quasi-differences will be different. The constant correction for the planets is

$$\Delta\alpha_{\oplus} - \Delta\alpha_{pl}^{max} = 2'.5 - 3'.5 = -1'.0$$

and the hourly change

$$\Delta t_{pl} = 14°59'.0$$

Analogously, for the moon we get

$$\Delta\alpha_{\oplus} - \Delta\alpha_{☾}^{max} = 2'.5 - 43'.5 = -42'.0$$

and the hourly change is

$$\Delta t_{☾} = 14°19'.0$$

Example 4. 13 September 1968 at $T_{gr}=01\text{h}\,48\text{m}\,24\text{s}$; λ 29°53′W. Find t_{loc} and δ of the moon.

From daily tables for 13.09, $T_{gr}=01\text{h}$	$t_T^{☾}$	164°18′.0 (14′.1)
	$\Delta_1 t$	11 32 .9
From interpolation table { at $\Delta T_{gr}=48\text{m}\,24\text{s}$; via $\bar{\Delta}$ and ΔT_{gr} }	$\Delta_2 t$	11 .4
	$t_{gr}^{☾}$	176°02′.3
	$-\ \lambda_W$	26 53 .0
	$t_{loc}^{☾}$	146°09′.3

δ_T	4°45′.9S ($\overset{+}{9'.9}$)
$\Delta\delta$	+8 .0
δ☾	4°53′.9S

Note: In actual practice, when picking out the hour angles of all celestial bodies the above schemes are simplified. Thus, the times T_{gr} are not given, and the designations of hour angles and declinations are frequently omitted.

SEC. 47. DETERMINING THE TIME OF TRANSIT OF BODIES, THE ARRIVAL TIME OF A BODY AT A GIVEN HOUR ANGLE AND OTHER PROBLEMS

In practical work, the navigator has to determine the time when a certain astronomical phenomenon occurs, for instance, the time of transit of a body, when it rises and sets, when it arrives at the prime vertical, and so forth. All of these problems are particular cases of a single general problem—a determination of the time of arrival of a body at a specified hour angle. Essentially, the problem is solved as follows: from the astronomical triangle we determine the local hour angle of the body for its given position; this hour angle is converted to the Greenwich meridian, and from the MAE via t_{gr} by *reverse entry* we select the mean (Greenwich) time T_{gr}, which is then converted by the zone description to T_{sh}.

However, for the most important cases (transits, sunrise, sunset, moonrise, moonset) the MAE gives *final solutions* of this problem for the *Greenwich meridian*. Thus, the meridian passage on the Greenwich meridian of the sun, moon, Aries and four planets, and also the time of sunrise and sunset to within 1m and of moonrise and moonset to within 0h.1 is given in the daily tables of the MAE.

I. DETERMINING THE TIME OF TRANSIT (MERIDIAN PASSAGE) OF THE MOON, SUN AND PLANETS

This problem is solved for the moon, the sun and the planets in exactly the same way. For this reason, we shall consider it for only one body, the moon, for which this phenomenon is more complicated and, besides, is very important in matters of navigation.

As already mentioned, the MAE gives the transits of celestial bodies on the Greenwich meridian; for the moon and sun, both upper and lower transits are given; for the planets, only the upper transit is given.

Due to the proper motion of the sun, planets and, particularly, the moon, the meridian passage on the Greenwich meridian varies. For the moon, the time of transit increases from 41m to 65m (about an average of 50m a day, or 2m per 15° of longitude, see Sec. 25). Therefore, the time of transit for observers situated in other longitudes will be different: for observers in east longitudes the moon transits earlier than at Greenwich; for observers in west longitudes, it transits later. Therefore, for east longitudes interpolate with the *preceding date*; for west longitudes interpolate with the *subsequent date*. To convert from meridian passage at Greenwich to local meridian passage T_{loc} at the given meridian, introduce the correction ΔT_λ computed from

$$\Delta T_\lambda = \frac{\Delta T}{360^\circ} \cdot \lambda^\circ \tag{9.11}$$

where ΔT is the time difference (formed from the MAE) between transit at the given date and the time of the preceding transit (if λ_E) and that of the subsequent transit (if λ_W). The transit time difference is taken with its sign.

Instead of computing (9.11), one can use a special table at the end of the MAE. For the moon, the correction ΔT_λ is always taken into account; for the planets, ΔT_λ is utilized only for appreciable longitudes of the position; for the sun, ΔT_λ is not ordinarily taken into account, since the differences $\Delta T < 1$m; thus $T_{tab} \approx T_{loc}$.

The time obtained after adding the correction ΔT_λ will be the local time of transit; we get T_{sh} in the ordinary way, "via Greenwich". Due to the fact that the lunar day is longer than the mean day and amounts to about 24h50m, on certain days there will be no transit of the moon on the Greenwich meridian and, possibly, on the local meridian. Indeed, if, say, on 24 July 1968 the moon transited (lower transit) at Greenwich at 23h 46m, then by adding 24h 50m we see that on the next day, 25 July, there will be no transit at Greenwich, and only on 26 July at 0h37m will there be the next lower transit of the moon at Greenwich. In the MAE there is a dash to indicate that there will be no transit. Then, to find T_{loc} do as follows: for east longitudes take the *succeeding* transit and interpolate with the previous date one day later; for west longitudes, on the contrary, the *preceding* transit is interpolated with the succeeding date, also a day later. For example, for east longitudes, between 26.07 and 24.07 and for west longitudes between 24.07 and 26.07. At times, one has to apply a simple choice of date

For planets, the interval between two successive upper transits may be smaller than 24h, then there will be *two* upper transits on that day.

Example 5. 23 September 1968 in $\lambda = 151°30'$E. Find T_{sh} (ZD $= -10$) for upper and lower transits of the moon.

Upper transit

T at Greenwich	0h 25 (—55m; difference between 25.09 and 23.09)
ΔT_λ	—22
— T_{loc}	0h 3m 25.09
λ	9 25
T_{gr}	14h 38m 24.09
ZD	9
T_{sh}	23h 38m 24.09
T_{leg}	0h 38m 25.09

Lower transit

T at gr	11h 58m (—56m)
ΔT_λ	—22
— T_{loc}	11h 36m 24.09
λ	9 25
+ T_{gr}	2h 11m
ZD	9
T_{sh}	11h 11m 24.09
T_{leg}	12h 11m

As will be seen from the MAE, there will be no lower transit of the moon across the Greenwich meridian on 22.09. Likewise, there will be no transit over the meridian with $\lambda = 151°31'$E local time; for zone time (T_{sh}) the transit will occur at 23h 53m on 22.09; the next lower transit will be at 0h 41m on 24.09.

To determine the time of lower transit of the planets not given in the MAE, add 12h to or subtract 12h from the time of upper transit on that day.

II. DETERMINING THE TIME OF UPPER TRANSIT (MERIDIAN PASSAGE) OF A STAR

This problem may be solved by exact and approximate methods.

(1) **Exact solution.** Determining the time of meridian passage is one of the particular cases of determining the arrival time of a body at a given hour angle, and is executed in the following order.

At the instant of upper transit of the star, its local hour angle is equal to 360° or 0°. From the formula $t^{\Upsilon}_{loc} = t^{*}_{loc} - \tau_{*}$ we determine t^{Υ}_{loc}; τ_{*} is taken out of the MAE from the name of the star, and

t^{Υ}_{loc} is converted to the Greenwich meridian by the formula $t^{\Upsilon}_{gr} = t^{\Upsilon}_{loc} \mp \lambda^{E}_{W}$. From the daily tables of the MAE, according to the date by reverse entry we choose the nearest smallest value of t^{Υ}_{T} and from that value we get T'_{gr}. Again, by reverse entry from the difference Δt^{Υ} we select ΔT_{gr} from the interpolation table with the accuracy required (1m or 1s).

Example 6. On 12 September 1968 in $\lambda = 38°31'$W. Find T_{loc} and T_{sh} of upper transit of the star α Tauri.

t^{*}_{loc}	360°00′.0
− τ_{*}	291 34 .3
t^{Υ}_{loc}	68°25′.7
+ λ_W	38 31 .0
t^{Υ}_{gr}	106°56′.7
t^{Υ}_{T}	96 2 .8...
Δt^{Υ}	10°53′.9...

	T'_{gr}	7h on 12.09
	ΔT_{gr}	43m 29s
	T_{gr}	7h 43m 29s
	− ZD_W	3
12.09	T_{sh}	4h 43m 29s

T_{gr}	7h 43m 29s
− λ_W	2 34 4
T_{loc}	5h 9m 25s

(2) **Approximate solution of the problem.**

(a) Without the MAE.

At the instant of upper transit, $t^{*}_{loc} = 0°$, hence, $t^{\Upsilon}_{loc} = \alpha_{*}$, but $t^{\Upsilon}_{loc} = t^{\oplus}_{loc} + \alpha_{\oplus}$. Whence we have $t^{\oplus}_{loc} = \alpha_{*} - \alpha_{\oplus}$ and, finally, $T_{loc} = t^{\oplus}_{loc} \pm 12$h.

Picking α_{*} from the list of stars or a globe or elsewhere and computing $\alpha_{\oplus}$ according to the rules given in Sec. 40, we get T_{loc} to 5-10m.

(b) With the MAE.

The MAE gives the time of transit (meridian passage) of Aries at the Greenwich meridian (T^{Υ}_{loc} at Greenwich). Taking this time for T^{Υ}_{loc}, we get an approximate (to within 5m) time of transit of the star, $T_{loc} = T^{\Upsilon}_{loc} + \alpha_{*}$, where α_{*} is extracted from the left column of the table for stars in the MAE.

Example 7. Taking the conditions of Example 6, we have

(a) Without MAE

$t^{\Upsilon}_{loc} = \alpha_*$	4h 34m	
$\alpha_\odot$	11 20	(reckoning from 22.09)
$t^{\odot}_{loc}$	17h 14m	
12	12	
T_{loc}	5h 14m	12.09

(b) With MAE

$+\ T^{\Upsilon}_{loc}$	0h 36m
α_*	4 34
T_{loc}	5h 10m

III. DETERMINING THE TIME OF ARRIVAL OF A CELESTIAL BODY AT A SPECIFIED HOUR ANGLE

The general order of solving this problem is shown at the beginning of this section and in the case of an exact determination of T_{sh} of transit of a star. Let us examine the peculiarities of solving it for the sun, moon and planets. After obtaining (via formulas, globe, or otherwise) t_{loc} of the body and converting it to t_{gr}, the procedure is to enter the daily tables in a *reverse entry* and obtain T'_{gr} and, from the interpolation table, ΔT_{gr}. Also pick out the quasi-difference Λ for the sun, planets and the moon. Using the quasi-difference and $\Delta T'_{gr}$ from the interpolation table, pick out directly the correction Δ and multiply by four for the planets and the sun, and by 4.2 for the moon in order to convert to seconds of time. The correction obtained is *always subtracted* from T_{gr}. This way a time accuracy to 1s is ensured.

Example 8. 12 September 1968 in $\lambda = 156°38'$E. Find T_{sh}(ZD=11E) and T_{loc} of arrival of sun at $t_{loc} = 87°12'$W to within 1s.

	$t^{\odot}_{loc}$	87°12′.0		
	$-\ \lambda_E$	156 38 .0		
On 12.09	$t^{\odot}_{gr}$	290°34′.0		
from MAE	$t^{\odot}_{T}$	285 56 .2 ($\overline{\Delta}$=0′.5)...	T'_{gr}	7h
by Λ 0′.5	$\Delta_1 t$	4°37′.8	ΔT_{gr}	18m 32s
and ΔT we get 0′.2		0.2×4=1s.	ΔT_{add}	−1s

$+\ T_{gr}$	7h 18m 31s	$+\ T_{gr}$	7h 18m 31s
ZD	11	λ	10 26 32
12.09 T_{sh}	18h 18m 31s	T_{loc}	17h 45 m03s

Note: For upper transit of sun, $t_{loc}^{\odot}$ is taken equal to 360°, otherwise the solution is similar.

Example 9. 20 September 1968 in $\lambda=48°27'E$. Find T_{loc} and T_z of the upper transit of the moon to within 1s.

	$t_{loc}^{☾}$	360°00′.0
	$-\lambda_E$	48 27 .0
	$t_{gr}^{☾}$	311°33′.0
On 20.09 from MAE	$t_T^{☾}$	307 28 .5 ($\bar{\Delta}=5.7$)...
By $\bar{\Delta}$ and ΔT...	$\Delta_1 t$	4° 4′.5
		1′.7×4.2=7s

T'_{gr}	17h
ΔT_{gr}	17m 5s
ΔT_{add}	— 7s
T_{gr}	17h 16m 58s
$+\lambda_E$	3 13 48
T_{loc}	20h 30m 46s 20.09

T_{gr}	17h 16m 58s
$+$ZD	3
T_z	20h 16m 58s

IV. MISCELLANEOUS SECONDARY PROBLEMS SOLVED BY THE MAE

(1) Finding the right ascension of the sun, moon and planets for a given date.

Problems of this kind are encountered when plotting celestial bodies of the solar system on a star globe or map, for which purpose we have to know α and δ of the body at a given time to within 0°.1. The problem of finding α is solved by the formula

$$t_{gr}^{\Upsilon}-t_{gr}^{body}=\alpha_{body} \tag{9.12}$$

For the planets, the values of α are given at the bottom of the column of ephemerides. It is ordinarily required to know α to within about 0°.5; for that reason, for all bodies except the sun and moon one can take the tabulated values of α. For the sun and the moon this quantity is obtained from formula (9.12); for the sun the values of t and δ are picked out for the middle of the day, and for the moon, for the given hour of T_{gr}.

Example 10. 19 July 1968 at about $T_{gr}=20\text{h}\ 40\text{m}$. Find the coordinates of the moon to plot on a star globe.

for $T_{gr}=21\text{h}$	t_T^{Υ}	252°42′.7
	$-t_T^{☾}$	201 28 .9
	$\alpha_{☾}$	51°13′.8 ≈ 51°.2
	$\delta_{☾}$	22° .1N

(2) Finding the right ascension of the mean sun for a given date. From formula (9.1) we have

$$t_{gr}^{\Upsilon} = T_{gr} \pm 12\text{h} + \alpha_{\oplus}$$

When $T_{gr} = 12\text{h}$, we have

$$\alpha_{\oplus} = t_{gr}^{\Upsilon}$$

Example 11. 19 July 1968. Find $\alpha_{\oplus}$ at $T_{gr} = 12\text{h}$.

$$\alpha_{\oplus} = t_{gr}^{\Upsilon} = 117° \approx 117°.3$$

(3) Finding the values of the equation of time for a given date. The equation of time $\eta = t_{gr}^{\oplus} - t_{gr}^{\odot}$ or $\eta = T_{gr} \pm 12\text{h} - t_{gr}^{\odot}$. At $T_{gr} = 12\text{h}$, $\eta = 360° - t_{gr}^{\odot}$, and at $T_{gr} = 0\text{h}$, $\eta = 180° - t_{gr}^{\odot}$.

Example 12. 19 July 1968. Find η at 0h and 12h Greenwich time.
(1) $T_{gr} = 0\text{h}$; $\eta = 180° - 178°27'.7 = +1°32'.3$;
(2) $T_{gr} = 12$; $\eta = 360° - 358°27'.2 = +1°32'.8$

SEC. 48. BASIC FACTS ON THE STRUCTURE AND USE OF NAUTICAL ALMANACS OF OTHER COUNTRIES

At the present time, the nautical astronomical almanacs of the largest sea-faring nations have identically constructed tables of the basic data and the same principles of interpolation. There are only slight differences in the arrangement of material and in secondary data. Let us examine a British-American almanac that began to be published in 1958 and has the largest circulation. This almanac has all the characteristic features of many other foreign almanacs. We shall call it the Nautical Almanac (NA).

In the NA, all basic data, that is Greenwich hour angles and the declinations of the sun, moon and four planets, the hour angle of Aries, the quantities τ_{*} and δ_{*} of 57 selected navigational stars, and also the time of sunrise and sunset, moonrise and moonset and the time of civil and nautical twilight are located on a single extensible sheet based on the arguments: Greenwich mean time (GMT) and date. Thus, nearly all the information a navigator needs is obtainable at one opening of the almanac.

Each open sheet of the NA accommodates three days. The ephemerides of the sun, moon and planets are given at intervals of one hour of Greenwich mean time during these three days; here also are the quasi-differences (v) and the variation of declination (d).

The coordinates τ and δ of stars, given under their proper names, and also the times of sunrise and sunset, moonrise and moonset and twilight (both for north and for south latitudes) are given once every three days.

The transits, semidiameters and parallaxes of the sun, moon and planets are tabulated for each day (for the moon, the parallaxes are given even for each hour).

The interpolation tables for the hour angles of the sun, planets, Aries and the moon and also for the declinations of these bodies are based on the same principle as that used in the Soviet MAE, that is, all corrections are given for each minute opening of the tables and without any interpolation.

Let us examine some peculiarities of the construction of the daily and other tables of the Nautical Almanac.

(1) Formula (9.8) for computing the hour angle of the sun is used in the NA transformed to

$$t_{gr}^{\odot}=\left(T_{gr}\pm 12\text{h}-\eta-\frac{\Delta\eta}{2}\right)^{\circ}+(\Delta T_{gr})^{\circ} \qquad (9.13)$$

where T_{gr} is the value of Greenwich time for every hour
$\Delta\eta$ is the change of the equation of time for that hour
ΔT_{gr} is the excess of Greenwich time over one hour.

All data are expressed in arc units.

The quantity $\frac{\Delta\eta}{2}$ is introduced into the basic value of the hour angle in order to eliminate the second correction of the hour angle of the sun. Here, an error in the hour angle is made intentionally. This error will never exceed —0′.15 and +0′10. In the intermediate values of $t_{gr}^{\odot}$ the error will be less, at the middle of the interval it will be zero.

(2) The formula for the hour angle of a planet is transformed so as to construct a single general table of corrections for the sun and planets. For this purpose, the quantity $\Delta\alpha_{pl}^{max}$ in the NA is taken equal to 2′.5 for one hour (instead of 3′.5 in the Soviet MAE), that is, equal to $\Delta\alpha_{\oplus}$; as a result, $\Delta\alpha_{\oplus}-\Delta\alpha_{pl}^{max}=0$ and there is no need for a special table for the planets. But then for Venus we encounter negative quasi-differences (up to —1′.0).

The resulting formula is

$$t_{gr}^{pl}=(T_{gr}\pm 12\text{h}+\alpha_{\oplus}-\alpha_{pl})^{\circ}+(\Delta T_{gr})^{\circ}+\left(\frac{2'.5-\Delta\alpha_{pl}}{60}\Delta T\right)^{\circ} \qquad (9.14)$$

where the first term is the hour angle of planet for every hour as given in the daily tables; the second term is the correction for excess ΔT_{gr} extracted from the same column as for the sun (see formula 9.13); the third term is a correction for the quasi-difference $v=2'.5-\Delta\alpha_{pl}$ given at the bottom of the column of hour angles once every three days. For the moon, v is tabulated for every hour. Thus, for the sun and planets, the NA uses a common interpolation table.

(3) The first correction for the altitude of Polaris to obtain latitude is increased by $1°$ in order to obtain a positive value (the first correction can have a value of about $-56'$). Therefore, subtract $1°$ from the sum of the corrections, or $\varphi = h_p + \mathrm{I} + \mathrm{II} - 1°$.

(4) The NA does not use the coordinate α at all; everywhere $360° - \alpha$ is given.

The Nautical Almanac makes use of the following designations:

(1) GMT (Greenwich Mean Time)—T_{gr}.
(2) GHA (Greenwich Hour Angle)—t_{gr}
(3) LHA (Local Hour Angle)—t_{loc}
(4) SHA (Sidereal Hour Angle)—τ_{*}
(5) Dec. (Declination)—δ
(6) Mer. Pass. (Meridian Passage)—culmination, or transit.

SEC. 49. USING THE MAE OF EARLIER YEARS

If there is no almanac of the current year available to obtain the basic data, earlier MAE's may be used (one of the four preceding years); in this case special corrections are introduced. It is thus possible to obtain the coordinates δ and t of the sun, the sidereal time and the coordinates δ and t of the stars. The coordinates of the planets and the moon are not thus obtainable.

The method under consideration is based on the fact that the coordinates of the true sun and the mean sun (which serve for obtaining S_{gr}) are practically the same for the same instants of several consecutive *tropical years*. If the calendar were reckoned in tropical years, these coordinates would be the same on the same dates and at the same T_{gr} for several years. However, the calendar years are *civil years* of 365 and 366 mean days, whereas the tropical year contains an average of 365d 5h 48m 46s. Obviously, the same instants of adjacent civil years will correspond to different instants of the tropical year. At the commencement of a new civil year equal to 365d, there will still be 5h48m46s to the end of the tropical year. Therefore, to obtain instants of the tropical year, add 5h 48m 46s to the civil year. And conversely from instants of the past civil year *subtract* this so-called correction for the date $\Delta d \approx$ 5h 49m. For the year before last, subtract $2\Delta d =$ 11h37m 32s and so on up to the fourth year. If the preceding year was a leap year, then up to 29 February add $\Delta d =$ 18h11m14s, after that date subtract 5h48m46s. Apply these date corrections with their signs to the instants of T_{gr} of the given year to obtain appropriate instants of earlier years, the almanacs for which we use to pick out coordinates. The magnitude of the correction Δd indicates the displacement of the earth in its orbit, but not its rotation on the axis; for that reason, the hour angles for these years must differ only by $\Delta\eta$ and $\Delta\alpha_{\oplus}$ for the time Δd. Thus,

after selecting $t_{gr}^{\odot}$, subtract from it the date correction in degrees with its sign.

A different procedure may be used that is based on the interpolation of the quantities η and δ for the time Δd.

The magnitudes of the approximate date corrections (Δd) and corrections of sidereal time (ΔS) for the Greenwich meridian for any four-year period are given in Table 6.

Table 6

Year MAE is used	Correction	Year for which MAE is available			
		Leap year	1st after leap year	2nd after leap year	3rd after leap year
Leap year	Δd	+ 0h 45m	+6h 34m (—17h 26m)	+12h 22m (—11h 38m)	+18h 11m (— 5h 49m)
	ΔS	+2′.2	+16′.4 (—42′.7)	+30′.6 (—28′.5)	+44′.8 (—14′.3)
1st after leap year	Δd	— 5h 49m (+18h 11m)	+0h 45m	+ 6h 34m	12h 22m
	ΔS	—14′.3 (+44′.8)	+2′.2	+16′.4	30′.6
2nd after leap year	Δd	—11h 38m (+12h 22m)	—5h 49m	+ 0h 45m	+ 6h 34m
	ΔS	—28′.5 (30′.6)	—14′.3	+2′.2	+16′.4
3rd after leap year	Δd	—17h 26m (+6h 34m)	—11h 38m	— 5h 49m	+ 0h 45m
	ΔS	—42′.7 (+16′.4)	—28′.5	—14′.3	+2′.2

Note: Corrections up to 29 February for leap years are indicated in brackets.

I. Obtaining the Coordinates t and δ of the Sun

(a) Give the observed instant of T_{gr} a correction of the date chosen from Table 6 for the year of the available almanac.

(b) With obtained T_{gr} and date enter MAE and pick out $t_{gr}^{\odot}$ and $\delta_{\odot}$ in the ordinary way.

(c) Convert date correction Δd to arc units $(\Delta d)^{\circ}$.

(d) Correct the obtained $t_{gr}^{\odot}$ with the correction $(\Delta d)^{\circ}$ with sign reversed: we get $t_{gr}^{\odot}$ and $\delta_{\odot}$ for the given year.

Example 13. On 10 February 1968 the sun was observed at $T_{gr}=$ 17h 18m 42s. Find $t_{gr}^{\odot}$ and $\delta_{\odot}$ from the MAE on 1967.

From Table 6 we have $\Delta d=+6$h 34m.

	T_{gr}	17h 18m 42s	10.02 68
	Δd	—5h 49m	
	T_{gr}	11h 29m 42s	10.02.67
From 1967 MAE	$t_{T}^{\odot}$	341°25′.7	(0′.3)
	$\Delta t_{1,2}$	7 25 .5	
	$t_{gr}^{\odot}$	348°51′.2	
	$\Delta d^{\odot}$	87°15′.0	
10.02 68	$t_{gr}^{\odot}$	436°06′.2 = 76°06′.2W	

δ_T	14°29′.7 (—0′.8)
$\Delta\delta$	0 .4
$\delta_{\odot}$	14°29′.3 S

From 1968 MAE we have $t_{gr}^{\odot}=76°06'.0$W; $\delta_{\odot}=14°29'.3$S

12.03.67	$T_{gr}=21$h...	t_{T}	124°48′.9
	$\Delta T_{gr}=16$m 34s...	Δt^{Υ}	4 9 .2
	On 1967	t_{gr}^{Υ}	128 58 .1
		ΔS	+44 .8
	12.03.68	t_{gr}^{Υ}	129°42′.9

32.12.67 MAE	62°42′.1	8°46′.8 N
1.01.67	62 42 .9	8 46 .6
Change of coordinates in year	—0′.8	+0′.2
On 12.III.67	62°46′.2	8°46′.5N
1 year × changes	— (1×0′.8)	+ (1 × 0′.2)
12.03.68	62°41′.8	8°46′.7N

From the 1968 MAE we have $\tau=62°41'.8$ $\delta_{*}=8°46'.7$N

II. Obtaining Sidereal Greenwich Time (t^{Υ}_{gr})

(a) Using the observed instant of T_{gr} and the given date, enter MAE of the available year and pick out t_{gr} in the usual way.

(b) From Table 6 choose a correction ΔS for the required year and apply it to t^{Υ}_{gr} with its sign.

Example 14. On 10 March 1968 at $T_{gr}=21$h 16m 34s. Find t^{Υ}_{gr} from 1967 MAE.

10.03 1967 $T_{gr}=20$h	t^{Υ}_{T}	291°18′.9
$\Delta T_{gr}=$ 1h 6m 34s	Δt^{Υ}	16 32 .7
		8 .5
1967	t^{Υ}_{gr}	308°00′.1
From Table 6	ΔS	+16 .4
1968	t^{Υ}_{gr}	308°16′.5

From MAE we have $t^{\Upsilon}_{gr}=308°16'.1$

III. Obtaining the Coordinates (τ and δ) of Stars

(a) From available MAE pick out values of coordinates of a star for 1 January and 32 December, subtract former from latter. We get the coordinate *changes* for one year (for some stars the changes are given in Table 5, Sec. 31).

(b) Choose values of τ and δ of given star for required date, using the same MAE.

(c) Multiply the changes obtained by the number of years and add to chosen τ_* and δ_* with their signs.

Example 15.

On 21.03.68 find τ_* and δ_* of α Aquilae (Altair) from 1967 MAE.

	τ_* Altair	δ_* Altair
32.12.58	245°42′.2	N 5°19′.7
1.01.58	245 43 .0	N 5 19 .9
Change of coordinates in year	—0′.8	—0′.2
On 20 August 1958	245°43′.1	N 5°19′.9
3 years × changes	—2′.4	—0′.6
On 20 August 1961	245°40′.7	N 5°19′.3

From MAE we have for 1961: $\tau_*=245°41'.0$; $\delta_*=$ N 5°19′.5

The greatest error in the coordinates obtained by these methods will occur when working with 4-year-back MAE's and may attain $\pm 0'.6$-$0'.7$. When using a MAE of the preceding year, errors will not exceed $\pm 0'.2$-$0'.4$.

The time of transit of the sun (to within 1m) and of the stars (to within 2m) may be taken from the MAE of any of four preceding years without corrections, just as the time of sunrise, sunset and twilight. For transits of stars, the errors will exceed 2 minutes in leap years prior to 29 February.

Examples on Chapter 9.

I. Determining t_{loc} and δ of a Celestial Body

16. 13.12.68 at $T_{gr} = 21h\ 27m\ 47s$; $\lambda_c = 123°38'.5E$; observed α Tauri.
17. 1.04.68 at $T_{gr} = 13h\ 58m\ 27s$; $\lambda_c = 43°\ 27'.5E$; observed the sun.
18. 23.12.68 at $T_{gr} = 19h\ 27m\ 10s$; $\lambda_c = 36°37'W$; observed Venus.
19. 25.08.68 at $T_{gr} = 20h\ 31m\ 20s$; $\lambda_c = 17°18'W$; observed Moon.

II. Determining the Time of Transit

20. 30 July 1968. Find T_{loc} and T_{sh} of upper and lower transit of the Moon in $\lambda_c = 171°43'E$ to within 1m.

21. 19 August 1968. Find T_Z of upper transit of sun to within 1m and 1s in $\lambda_c = 163°20'W$.

22. 10 June 1968. Find T_{sh} of upper transit of Saturn in $\lambda_c = 19°45'E$.

23. 12 September 1968. Find T_{sh} of upper transit of the star Aquilae (Altair) in $\lambda_c = 18°12'W$ by exact and approximate methods.

III. Miscellaneous Problems

24. 30 October 1968 in $\varphi = 43°6'.7N$; $\lambda = 131°53'.0E$ at $T_{sh} = 9h\ 30m$; $T_{ch} = 11h\ 27m\ 56s$; $u_{ch} + 2m\ 18s$*. Find the altitude h and azimuth A of the sun.

25. 24 August 1968 in $\varphi = 59°58'7N$; $\lambda = 30°19'E$ at $T_{sh} = 19h35m$; $T_{ch} = 4h39m24s$; $u_{ch} = -3m42s$. Determine altitude and azimuth of Jupiter.

26. 12 April 1968 in $\varphi = 38°42'.5S$; $\lambda = 24°30'.6W$ at $T_{sh} = 7h40m$; $T_{ch} = 9h\ 32m\ 51s$; $u_{ch} = +7m\ 2s$. Find azimuth of sun.

* See Sec. 52.

PART TWO

INSTRUMENTS AND TOOLS USED IN NAUTICAL ASTRONOMY

CHAPTER 10

THE CHRONOMETER AND TIMEKEEPING

SEC. 50. TIMEKEEPING AT SEA

On board ship, exact time is needed for purposes of navigation, the solution of problems in nautical astronomy, communication with land, performing of duties and the organization of activities on the ship. Highest accuracy (up to 0s.5-1s.0) is required for the solution of problems in nautical astronomy; for other purposes, the accuracy may be reduced to 15s and even to 1m.

The problems of ship timekeeping involve:

(1) keeping exact time, a so-called standard of time on the ship;

(2) receiving time signals and obtaining watch corrections;

(3) care of timepieces on the ship and provision of accurate time readings for observers, posts and living quarters.

For these purposes, a ship usually has a number of timepieces that range from high-precision chronometers to simple clocks and watches. In addition, there are stop-watches for measuring intervals of time. Time signals are received by radio receivers and relayed to the navigation room.

Ship timekeeping is in the charge of the third navigator, chief of navigation outfit, and is supervised by the captain.

SEC. 51. DESIGNATION AND CONSTRUCTION OF CHRONOMETER AND OTHER SHIP TIMEPIECES

Nearly all problems in nautical astronomy require a knowledge of the time on the initial meridian, which at present is Greenwich mean time. This time is used to choose the coordinates of a body in almanacs, and in some problems it is utilized directly. Greenwich time is measured by a special timepiece, the chronometer.

A marine chronometer, or simply **chronometer**, (Fig. 77), is an accurate spring-driven clock of special construction. The chronometer is adjusted at the factory to mean units of time and on board ship it is set approximately to the time of the Greenwich meridian, so that it can be used at any time to compute T_{gr}. For this reason, the

Fig. 77

chronometer is often called the keeper of Greenwich (universal) time on board ship. Thus, the most accurate time standard at sea is the chronometer.

Due to the fact that the mean time changes continuously and uniformly, an instrument measuring such time must have a continuous uniform motion. All the features of chronometer design that distinguish this timepiece from ordinary clocks and watches are due to the attempt to ensure a uniform rate over long periods of time.

The chronometer consists of the following basic components: (1) regulator; (2) escapement; (3) motor; (4) counter.

The main peculiarities of construction are:

(1) A specially designed regulator (spring-driven pendulum) that reduces the effects of temperature variations on the rate, so-called balance with temperature compensation.

For this purpose, ring *a* (Fig. 78) of the balance is made of two strips of metal with different coefficients of expansion and is cut in two places. The oscillation period of the balance is expressed by the formula

$$T = \pi \sqrt{\frac{I \cdot L}{N}} \qquad (10.1)$$

where I is the moment of inertia of the balance relative to the axis of rotation

L is the length of the cylindrical spring p, which executes reverse oscillation of the pendulum (it is analogous to the "hair spring" of watches)

N is a coefficient that depends on the elasticity of this spring.

When the temperature increases, the centre bar b of the balance (like the other parts) expands, and weights q_1, q_2 should move away

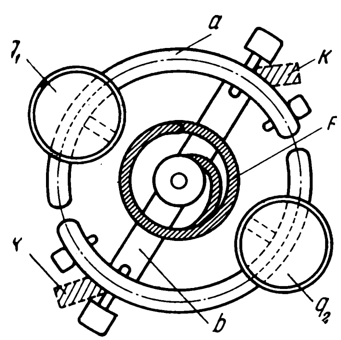


Fig. 78

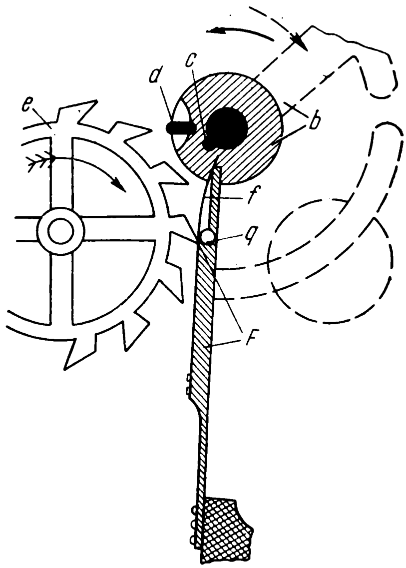


Fig. 79

from the centre of rotation. However, the outer strip of the ring is of a metal with a large coefficient of expansion and will again bring the weights towards the centre of rotation, so that the moment of inertia I does not change and the oscillation period T hardly changes at all.

(2) The escapement of the chronometer, unlike anchor escapement and cylinder escapement, operates without friction, exclusively by beats, the so-called chronometer escapement (Fig. 79).

In the lower part of the balance b is a disc with two cams, d and c, at different levels. The power of the mainspring is conveyed to the balance via impacts of the teeth of the travelling wheel e against the impulse stone d, which occurs when the travelling wheel is released from the stop at the rest stone q located on the spring F. The latter is deflected to release a tooth of the wheel. This is done via a thin spring f by the pressure of the disengagement stone c. Hence, when the balance moves towards the solid hand, c exerts pressure

on f and momentarily deflects F above with q; one of the teeth of the travelling wheel will slip by, but the next one will be stopped by q, which—due to F—will return to its original position. This will be heard as an audible beat. At the same time, another tooth will strike d, setting the balance in motion. On the return (idle) stroke, effected by the cylindrical spring p (Fig. 78), c will pull away spring f without moving the big spring F and without perceptibly slowing down the vibrations of the balance. This ensures free oscillations of the balance and a uniform motion of the travelling wheel e.

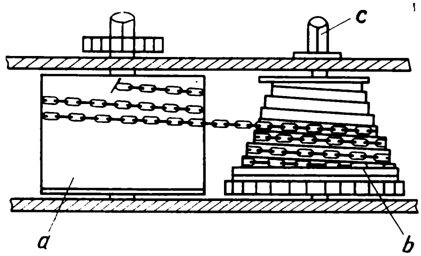


Fig. 80

(3) Design of motor. The action of the mainspring a (Fig. 80) is not transmitted directly to the toothed wheels but via a conical fusee b that equalizes the action of the spring for various degrees of winding. Winding is done by the winding head c.

The chronometer differs from ordinary clocks in other ways, namely:

(1) an additional scale and hand on the dial to indicate the total number of hours the spring has been operating since winding (usually 56h, sometimes 7 days);

(2) the second hand moves in half-second jumps; these jumps are accompanied by sharp, easily countable beats of the escapement mechanism. A faint click is heard in the intervals between beats as the pendulum reverses direction.

The precision with which it is manufactured and the peculiarities of construction of the chronometer impose strict requirements on storage and operation, which we shall deal with later on. Under the proper conditions, the chronometer ensures T_{gr} to within ± 1 second at time of sight. Large ships on long voyages keep two chronometers, other ships have one. Smaller ships sometimes have deck watches instead of chronometers.

A deck watch (Fig. 81) is an accurate watch with a large second hand moving in jumps of 0s.2. These watches are usually set for Greenwich time and are designated for astronomical and certain

navigational observations. They also replace chronometers. Deck watches are kept in a special wooden box with two lids, one of which is glass.

Ship (marine) clocks are round, wall-mounted clocks with a face divided into 24h, sometimes into 12h. Modern Soviet clocks (like foreign makes too) have hour, minute and second hands usually mounted on a single central axis. The face is divided into 12h, the hands and figures are coated with a luminescent paint. Older clocks do not have second hands and the face is divided into 24h. Ship clocks are set to ship time (T_{sh}), are located in the principal service and living quarters, and serve as timekeepers for the daily affairs of life on board ship. Ship clocks located in the radio room are usually set to Greenwich (on long voyages) or Moscow time and must have a second hand.

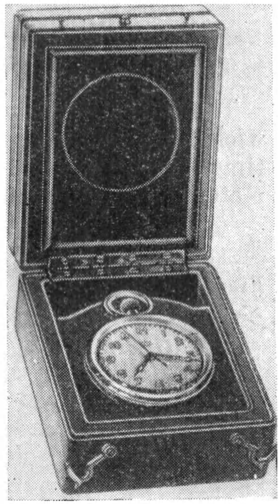


Fig. 81

Since ship clocks do not have a compensating device and are subjected to the effects of external factors, their rate must be adjusted regularly, for which purpose there is a special regulator on top of the dial.

Stop watches are pocket watches with second and minute hands that are equipped with a starting, stopping and hand-return device for reckoning from 0m0s. Some stop watches have two second hands. Stop watches are designed for measuring short intervals of time in navigation, astronomical and other observations. Ships ordinarily have two or three stop watches.

SEC. 52. CHRONOMETER AND WATCH ERROR (CORRECTION)

The dial of a chronometer is divided into 12h instead of 24, therefore, in the second half of the day the chronometer readings differ from Greenwich time by 12h. For example, $T_{gr} = 13h25m10s$, and the chronometer reads 1h25m10s. For this reason, it is sometimes necessary to add 12h to the reading of the chronometer. To determine whether it is necessary to add 12h and to determine the Greenwich date, which often differs from the date at the given place, the first step is to make an approximate computation of T_{gr}. Given the ship time T_{sh} and the zone description of the ship's clock, we get the following approximation:

$$T_{gr} \approx T_{sh} \mp ZD^{E}_{W} \tag{10.2}$$

In addition, despite the care taken in manufacturing and maintaining the chronometer, an ideal mechanical chronometer with absolutely uniform movement is impossible. For this reason, at a given instant a marine chronometer will read differently from T_{gr} by a certain error which varies with time.

The chronometer correction u_{ch} relative to Greenwich time is the difference between Greenwich time T_{gr} and the chronometer reading T_{ch} at the same instant; *thus*,

$$u_{ch} = T_{gr} - T_{ch} \tag{10.3}$$

The correction is *positive* (+) if the chronometer is slow on Greenwich time, and *negative* (—) if the chronometer is fast on Greenwich time. Since the dial of the chronometer is divided into 12h, chronometer correction cannot be greater than ±6h.

Example 1. 5.08 at $T_{sh} = 8\text{h}$ (ZD=10E); $T_{gr} = 22\text{h}\,00\text{m}\,00\text{s}$. $T_{ch} = 9\text{h}\,49\text{m}\,22\text{s}$. Find u_{ch}.
Solution. Since $T_{gr} = 22\text{h}$, obviously $T_{ch} = 21\text{h}49\text{m}22\text{s}$.

Then:	
T_{gr}	22h00m00s 5.08
$-T_{ch}$	21 49 22
u_{ch}	+0h10m38s

In practice it is more convenient if the readings of the ship chronometer are closer to Greenwich time; therefore, if a large correction is obtained, it may be reduced by turning the hands of the chronometer to the required reading. To set the minute and hour hands of the chronometer, unscrew the lid of the dial, put the winding key on the head of the hand axis and slowly move the hands forward to the right of the required reading. In doing so, take care to bring the minute hand in accord with the second hand, but never touch the second hand. Opening the mechanism and moving the hands of the chronometer can affect its rate. Therefore, this should be done only in exceptional cases.

If at a given instant we know the chronometer correction and the chronometer reading, it is easy to determine the required Greenwich time from the formula

$$T_{gr} = T_{ch} + u_{ch} \tag{10.4}$$

First determine the date and T_{gr} approximately (to at least within 1h or 2h) from ship time and the zone description using formula (10.2).

Example 2. 20.09 at $T_{sh} = 10\text{h}35\text{m}$; $\lambda = 38°50'\text{W}$; $T_{ch} = 1\text{h}23\text{m}7\text{s}$; $u_{ch} = +12\text{m}34\text{s}$. Find T_{gr}.

Solution.

$+\ T_{sh}$	10h35m 20.09	T_{ch}	1h23m07s
$\ \ \ ZD_W$	3	u_{ch}	+12 34
$T_{gr} \approx$	13h35m 20.09	T_{gr}	13h35m41s

Example 3. On 8.10 at $T_{sh} = 8\text{h}55\text{m}$ ($ZD = 10E$); $T_{ch} = 10\text{h}50\text{m}48\text{s}$; $u_{ch} = +2\text{m}11\text{s}.5$. Find T_{gr}.

Solution.

$-\ T_{sh}$	8h55m 8.10	T_{ch}	10h50m48s
$\ \ \ ZD_E$	10	u_{ch}	+ 2 11 .5
$T_{gr} \approx$	22h55m 7.10	T_{gr}	22h52m59s.5

The computation of T_{gr} must be thoroughly mastered, since this operation is the beginning of practically all problems in nautical astronomy. Any blunder in the determination of T_{gr} will lead to a wrong solution of what is at times an involved problem, since T_{gr} is ordinarily obtained at the very beginning of the solution.

Everything that has been said above concerning the chronometer correction also applies to deck watches and ordinary watches and clocks. The watch correction u_{wat} relative to Greenwich time is the difference between Greenwich time and the reading of the watch T_{wat} at the same instant, or

$$u_{wat} = T_{gr} - T_{wat} \tag{10.5}$$

In a similar fashion we determine the watch correction relative to Moscow time or any other zone time; thus,

$$u_{wat} = T_{Mos} - T_{wat}$$

or

$$T_{Mos} = T_{wat} + u_{wat} \tag{10.6}$$

and, generally,

$$T_z = T_{wat} + u_{wat} \tag{10.7}$$

From the foregoing it is clear that the reading of any timepiece must be corrected in order to give the proper time. In nautical astronomy, one has to do exclusively with the chronometer correction or watch correction relative to Greenwich time.

SEC. 53. THE CHRONOMETER RATE AND ITS VARIATIONS

Due to the nonuniform movement of the chronometer mechanism its error relative to Greenwich time is constantly changing or, technically speaking, the chronometer is gaining or losing.

The amount of change in the chronometer error for some interval of time is known as the **rate** during that interval. The chronometer rate is positive if the chronometer is losing, and negative if it is gaining relative to Greenwich time.

The chronometer error for one day is the daily rate of the chronometer ω.

The daily chronometer rate ω with its sign is obtained if we subtract from the subsequent chronometer correction (error) u''_{ch} the preceding correction u'_{ch} and divide the difference by the interval ΔT^d between determinations expressed in days and fractions of days (to within 0d.01); thus,

$$\omega = \frac{u''_{ch} - u'_{ch}}{\Delta T^d} \tag{10.8}$$

If a positive correction decreases (or a negative one increases), then ω will be negative and the chronometer will be gaining; if, on the contrary, the positive correction increases (the negative decreases), then ω will be positive and the chronometer will be losing. The daily rate ω is computed with an accuracy up to 0s.01 and is rounded off to 0s.1.

Example 4. On 11.07 at $T'_{gr} = 14\text{h}$ determined $u'_{ch} = -8\text{m}\ 34\text{s}.5$; on 16.07 at $T''_{gr} = 19\text{h}$ determined $u''_{ch} = -8\text{m}\ 45\text{s}.0$ Find the daily rate of the chronometer.

Solution: (a) $T''_{gr} - T'_{gr} = \Delta T^d = 5\text{d}.21$ (Table 40a, MT-53);

$$\text{(b)}\ \omega = \frac{-8\text{m}\ 45\text{s}.0 - (-8\text{m}\ 34\text{s}.5)}{5\text{d}.21} = -2\text{s}.02 \approx -2\text{s}.0$$

The daily rate should be computed for every determination of the chronometer correction. However, in practical use, when extrapolating the correction (see below), it is necessary to determine the mean daily rate for several days.

As has been noted, the rate of a chronometer is not constant. It is affected by temperature variations, changes in humidity, pressure, the wear of parts of the mechanism, their oxidation, thickening of oil, and so forth. Of all these factors, of prime importance are temperature variations. Notwithstanding the special construction of the pendulum, it has not been possible to compensate fully for the effects of temperature variations. Therefore, corrections have to be introduced to account for these effects. The amount of change

in the daily rate of the chronometer that remains after compensation may be expressed with sufficient accuracy by the "temperature formula of rate":

$$\left.\begin{aligned} \omega_t &= \omega_0 + \alpha(t - t_0) + \beta(t - t_0)^2 \\ &\text{or} \\ \omega_t &= \omega_0 + c(t - t_0) + \frac{s}{200}(t - t_0)^2 \end{aligned}\right\} \qquad (10.9)$$

where ω_t is the daily rate at a temperature t

ω_0 is the daily rate at a temperature $t_0 = +18°C$

α and β are linear and quadratic temperature coefficients determined in the laboratory and recorded in the certificate

c is the temperature coefficient

s is the coefficient of deviation of rate from proportionality to temperature.

The coefficient β depends on the method of compensation and the construction of the chronometer; it should not exceed $\pm 0.01 \frac{\text{sec}}{\text{days}\cdot\text{degrees}}$; the coefficient α differs in different chronometers and should not ordinarily exceed $\pm 0.1 \frac{\text{sec}}{\text{days}\cdot\text{degrees}}$. Under factory laboratory conditions, the above quantities and the daily rate of the chronometer ω_0 are determined at $t = +18°C$, or several rates are given, say, for $t = +4°$, $+18°$ and $+32°C$; this is accompanied by a table (or only a formula) for correcting the chronometer rate:

$$\Delta\omega = \alpha(t - t_0) + \beta(t - t_0)^2 \qquad (10.10)$$

for temperatures between 5° and 35°C. If only the formula is given, then the navigator computes the corrections $\Delta\omega$. Formulas (10.9) may be written in the form

$$\omega_t = \omega_0 + \Delta\omega \qquad (10.11)$$

This formula serves for computing ω_t for intermediate temperatures.

All these findings, and also the results of laboratory tests of the chronometer are given in the **certificate** that comes with each chronometer.

The chronometer rate is also affected (though very slightly) by pressure changes. Experiment has demonstrated that an increase of 1 mm in pressure slows the rate of a chronometer by about 0s.01 per day, which means that even for a change of 30 to 40 mm in pressure in one day, the rate will change by only 0s.3 to 0s.4 which is within the limits of random fluctuations of rate.

Changes in humidity influence the rate to a much greater degree. Judging from the observations of the Kronstadt anstronomer

V. E. Fus, carried out at the close of last century, humidity fluctuations from 45% to 85% caused changes of rate of up to 4s; as a rule, the chronometers lost time. However, it has not been possible to make an accurate check on the effect. For this reason, it is advisible to protect the chronometer box with a woolen or plastic case, particularly during voyages in regions of enhanced humidity.

Changes that occur in the mechanism itself (thickening of oil, wear of axles and jewels, oxidation, etc.) cannot be taken into account, but they gradually alter the rate of any chronometer. That is why it is important to observe the rate of a chronometer at all times and to have it checked in the navigation chamber whenever irregularities are found. An index of the quality of performance of a chronometer (or clock) is the constancy of its daily rate. The difference of two successive daily rates is known as the variation Δ. The quantity $\Delta = \omega_2 - \omega_1$ and is the most important index of the quality of performance of a chronometer. Ideally, the variation is zero.

For marine chronometers, the mean daily rate for various temperatures should not exceed $\pm 4^s$, while the mean variation should be within $\pm 0^s.5$. For various temperatures, the extreme values should not exceed $\pm 2^s.5$. Under these conditions the performance of a chronometer is said to be satisfactory.

One should bear in mind that when a chronometer runs down due to someone forgetting to wind it, the rate may change, and so the daily rate must be derived once again. Observe the performance of the chronometer carefully for about two weeks.

SEC. 54. DETERMINATION OF CHRONOMETER ERROR BY RADIO TIME SIGNALS. PROGRAMS OF RADIO SIGNALS. CHRONOMETER RECORD BOOK

Just recently, determining chronometer errors (corrections) on board ship was a rather involved problem and was handled by a variety of methods: from astronomical observations of celestial bodies for a given known longitude, by comparing with chronometers at an observatory, by visual time signals delivered in ports, etc.

At the present time, radio time signals are the only ones used to determine chronometer corrections. A number of radiostations throughout the world broadcast time signals of Greenwich time at specified hours. They are given in a definite sequence, or according to a specific **program**. These signals are broadcast by powerful radio stations on all wavelengths, thus making it possible to receive them anywhere in the world, by any kind of receiver and at any hour of the day or night. For navigational purposes, special signals

are given that are more precise than the signals of general broadcasts. Prior to the reception of special time signals, the navigator should be acquainted with the programs, the radio stations, their call signals, time of operation and the frequencies on which they operate. This information is given in the "Program of Time Signal Broadcasts" published in the U.S.S.R. by the All-Union Research Institute of Measurements and published in the Notices to Mariners, in "The Admiralty List of Radio Signals", Volume V (British), the American "H.O. No. 205", and other publications.

Of late, the chief programs of special time signals are:

I, program* known abroad as the "English system" (the program of the Rugby radio station);

II, or new international, program;

III, the American program; and

the Japanese program.

(1) **On the first program**, five or fifteen minutes prior to special hours T_{gr}, short dashes (0s.1) are given every other second from one to 59s. The beginning of each 5 minutes is distinguished by a long dash (signal) (0s.6). Thus, the first program delivers five series of signals consisting of 59 short dashes and six long dashes. This program is most convenient because it permits receiving several signals every minute and obtaining a number of corrections; besides, it is very simple. This program is used for sending out time signals by nearly all Soviet and European radio stations, some radio stations of South America, and others.

(2) **On the second, or new international program**, the time signals are in the form of six short dashes:

57m 00s-57m 50s	Warning signals for group I (*X* every 10 seconds) —··— —··—	
57 55-58 00	Group I time signals	55 56 57 58 59 0 sec · · · · · ·
58 08-58 50	Warning signals for Group II (*N* every 10 seconds) —·—·—·	
58 55-59 80	Group II time signals	55 56 57 58 59 0 sec · · · · · ·
59 06-59 50	Warning signals for Group III (*G* every 10 seconds) ——·——·——·	
59 55-00 00	Group III time signals	55 56 57 58 59 0 sec · · · · · ·
00 15-00 30	16 short signals	
or		
00 10-00 30	21 short signals	

* Recommended by the International Astronomical Union on 5.09.55 as a unified system of signals replacing all other systems. However, we still have a variety of programs.

Signals on this program are sent out by certain European stations, for instance, French, Italian, Turkish, South African, Indian, Chinese, Indonesian, New Zealand, etc.

(3) **The United States** system of time signals begins at 55m00s, that is, 55 minutes after the four or five minutes prior to the onset of the subsequent hour; during this time, short dashes are sent out every second; the beginning of the dashes mark seconds, and their duration is not fixed. Exceptions are the 29th seconds of each minute (which are missed) and some of the last ten seconds of each minute, which are sent out as follows:

	50s	51s	52s	53s	54s	55s	56s	57s	58s	59s	60s
55m	—		—	—	—	—					—
56	—	—		—	—	—					—
57	—	—	—		—	—					—
58	—	—	—	—		—					—
59	—										—

The beginning of the long dash at the end (1s.3) coincides with 0m00s of the new hour.

The signals of this program are sent out by stations in Washington (every even hour of T_{gr}), San Fransisco (6 times a day), Balboa (twice a day), Honolulu (3 times), etc.

In addition to these time signals, the United States broadcasts technical tuning signals (standard carrier frequencies) that are exactly oriented in time. These signals may also be used to determine chronometer corrections, every 5 minutes in the 24-hour day. The transmissions are by radio stations WWV (near Washington) on six frequencies and WWVH (Hawaiian Islands) on three frequencies. The signals are heard in most areas of the Atlantic and Pacific Oceans. The transmissions of WWV are interrupted for four minutes beginning with the 45th minute of every hour, of WWVH for four minutes at the beginning and in the middle of every hour.

The signal begins on every fifth minute and continues three minutes, then follow numbers in telegraphic code that indicate T_{gr}, at which time the next signal begins: for example: 0—9—2—5, which means that the next signal will be at $T_{gr} = 09h\,25m\,0s$. This is followed by a voice announcing "this is radio station WWV; the signal will be resumed at 4h 25m a.m. Eastern Standard Time*; 4h 25m a.m."

* In the United States, *zone time* on the continent is called Standard Time (Eastern, Central, Mountain, Pacific; from 5th to 8th zone westwards).

(4) **Japanese program:**

0m 05s-0m 15s	Ten-second dash
0 30-0 55	Warning signals for Signal I (*N*) — • — • — • etc.
1 00-1 01	Time Signal I —
1 30-1 55	Warning signals for Signal II (*D*) — • • — • • — • •
2 00-2 01	Time signal II —
2 30-2 55	Warning signals for Signal III (*B*) — • • • — • • • — • • •
3 00-3 01	Time Signal III —
3 05-3 15	Long dash if signals are correct; a series of dots if incorrect

This program is broadcast by radio stations of Japan and is clearly audible in all seas of the Far East.

Radio stations broadcast call signals and tuning signals prior to the time signals. Some stations follow the ordinary time signals with so-called rhythmic signals constructed on the principle of the vernier of time (61 rhythmic signals during 60s) and serve to determine the chronometer corrections in coastal astronomical studies where T_{gr} is required accurate to within 0s.01 and 0s.001.

Occasionally, for navigational purposes, the chronometer correction is determined from ordinary time signals broadcast by standard broadcasting programs. At the present time, the time signals broadcast by Moscow stations ensure an accuracy of up to $\pm$0s.1. But in other cases when the accuracy is not known, one should not work from such signals.

When sailing in a limited region, the navigator should determine the chronometer correction using the same radio station and the same program. This develops habits that reduce the probability of errors. A record is made of the tuning and the broadcast so as to be able to locate the station as fast as possible and tune the receiver to reception of time signals.

Time signals may be received:

(a) directly with the chronometer, if there is a loudspeaker or earphones near the chronometer,

(b) with a stop watch,

(c) with a deck watch.

To avoid mistakes and inaccuracies, do not confine yourself to only one signal. It is better to record 3 to 6 signals and then deduce an average correction. To do this, first organize a system of recording the desired signals.

When receiving signals with a stop watch, the second hand is started at the instant the first signal is given, subsequent signals being received with the second hand in motion. When the stop watch is stopped at the desired instant of the chronometer and the reading is subtracted from this instant, we get T_{ch} for the first signal. Adding to this T_{ch} the recorded instants of the stop watch,

we obtain the remaining instants of the chronometer (Example 5, Scheme 2). It is still better not to stop the stop watch, but to compare it with the chronometer two or three times (see Sec. 56).

When receiving signals with a deck watch, the latter should be compared with the chronometer both prior to and following reception; then determine the correction of the deck watch in accord with Scheme No. 1, and after that determine u_{ch} from the correction and the mean comparison.

For the first program and the United States program it is more convenient to receive the signals of the 20th and 40th seconds, counting the dashes continuously.

Example 5. Determine u_{ch} from the time signals.

Scheme 1

Signals Received with Chronometer

T_{gr} of signal	T_{ch}	u_{ch}
17h56m20s	18h2m31s.5	—6m11s.5
56 40	2 52 .0	—6 12 .0
57 20	3 32 .0	—6 12 .0
57 40	omitted	—
58 20	4 32 .0	—6 12 .0
17 58 40	18 4 52 .0	—6 12 .0

Average u_{ch} = —6m11s.9

Scheme 2

Signals Received with Stop Watch

T_{gr} of signal	Sec.	T_{ch}	u_{ch}
17h56m20s	0m0s.0	18h2m31s.7	—6m11s.7
56 40	0 20.1	2 51 .8	—6 11 .8
57 20	omitted	—	—
57 40	1 20.1	3 51 .8	—6 11 .8
58 20	2 0.1	4 31 .8	—6 11 .8
17 58 40	2 0.2	18 4 51 .9	—6 11 .9

Average u_{ch} = —6m11s.8

Stop watch stopped at instant of chronometer	18h8m 0s.0
Stop-watch reading	—5m 28 .3
First instant by chronometer (T'_{ch})	18h2m31s.7

On the international program it is more convenient to note the chronometer reading at the instant of the last dot in each series of signals; on the Japenese program, at the beginning of the dash of all three series.

The accuracy of the chronometer correction determined by radio is sufficiently high, particularly if one uses a magnifying glass when receiving with the second hand of a deck watch. The error in the chronometer correction obtained by radio time signals does not exceed $\pm 0^s.3$.

The corrections, like all other data concerning operation of the chronometer, are taken down in the chronometer record book, a sample of which is given below.

Date	GMT of radio signal and name of station	Chronometer No.			Chronometer No.			Time by ship's clock	Signature of observer	Remarks
		Chronometer time	Chronometer correction	Daily rate	Chronometer time	Chronometer correction	Daily rate			

In this form (which is the accepted form for the merchant marine) there is no column for changes in the daily rate and for temperature fluctuations. These findings should therefore be recorded in the column "Remarks". Changes in the daily rate, as we have noted, characterize the performance of a chronometer, and the temperature is needed to deduce the daily rate for conditions that differ from those indicated in the certificate.

In the "Remarks" column one records any other circumstances like stormy weather, excessive humidity, and so forth; here also are recorded cases of chronometers stopping due to their running down. If there is a second chronometer, use the opposite page. Its correction is best derived by *comparison* with the first chronometer, whose correction was determined as indicated above.

SEC. 55. CARE OF CHRONOMETERS

(1) **Storage of chronometer.** A chronometer should be stored in a special box in the navigator's desk and *never removed*, with the exception of very special cases (demagnetization of the ship and the like). The outer casing of the chronometer should be firmly wedged in the drawer of the desk and covered with a woolen fabric. The locking device of the gimbals of the chronometer is released. Take care that there should be no steam pipes, water pipelines, strong magnetic or electromagnetic fields and vibrating components near the chronometer. A *constant* temperature of the order of $+18°C$ should be maintained in the vicinity of the chronometer. When working with the chronometer, open only the upper wooden lid of the casing, and take observations through the inner glass lid.

(2) **Winding the chronometer.** The chronometer should be wound *every day* between 7 and 8 in the morning by the same person (usually the third navigator). To wind, with the left hand gently turn the chronometer in its gimbals on its face, pushing back the slide and entering the key in the keyhole; then turn it slowly *counterclockwise* making from 7.5 to 8 half turns, since each half turn yields a wind of 3h. It is desirable not to wind the chronometer to the stop, but to the mark 8h or 4h, for then there will be no danger of tearing the chain from the fusee. Chronometers should always be wound to the same marks so that they perform with the same spring tension throughout the 24-hour period.

(3) **Transportation of the chronometer** for short distances is done by hand. The instrument should be rigidly clamped in its gimbals.

For transportation over considerable distances, the balance should be stayed to prevent breaking it. To do this, take the chronometer out of the box, unscrew the lid and, holding the mechanism by the edge of the dial, turn it over and extract it from the mechanism. Then put the mechanism on the frame, dial downwards, and with tweezers carefully insert two wedges under the balance (at the two ends of the centre bar) (Fig. 78). The wedges should be made of pure dry cork 9-10 mm in length, 4 mm in width and 1.2 mm thick. When the instrument reaches its destination, remove the wedges in the same manner. This operation has to be done when starting a new chronometer or one received after repairs or being sent for repairs. When assembling or disassembling a chronometer, never allow your fingers to touch the parts of the mechanism.

(4) **Starting the chronometer.** To start a chronometer that has run down, wind it, clamp rigidly, and turn the whole box rather slowly one-quarter turn in azimuth. To start up a new or repaired chronometer, first remove the stop wedges (see Item 3), then wind and start as described above.

(5) **Chronometer trouble.** If the chronometer has suffered mechanical damage, or if the beat is heard to be irregular, or the change in daily rate is large and irregular (see Sec. 53) the chronometer should be sent to the navigation chamber for checking and repairs. Do not, under any circumstances, try to repair it with the facilities on board ship.

If the faults are detected en route, stop the chronometer and go over to the second one, or use a deck watch or any good timepiece available. However, in this latter case clock corrections should be made several times a day, if possible, close to the time of observations. Observation accuracy may in this case be somewhat below the usual level.

SEC. 56. WORKING WITH CHRONOMETER AND WATCH

(1) **Comparison.** Since the chronometer on board ship is never removed from its box and all observations are done with a clock or a stop watch, one must learn to *compare* a watch with the chronometer.

A comparison (com) is the difference between the simultaneous readings of a chronometer (T_{ch}) *and a watch* (T_{wat}):

$$\text{com} = T_{ch} - T_{wat} \tag{10.12}$$

The comparison is positive (+) if the watch is slow on the chronometer, and negative (—) if fast.

Having obtained com, we can, from the watch instants obtained in observations, compute the appropriate instants of the chronometer from the formula

$$T'_{ch} = T'_{wat} + \text{com} \tag{10.13}$$

In addition, knowing the comparison, one can use the known chronometer correction u_{ch} to compute u_{wat} and conversely. Indeed, combining two equations for one and the same instant, we get

$$\begin{array}{c} T_{gr} - T_{ch} = u_{ch} \\ + \\ T_{ch} - T_{wat} = \text{com} \\ \hline T_{gr} - T_{wat} = u_{ch} + \text{com} = u_{wat} \end{array}$$

that is,

$$\left.\begin{aligned} u_{wat} &= u_{ch} + \text{com} \\ u_{ch} &= u_{wat} - \text{com} \end{aligned}\right\} \tag{10.14}$$

Example 6. $T_{wat}=8h\,14m\,45s$ was obtained at time $T_{ch}=10h55m20s$. The correction $u_{ch}=+8m\,45s$. Find u_{wat}.

Solution.

T_{ch}	10h 55m 20s	$+\ u_{ch}$	+0h 08m 45s
$-T_{wat}$	8 14 45	com	+2 40 35
com	+2h 40m 35s	u_{wat}	+2h 49m 20s

The simplest way to compare a watch and chronometer is as follows. The observer takes the chronometer instant (T_{ch}) 30 to 40 seconds in advance (some multiple of 5 or 10 seconds) and writes it down. Ten seconds before the desired instant, the observer listens to the beats of the chronometer counting half seconds: "mark", "mark", "mark", ..., then five seconds before the selected time begin the main count out loud: mark (on the fifth second), half, one, half, two, half, three, etc., at the same time looking at the watch; and at the count of "five" note and immediately record the reading of the second hand, then the minute hand, and the hour hand; in doing so, note the correctness of the minutes when the hand passes through 0s, that is, in positions close to 55s-60-5s. This reading of the watch will be simultaneous with the recorded T_{ch}.

This method of counting is convenient in that whole seconds are counted (the count "half" falls on half-second beats). What is more, a certain skill is developed in counting seconds mentally without a chronometer, and this is needed in astronomical observations.

It is not wise to confine oneself to one comparison. It is best to make two or three, selecting chronometer times in different parts of the second-hand dial, and then take the arithmetic mean of the comparisons. If the ship carries two chronometers, a comparison of the second one with the first is done in a similar fashion. Stop watches are compared with chronometers in the same way, except that the stop watch is stopped for the comparison.

A watch may be compared with a chronometer by using a stop watch, starting at a certain recorded instant of the chronometer and stopping it at a specified instant on the other timepiece. Then by subtracting the stop-watch reading from the watch reading, we get the simultaneous readings T_{ch} and T_{wat}.

When performing astronomical observations with a watch or clock or stop watch, the rate of which is unknown, make two comparisons: one before and one after the observation, particularly if the time interval between comparison and observation is great. One comparison will suffice if the timepiece has been carefully adjusted.

(2) **Working with watch and chronometer during observations.** When measuring the altitude of a celestial body, the observer should also note and record the chronometer time or watch time.

Later on we shall see that at sea it is often required to measure from three to five altitudes of each celestial body, so that in the observation of say, four stars, one has to record 12 instants of time.

If the observations are made with the help of an assistant, this operation is very simple and is performed by the assistant at the command: "stand-by-mark", the order "mark" is given at the instant the image of the body touches the horizon, and "stand-by" is given a few seconds in advance.

If one person is making the observation, it is necessary to learn to count seconds so as to record several instants of time. The purpose is to allow the observer to shift sight from sextant to watch. The counting is done as follows: when the body touches the horizon, the observer says "mark" and begins to count "half, one, half, two, half, three", etc., as in the comparison procedure. Shifting his eyes to the timepiece, the observer notes the closest instant (the best thing is a multiple of 5s) in advance and counts up to that point. Subtracting from the indicated instant the number of seconds counted and recording the result, we get T_{wat} at the time the altitude of the body is measured. The accuracy of the results depends on the training of the observer and the length of the interval. For short intervals of the order of 5 to 10 seconds and with a small amount of training, accuracies are obtainable up to $\pm$1s and even $\pm$0s.5.

Instants of single-hand stop-watch time are recorded in the same manner. The first instant is obtained when the stop watch is started (0m 0s), and subsequent instants are obained by the "counting method".

After comparing the stop watch with the chronometer or stopping it at T_{ch}, instants of chronometer time are obtained by adding instants of stop-watch time to the first chronometer count (Example 7).

Example 7. Observations of stop watch yield the instants $T_{stop\,w}$. Find the corresponding T_{ch}.

Solution.

(a)

T_{ch}	9h28m40s	set time
	—	
$T_{stop\,w}$	7m13s	stop-watch reading at T_{ch}
T'_{ch}	9h21m27s	

(b)

No.	Sextant readings	$T_{stop\,w}$	T_{ch}
1	. . . .	00s	9h21m27s $= T'_{ch}$
2	. . . .	+53s	9h22m20s
3	. . . .	+1m38s	9h23m 5s

(3) **Reducing chronometer corrections to observation times.** In all the foregoing cases, to obtain T_{gr} it is necessary to know the *chronometer correction* (u_{ch}) *at the time of observation*. Computation of the correction u_{ch} is performed by changing the last u'_{ch} correction obtained from radio time signals by the magnitude of the chronometer rate during the interval ΔT^d that has elapsed since its determination, i. e., by extrapolation of the quantity u'_{ch} by means of the formula

$$u_{ch} = u'_{ch} + \omega \Delta T^d \tag{10.15}$$

where ω is the daily rate of the chronometer at average temperature in the time interval ΔT; the quantity ΔT^d is expressed in days and fractions of days to within 0d.01.

If the temperature near the chronometer has not changed, for ω we can take the last daily rate determined from u_{ch}.

Do not disregard computation of u_{ch}; a new value should be computed for every round of observations.

Example 8. A determination of $u'_{ch} = -6\text{m}\ 57\text{s}.5$ was made on 13.09 at $T_{gr} = 14\text{h}$; $\omega = -2\text{s}.4$. Find u_{ch} at $T_{gr} \approx 6\text{h}$ on 14.09.

(a) $\Delta T^d = 6\text{h} + 24\text{h} - 14\text{h} = 16\text{h} = 0\text{d}.67$,

(b) $\omega \Delta T^d = -1\text{s}.6$.

(c) $u_{ch} = -6\text{m}\ 57\text{s}.5 + (-1\text{s}.6) = -6\text{m}\ 59\text{s}.1 \approx -6\text{m}\ 59\text{s}$.

SEC. 57. CARE OF SHIP TIMEPIECES

Ship timekeeping includes constant care of all ship timepieces no matter where they are located. This is ordinarily in the charge of one of the navigators (usually the third). All ship timepieces must keep ship time to within $\pm 1\text{m}$, and $\pm 15\text{s}$ in certain rooms, such as the navigator's room, the engine room. As already mentioned, the timepiece in the radio room keeps Greenwich or Moscow time with an accuracy of 5 to 10s.

All ship timepieces are wound and checked daily between 7 and 8 a.m. by the navigation officer or a trained sailor under the guidance and responsibility of the navigation officer. The clock in the navigator's room is additionally compared with that in the engine room prior to maneuvres (docking, casting anchor, and so forth) and passage of narrows. Timepieces are checked (put forward or back) when crossing time zone boundaries by means of a "reference timepiece", which may be any good watch or clock ajusted to $\pm 5\text{s}$ by the chronometer.

When checking ship timepieces, it is necessary to derive their daily rates, on the basis of which they are regulated with a regulator.

CHAPTER 11

THE THEORY AND CONSTRUCTION OF THE MARINE SEXTANT

SEC. 58. PECULIARITIES OF MARINE ANGLE-MEASURING INSTRUMENTS

In the solution of many problems in navigation and astronomy, it is required to measure the angles between various objects. These measurements are made from a moving base (deck) and ordinary "land-based" instruments are unsuitable.

Marine angle-measuring instruments have to be made for operation by hand without a stable mounting when the ship is rolling or pitching, and the measurements have to be made rapidly and in a simple manner. These conditions are best complied with by means of a "reflecting" optical system consisting of two mirrors or prisms. By turning one of the mirrors, the observer can see at once both objects between which the angle is being measured, the position of the mirrors corresponds to the magnitude of the angle. This system was incorporated in an instrument, called the **sextant,** in the 18th century.

Compared with land-based instruments, the sextant has the advantage of being able to measure angles quicker and without a stable mounting (by hand), but it has certain drawbacks: limited angle of measurement (120° to 140°), reduced accuracy and reliability of measurement.

At the present time, the altitudes of celestial bodies above the visible horizon and angles between objects on board ship are measured by the *marine sextant*; besides, at sea use is also made of the *artificial-horizon sextant with integrator*, designed exclusively for measuring the altitudes of celestial bodies in the absence of a visible horizon. Finally, in the early 1950's the *radio sextant* appeared. This instrument permits measuring the altitudes of bodies by their radio emission. **Dipmeters** are used to measure the dip of the visible horizon at sea. These instruments are based on the same reflection principle, but are only made for measuring angles close to 180°.

SEC. 59. THEORETICAL PRINCIPLES OF THE MARINE SEXTANT

The fundamental optical system of the sextant is shown in Fig. 82. Let *RO* be the direction of a ray of light from a right-hand object, *LO*, from a left-hand object. It is required to measure the angle σ between the directions towards these objects. Place a mirror *A* in the path of ray *LO*, and mirror *B* in the path of *RO* so that their planes are perpendicular to the plane of the angle σ, and the reflecting surfaces are directed into the instrument. Turning mirror *B*

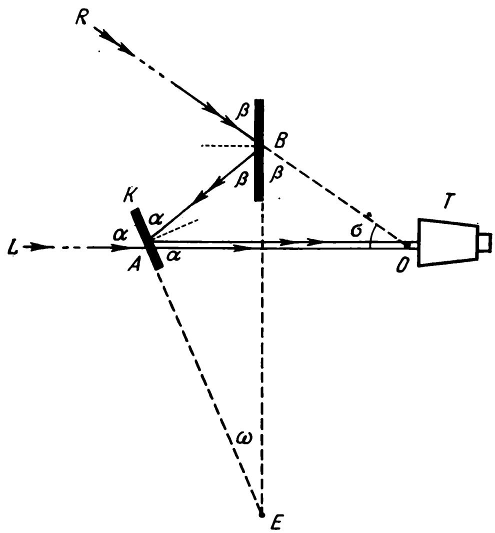


Fig. 82

on its axis perpendicular to the drawing, we can achieve a position such that a light-ray from the right-hand object will be reflected from the mirror surfaces and will move in the direction *AO*. The left-hand object *LO* will be seen above mirror *A*. Superposing these images in the field of view of telescope *T*, we obtain a definite and unique position (for the given angle) of mirror *B* relative to *A*. Obviously, a strict relationship may be obtained between the angle being measured σ and the angle ω between the planes of the mirrors.

On the basis of one of the laws of light reflection (the angle of incidence equals the angle of reflection), we get the angles β and α between the planes of the mirrors and the light-rays *RO* and *LO*. Applying the geometric theorem that "an exterior angle of a triangle is equal to the sum of the two interior opposite angles" to triangle *OAB* containing angle σ, and triangle *EAB* containing angle ω,

we get

$$\text{angle } BAL = 2\alpha = 2\beta + \sigma \tag{*}$$

and

$$\text{angle } BAK = \alpha = \beta + \omega \tag{**}$$

whence

$$\sigma = 2(\alpha - \beta) \tag{11.1}$$

and

$$\omega = \alpha - \beta \tag{11.2}$$

After substitution of (11.2) in (11.1), we have

$$\sigma = 2\omega \tag{11.3}$$

thus, *the angle being measured is equal to twice the angle between the planes of the mirrors of the sextant for their position when both images of the objects* (*direct L* and *doubly reflected R*) *are aligned in the field of view of the telescope.*

Expression (11.3) may be written in the form

$$\omega = \frac{\sigma}{2} \tag{11.4}$$

or the angle between mirrors is equal to half the angle being measured.

By this reasoning, we can replace the *measurement of angles between objects* by *measurement of the angle between the planes of*

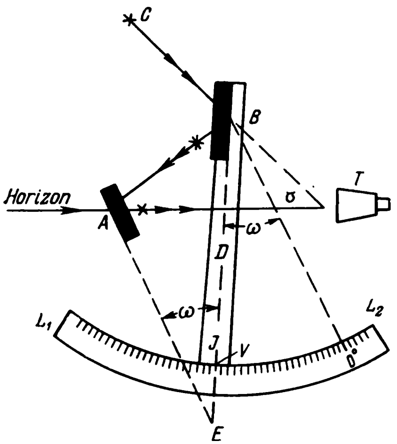


Fig. 83

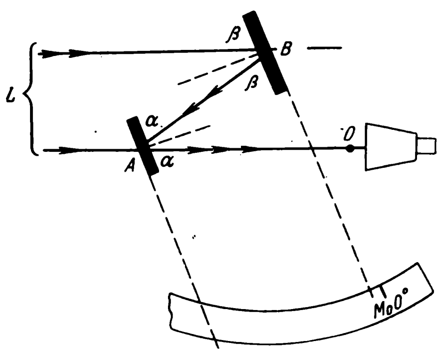


Fig. 84

the mirrors. To do this, place mirror B (called the *index mirror* or *glass*) on a metal arm D (Fig. 83) called the *index arm.* The index arm is capable of turning on its axis, which passes through the centre of mirror B. The other end of the index arm with index J moves

along the arc of the *limb* L_1L_2, which is graduated in *half-degree* divisions [on the basis of formula (11.4)]; the numbers, however, indicate values of whole degrees so that the reading need not be doubled. On the right-hand side of the arc is a zero mark. To the left of it are divisions marked up to 140°-150° (at one-degree intervals); to the right are only five degrees up to 355°. The second mirror A, called the *horizon glass*, is fixed rigidly to the frame of the sextant.

Obviously, angles will be read by the angle of rotation of the large index mirror relative to the small horizon glass, more precisely, relative to the line $BO°$, which is parallel to AE. Indeed, the arc $0°v$ of the limb will be measured by the central angle ω, which is equal to the angle between the mirrors. Hence, the reading v of the limb will be equal to the angle σ being measured.

Let us consider some special cases of positions of the index mirror.

(1) **One object L is observed at infinity.** When an object L recedes to infinity (Fig. 84), the light-rays coming from it will be parallel ($\sigma = 0°$) and the alternate angles ABL and BAO will be equal, that is, $180° - 2\beta = 180° - 2\alpha$, whence $\alpha = \beta$. Hence, mirrors B and A will be parallel, and the angle $\omega = \alpha - \beta = 0°$. For this position of mirror B, there should be a reading of 0° on the limb opposite the index of the index arm. But for technical reasons, it is hardly ever equal to 0°.

$M_0^{\bullet}$ *on the limb of the sextant corresponding to the parallel position of the mirrors is called the zero mark on the limb.*

When a sextant is manufactured and adjusted, the zero mark (0°) coincides with the zero on the limb, but in the process of operation, the mirror A moves and the point of zero differs from the zero mark; in other words, the zero mark on the limb may turn out different in each observation. It is obvious, then, that a certain correction has to be introduced into the readings.

(2) **Observing an object R situated at a limiting angle.** The greatest possible angle for observation σ_{max} (Fig. 85) will be for a position of the index mirror such that the ray from the right-hand object will no longer be reflected, but will slide over the surface of the mirror and yet will still be incident on the horizon glass A. Here, angle $\beta = 0°$ and from formulas (11.2) and (11.1) we get

$$\omega_{max} = \alpha \text{ and } \sigma_{max} = 2\alpha \tag{11.5}$$

From expression (11.5) it is clear that the limiting angle under measurement depends on the position of the horizon glass. In modern sextants, the angle α is equal to 70°-75° and, consequently, the limiting angle σ_{max} will be 140°-150°. In recent Soviet sextants, $\sigma_{max} = 140°$, in older ones and in many foreign makes, $\sigma_{max} = 150°$ or 145°. So that the index arm should not prevent the mirror B from being put in the extreme left-hand position, the index mirror

is placed at an angle to the longitudinal axis of the index arm (Fig. 85). For this reason, the zero mark 0° is shifted by the same angle to the right of the theoretical position (O_1).

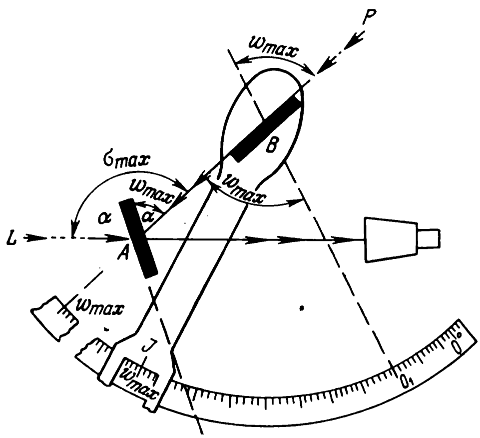


Fig. 85

(3) **Observing an object L situated close to the sextant.** If an object L is situated close to the sextant, rays from it proceeding to mirrors A and B will not be parallel, but will form a slight angle y (Fig. 86) due to the fact that the angles in a sextant are not measured from a single point—the centre of the azimuth circle; actually, the angle y is the parallax of the object; the closer object L is, the more will be the effect of the AB distance between the mirrors. Considering the triangles LBA and EBA by the same reasoning as at the beginning of this section, we get

$$2\beta = y + 2\alpha \text{ or } y = 2(\beta - \alpha)$$

and

$$\beta = \omega' + \alpha \text{ or } \omega' = \beta - \alpha \qquad (11.6)$$

whence

$$y = 2\omega' \qquad (11.7)$$

Expression (11.7) shows that despite the changed position of the angles, the basic law (11.3) continues to hold. Howerer, from a comparison of (11.6) and (11.2) it is obvious that the angle ω' will be negative, that is, it will be reckoned from 0° in the other direction. This is also evident from Fig. 86, where the angle ω' is equal to the arc $0°v_1$. It is for the measurement of such excessive angles that

the limb has divisions to the right of 0° (360°): 359°, 358°, to 355°. Arcs greater than five-degree divisions will not be needed. Indeed,

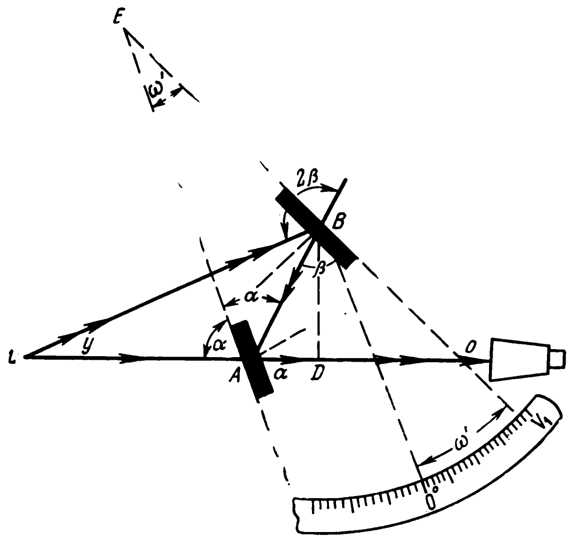


Fig. 86

from the triangle BAD, where BD is perpendicular to the line LO, we have

$$BD = AB \times \sin(180° - 2\alpha) \qquad (11.8)$$

In modern sextants, $AB = 10$ cm, $\alpha = 75°$, therefore $BD = 5$ cm. From the triangle BDL we have

$$\tan y = \frac{BD}{LD} = \frac{5\,\text{cm}}{LD} \qquad (11.9)$$

For an angle $y = 5°$ the distance LD to object L will be only 57 cm. It is quite obvious that such close distances are not encountered in actual practice.

REDUCING ANGLES TO CENTRE OF INDEX MIRROR

Due to the fact that the angles in a sextant are not measured from a single central point B, their vertices will move along the line of ray LO_1 (Fig. 87) as the angle changes, and it becomes impossible to compare them. To reduce all angles to the point B (the centre of the index mirror), add to each the **angle of reduction** y

which expresses the effect of the distance AB between mirrors on the angles being measured.

From the triangle BO_1L we get the magnitude of the reduced (exterior) angle σ_1', or

$$\sigma_1' = \sigma_1 + y \tag{11.10}$$

As is evident from (11.9) and Fig. 86, the magnitude of the angle y depends on the distance $LD \approx LA$ between the direct object and the sextant. Obviously, the angle y diminishes as the distance

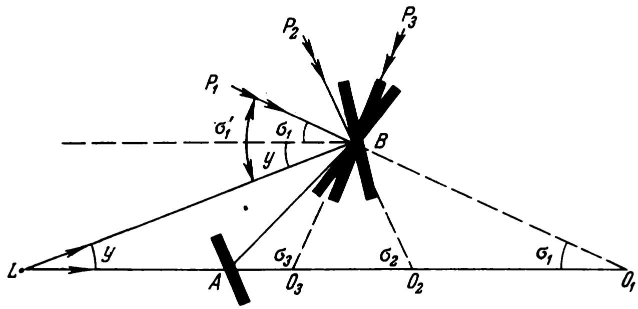


Fig. 87

increases, and if the object L is at infinity, the angle y is zero. Let us agree to disregard the angle y if it is less than $0'.1$. From (11.9) for a small angle we have

$$\left.\begin{aligned} &\tan y \approx y' \text{ arc } 1' = \frac{5 \text{ cm}}{LD} \\ &\text{whence} \\ &LD = \frac{5 \text{ cm}}{0'.1 \text{ arc } 1'} \approx 1.72 \text{ km} \approx 1 \text{ nautical mile} \end{aligned}\right\} \tag{11.11}$$

Thus, if a left-hand object is at a distance of 1 nautical mile or more, the angle $y < 0'.1$ and it may be neglected. Under ordinary conditions at sea, the visible horizon is more than 1 mile distant, so only in rare cases when the object L is close to the observer does one have to reduce angles to the centre of the index mirror (that is, add angle y).

SEC. 60. ZERO-POINT CORRECTION. INDEX CORRECTION

In the preceding section it was noted that the actual **position of zero on the limb** does not coincide with the zero mark 0°, as a result of which each reading of an angle has to be corrected by the magnitude of the arc $0°M_0 = \Delta$ (Fig. 88), or the so-called **zero-point correction.**

A correction, as we will recall, is the difference between the true value of some quantity and its measured value. On this basis, the *zero-point correction is equal to the algebraic difference between* 0° (*or* 360°) *and the reading of the zero mark on the limb*, that is,

$$\Delta = 0° - M_0 \tag{11.12}$$

Here, M_0 is the reading of the zero mark, or the reading obtained for a parallel position of the mirrors. This mirror position may be

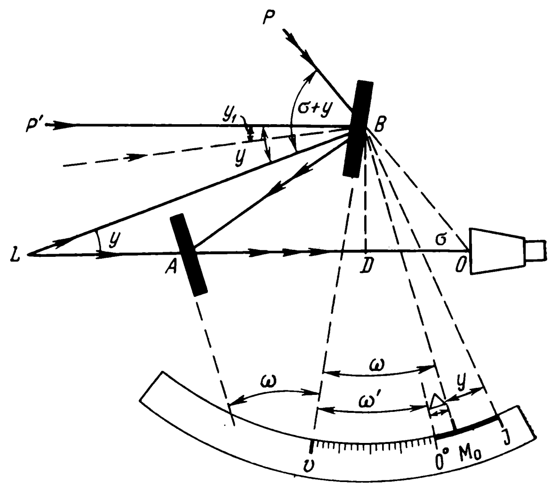


Fig. 88

obtained for coincidence, in the field of view of the telescope, of a doubly reflected (P') and direct (L) image of the same object at infinity.

From Fig. 88 it is seen that the angle being measured σ will be equal to the arc $\omega = \omega' + \Delta$ and not the reading $v = \text{arc } \omega'$, and so

$$\sigma = 2(\omega' + \Delta) \tag{11.13}$$

or in half-degree graduations of the limb:

$$\sigma = \omega' + \Delta$$

However, to reduce angles to the centre of the index mirror, transfer the vertex O of the angle to the centre of the mirror B; to do this, add angle y to angle σ in (11.10). As a result, the reading v, which corresponds to the measured angle, must be corrected by

two corrections: the zero point on the limb (Δ) and the magnitude of angle y, or

$$\sigma = \text{arc } vJ = \omega' + (\Delta + y) \tag{11.14}$$

The total correction $\Delta + y$ by which the reading on the limb is corrected is known as the **index correction of the sextant** (i).

The index correction may be obtained from direct observations. Indeed, if in the field of view of the telescope we bring to coincidence both direct images of an object L not at infinity (Fig. 88), the index mirror will occupy position BJ. The reading of the measured angle ω' must be corrected by a correction equal to $\Delta + y$, but arc $0°J = \Delta + y$, hence J is the reading that corresponds to the index correction. Therefore *the reading of a sextant with coincident direct and doubly reflected images of a single close-lying object L is called the reading of the index correction* (*oi*), *and the difference between the reading* 0° (*or* 360°) *and the reading oi is called the index correction* i, that is,

$$i = 360° - oi \tag{11.15}$$

The index correction will have the sign (+) if $oi < 360°$, and (−) if $oi > 360°$.

As we see, the entire difference between the zero-point correction (Δ) and the index correction (i) has to do with the object that is being brought into coincidence in the field of view of the telescope: one located at a large distance (Δ) or nearby (i).

When measuring altitudes at sea, the direct object is the visible horizon, the distance to which is always greater than one mile. In this case, as we have already established, the angle $y < 0'.1$ and the index correction i is practically equal to the zero-point correction. Therefore, instead of bringing into coincidence images of the *horizon* (object L), which is ordinarily not clearly visible, the images of a celestial body (which for practical purposes is at an infinite distance) are brought into coincidence, and we get the *zero-point correction*, which is generally also called the index correction.

Exceptions are cases when the horizon is closer than one mile. They occur in the following instances:

(1) when observing the sun or some other body above the shoreline;

(2) when observing a body above the water-line of another ship whose distance away is known.

In such cases, that is, when the horizon or object is closer than one mile, the index correction should be determined from the *direct object* (L) and not from a celestial body.

SEC. 61. ELEMENTS OF THE MARINE SEXTANT

1. ESSENTIALS OF CONSTRUCTION

At the present time, Soviet ships are equipped mainly with Soviet-made micrometer sextants (CH) and subsequently improved

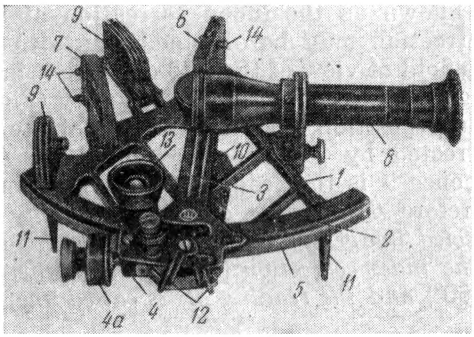


Fig. 89

1—frame, *2*—limb, *3*—index arm, *4*—reading device, *4a*—micrometer drum, *5*—gear rack, *6*—index mirror (mobile), *7*—horizon glass (stationary), *8*—telescope, *9*—coloured shades, *10*—arm of sextant, *11*—supports, *12*—levers of connecting device, *13*—magnifying glass with lamp for night readings, *14*—adjusting screws of index mirror and horizon glass.

models (CHO, CH-2M, CHO-M, etc.). The general appearance of the navigation sextant CHO-M is shown in Fig. 89.

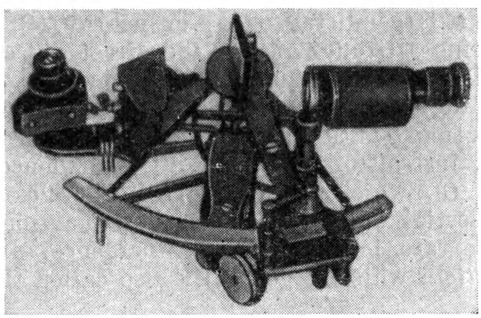


Fig. 90

Basic performance data of CH sextants are:
1. Reading accuracy $\pm 0'.1$.
2. Limiting measurable angle 140°.
3. Field of view and telescopic magnification:
(a) day—4°.5; magnification 6 to 8 $\times$;
(b) star—6°-8°; magnification 3.5 $\times$;
(c) general—8°; magnification 7 $\times$.

4. Reliable performance guaranteed: one year for first models, five and more years for latest models.
5. Weight of sextant: first models 1,082 g; latest models 1,350 g.

In addition to the CH sextants, Soviet ships also have sextants of foreign make: Hughes and Kelvin (Fig. 90), Platt and others. The design of these instruments is similar to the CH sextants and they differ only in the design of various components, size, weight and optics. Some ships have vernier sextants of earlier make than the CH models, or foreign-made sextants. With the exception of the reading system, their designs differ but slightly from the CH sextants. We shall therefore discuss only the vernier.

2. SEXTANT READING DEVICES

A. Micrometer Drum (or Micrometer Screw)

For small movements of the index arm and accurate readings, Soviet sextants make use of a worm-gear engagement consisting of the **reading device** and a gear rack cut in the frame of the sextant (Fig. 89). The reading device consists of an endless *tangent* screw (tangent to the limb) of conical shape on the same axis as the micrometer drum, and a releasing clamp in the form of two levers and a spring that presses the screw to the gear rack. The conical shape of the screw is for the purpose of keeping the drum as far as possible from the limb to increase the diameter of the drum.

The gear system is such as to ensure that for one complete rotation of the drum the index arm moves exactly one division on the limb of the sextant (that is, one-half degree). The drum is thus divided into 60 divisions, each representing one minute of arc. If the index falls between minutes, their fractions are obtained by eye to within $\pm 0'.1$. For greater precision, some sextants have a vernier by the drum. The index arm is moved to another position on the limb after slackening the screw and disengaging it from the gear rack. This is done by pressing the levers of the releasing clamps.

This device permits measurements and angular readings to be made more simply and rapidly than by means of a vernier, yet it has a number of drawbacks that are peculiar to gear systems: "backlash" of the screw, errors of the worm gear, and dependence of reading on the mechanical state of the gear rack and the screw. Dust, dirt, frozen droplets of water or hardened oil, and also wear of the gear rack or damage may give rise to reading errors. The backlash of the drum does not generally exceed $\pm 0'.2$, but in old sextants it may turn out to be greater, and it is eliminated by special techniques.

In the latest Soviet models (CH-2M and CH-M), the gear rack is made of bronze or a special alloy, the screw is steel, thus reducing wear of these parts and increasing the reliable lifetime. From the foregoing it is evident that micrometer sextants require greater care and special observational techniques.

B. The Theory and Design of the Vernier

The **vernier** of a sextant is an arc-shaped plate attached to the index arm concentric with the limb. It is designed for reading angles down to tenths of a minute (Fig. 91). On the right end is

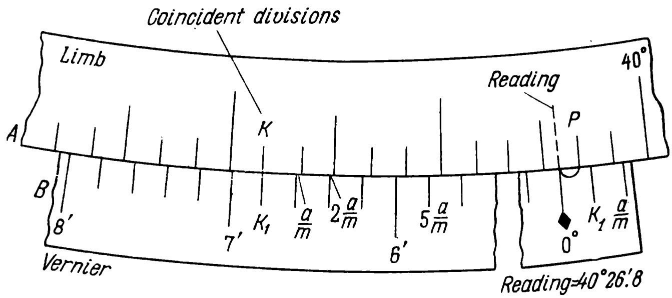


Fig. 91

a mark 0°, which is the *index* for reading degrees and tens of minutes on the arc of the limb (vernier sextants have the limb graduated into 10′ divisions). Left of this mark (towards increasing readings) the vernier scale has equal graduations that are slightly smaller than the ten-minute divisions of the limb. The vernier principle is based on the property of marks of two different uniform scales to coincide at periodic intervals.

For example, in Fig. 91 the coincident marks are K_1 and K, the arc $OK_1^\circ = \frac{a_l}{m} \times K_1$, where a_l is the value of a division of the limb, m is the number of divisions ($m = 60$ if fractions of seconds are obtained and $m = 50$ if fractions of minutes), and K_1 is the division number. The sextant reading will be $OC = 40°20' + 6'.8 = 40°26'.8$. The vernier provides readings to an accuracy of $0'.1$.

Verniers are used in modern sextants (British makes, for example), but more often as a supplementary device to the micrometer drum. Vernier sextants have advantages that stem from the absence of a gear system: high accuracy and reliability and long lifetime. But due to the complex reading that the vernier involves, most navigators prefer the micrometer sextant.

The vernier principle is widely used in measuring techniques.

SEC. 62. BASIC INSTRUMENTAL ERRORS OF THE SEXTANT AND THEIR REDUCTION. FINDING THE INDEX CORRECTION

All instrumental errors of the sextant may be broken down into the following basic groups:

(1) Errors of manufacture (investigated in the laboratory). The effect of these errors on readings is given in the certificate of the sextant in the form of a table of instrument corrections (s).

(2) Mechanical damage to parts, wear and other factors affecting the readings. These defects require that the sextant be returned for repairs. They do not concern us here.

(3) Adjustable errors of mirrors and telescope that result in reading errors. These errors may be determined and corrected by the observer under ship conditions.

1. ERRORS OF MANUFACTURE

Prismatic error of index mirror. In the theory of the sextant, it is assumed that the surfaces of the mirrors are strictly parallel surfaces, and rays reflected by them are not distorted. However, in reality the mirrors always have a certain lack of parallelism of the faces, the so-called **prismatic error of the mirror.** It is characterized by a "wedge angle" γ, which should not exceed $1''$ in sextant mirrors.

Prismatic error in the horizon glass gives rise to a constant error in readings, which enters into the index correction and, therefore, will be taken into account by the observer. Prismatic error of the index mirror, however, gives rise to reading errors that increase with increasing angle being measured. If the wedge angle $\gamma = 1''$, an error of $0''.1$ will result when measuring an angle of 20°, $10''$ for an angle of 130°, which means that the error depends on the size of the angle being measured. The prismatic error is ordinarily determined jointly with the other sextant errors.

Errors of threading in the gear rack. Inaccuracies in gear-rack threading give rise to reading errors. These are not ordinarily large, but are variable in the different parts of the limb. For this reason, when determining the total error due to a variety of causes, one should choose small angle intervals (10° or, better still, from 3° to 5°). These errors are similar to limb *division errors* in the vernier sextant. The gear rack and its screw gradually wear down and the error changes necessitating periodic examination of the sextant.

Eccentricity of the index arm. By *eccentricity* is meant noncoincidence of the centre of the limb with the rotation axis of the index arm, due to inaccurate manufacturing of the sextant. Let C_1 (Fig. 92)

be the centre of the arc of the limb whose radius is R (the angle at C_1 is equal to a limb reading, that is, $\frac{C_1}{2}$ in limb graduations); C is the centre of rotation of the index arm (the angle at C is equal to the angle between the mirrors in limb graduations, or one half the desired angle); e is the CC_1 distance in millimetres or the linear magnitude of eccentricity; angle p is the direction of the eccentricity (angle $M_0CE \approx$ angle M_0C_1E). Let us find the magnitude of $\Delta C =$ angle C — angle C_1, wich is the correction to the sextant reading for eccentricity. For angle D we have (Fig. 92)

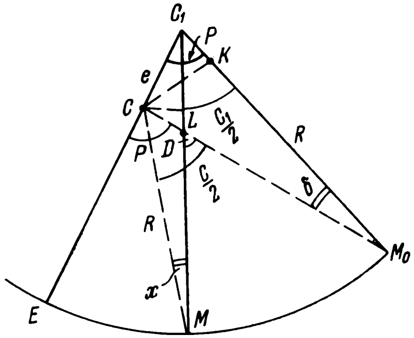


Fig. 92

$$\frac{C}{2} + x = \frac{C_1}{2} + \delta$$

or

$$\frac{1}{2}(C - C_1) = \delta - x$$

Substitute e and p for δ and x, for which purpose drop perpendiculars CK and CL; from the triangles obtained we have

$$CK = e \cdot \sin p \text{ and } CL = e \cdot \sin\left(p - \frac{C_1}{2}\right)$$

$$CK = R \cdot \sin \delta \text{ and } CL = R \cdot \sin x$$

or

$$R \cdot \sin \delta = e \cdot \sin p$$

and

$$R \cdot \sin x = e \cdot \sin\left(p - \frac{C_1}{2}\right)$$

Due to the smallness of the angles δ and x we get

$$\delta'' = \frac{e}{R \text{ arc } 1''} \times \sin p$$

$$x'' = \frac{e}{R \text{ arc } 1''} \times \sin\left(p - \frac{C_1}{2}\right)$$

Introducing, in place of the linear quantity e_{mm}, the angular ε expressed in seconds,

$$\varepsilon'' = \frac{e_{mm}}{R_{mm} \text{ arc } 1''}$$

and changing the sign of x, we finally get

$$\Delta C'' = 2(\delta - x)'' = 2\varepsilon'' \left[\sin p + \sin\left(\frac{C_1}{2} - p\right)\right] \qquad (11.16)$$

where the quantities ε'' and p° characterize the eccentricity and are listed in the certificate of the sextant.

Generally, eccentricity errors exceed other errors, for example, when $e = 0.1$ mm, $p = 0^\circ$, $R = 165$ mm *, we have: for $C_1 = 60^\circ$, $\Delta C = 2'.1$ and for $C_1 = 130^\circ$, $\Delta C = 3'.8$.

Errors involved in the prismatic quality of the mirrors, the threading of the gear rack, and eccentricity are ordinarily determined jointly and accounted for in the form of the **instrument correction** s, which is given in the certificate of the sextant in the table "reading corrections". The instrument correction is determined in the laboratory on a special testing device (Fig. 93) consisting of a precise azimuth circle a with mobile top table and collimator b with a reference mark that reproduces an object at infinity. The sextant c is mounted on the table and is adjusted and clamped securely. Setting the index arm at 0°, 10°, 20°, etc., and bringing the reference mark of the collimator to the centre of the field of view (by turning the table), we get readings on a precise circle (by means of microscope micrometers). A comparison of the exact angle with that obtained by the sextant yields the correction of the sextant reading. The measurements are repeated a number of times and then averaged.

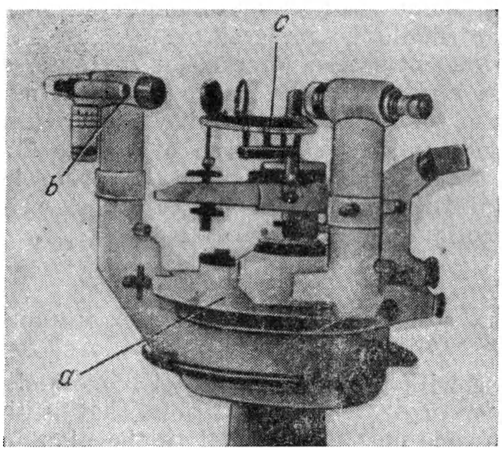


Fig. 93

* In the old ЗМИ sextants $R = 185$ mm.

The quantities s, p° and $2\varepsilon''$ computed from (11.16) are then entered in the certificate of the sextant.

Depending on the magnitude of the correction s, sextants are divided into two classes: if the correction s is less than $40''$ (in other countries, $< 0'.8$), the sextant is "Class A", for s from $40''$ to $2'$, the sextant is "Class B". For $s > 2'$, the sextant is pronounced faulty and is not issued to ships.

The correction s may change with time due to wear of rack, deformations, ageing of the metal, and transportation (these factors affect the eccentricity). For this reason, the instructions call for an annual checkup of sextants.

Corrections in the case of gear-rack sextants are particularly unstable, so that one should not use certificates of more than three years.

Errors of the micrometer drum. In most sextants, both of Soviet and foreign make, the gear system adds an error which depends on the angle of rotation of the drum. In this connection, a correction s_1 per 10 drum divisions is obtained in sextant tests. This correction is added to the basic correction s.

Prismatic error of coloured shades. Under specifications, the prismatic error of coloured shades should not exceed $10''$. Provided that the shades are placed at an angle of 90° to the ray of light, this quantity does not yield a perceptible error in the reading. However, faulty sextants have large prismatic errors occasionally. This error may be detected when determining the index correction by the sun with different coloured shades. If the difference of readings $oi_2 - oi_1$ for different shades is different, the shades are faulty and the sextant should be replaced.

Backlash of tangent screw. Micrometer sextants, especially used ones, sometimes reveal that the index arm remains stationary for small rotations of the drum. This is known as backlash. Tests have detected in some sextants a backlash of $1'.5$ to $2'$. The effect of backlash may be reduced by rotating the drum in *one direction* only (for example, towards increasing readings) when determining the index correction and in observations.

II. ERRORS IN POSITION OF TELESCOPE AND MIRRORS. ADJUSTMENT OF THE SEXTANT

When working with the sextant, its telescope and mirrors should occupy the theoretically correct position, that is the line-of-sight axis of the telescope must be parallel to the plane of the limb, to which the planes of the mirrors should be perpendicular. These positions may be upset in the course of work, resulting in errors in sextant readings. Let us consider the effect of these errors and adjustment of the sextant.

(1) **Line-of-sight axis not parallel to plane of limb.** An analytical study of the error in reading due to inclination of the axis of the telescope is performed on an auxiliary sphere and results in the following relationship:

$$\Delta C'' = 0.0175\,(K')^2 \times \tan\frac{C}{2} \qquad (11.17)$$

where $\Delta C''$ is the error in angle reading (in seconds)
K' is the inclination of the line-of-sight axis of the telescope (in minutes)
C is the angle measured by the sextant.

From (11.17) it is seen that the reading of the measured angle will be greater than the actual angle and the error depends on the magnitude of the angle being measured. Errors in readings for small angles of inclination (up to 10′) do not exceed 0′.1 and may

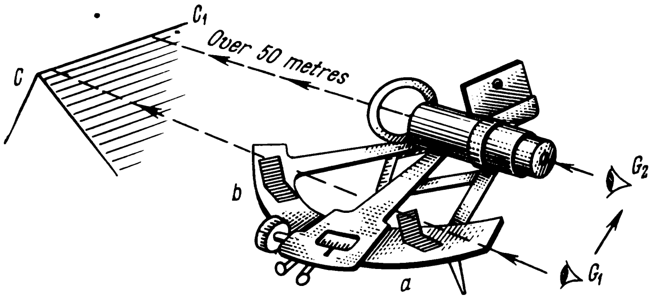


Fig. 94

be neglected; if the angle of inclination of the axis is of the order of 1°, the errors may reach 1′.5 to 3′. It is therefore advisable, before taking observations, to periodically check the parallelism of the telescopic axis and to remedy any inclination.

To check position of telescope, position the sextant (readied for day observations) horizontally on a stable support and place two sighting vanes (a and b in Fig. 94) on the edges of the limb so that the straight line connecting them is parallel to the sight axis of the telescope. Sighting vanes are angle brackets of identical size used for testing a sextant and included in the set of items that come with it. Then choose an object at least 50 metres distant and turn the sextant so that some horizontal line (CC_1) of the object is aligned with the tops of the sighting vanes (eye position G_1 in Fig. 94); then switch the eye to the telescope (position G_2). If the horizontal line of the object is in the centre of the square of cross-wires of the telescope, then the sight axis is parallel to the plane of the limb.

Otherwise, correct the position of telescope by means of the upper and lower screws of the telescope collar. By slackening one screw and tightening the other, the object can be brought to the centre of the cross-wire square. The remaining lack of parallelism will not exceed 5′ to 10′, and will not give rise to errors in practical readings if the images of the objects are brought into coincidence in the centre of the cross-wires.

In the absence of sighting vanes, the line of sight to the object may be put in the plane of the limb itself, in which case the object should be at a greater distance.

Another method for a telescope check, and one that is convenient at sea, is as follows. After adjusting both mirrors, set two wires parallel to the limb, find two stars with an angular distance apart of about 90° and bring their images into coincidence with each other and with the lower wire. Tilt the sextant and bring their images to the upper wire. If the images have not separated, the line-of-sight axis of the telescope is parallel to the plane of the limb, otherwise it is not.

(2) **Index mirror not perpendicular to plane of limb.** When perpendicularity of index mirror is upset, a ray reflected from its surface emerges from a plane parallel to the plane of the limb, which forces the observer (when trying to align objects) to bring the limb out of the plane of the angle being measured and to observe not the actual magnitude of the angle, but its projection on the inclined plane. The resulting reading will be in error.

An analytical investigation of this error yields the following relationship:

$$\Delta C'' = -0.035\,(l')^2 \frac{\sin\left(\alpha + \frac{C}{2}\right) \times \sin\left(\alpha - \frac{C}{2}\right)}{\sin C} \qquad (11.18)$$

where $\Delta C''$ is the error in angle reading (in seconds)

l' is the angle of inclination of the normal to the index mirror to the plane of the limb (in minutes)

C is the angle measured by the sextant

α is the angle between the axis of the telescope and the plane of the horizon glass ($\alpha = 70°$ to $75°$).

From formula (11.18) and Table 7 it will be seen that the magnitude of the error in a reading increases with diminishing angle C.

For an angle of 0° being measured, that is when determining the index correction via a distant object, the error (as may be seen from the table) is equal to infinity; this is evident in observations when we find that it is impossible to bring the direct object and its reflected image into coincidence. When the index arm is moved close to zero, the images will pass one another without becoming coincident or will even fail to overlap at all.

Table 7

C / l	0°	5°	10°	30°	50°	70°	90°	140°
	—	—	—	—	—	—	—	—
5′	∞	9″.3	4″.7	1″.5	0″.9	0″.6	0″.4	< 0″.1
10′	∞	37 .3	18 .6	6 .0	3 .4	2 .2	1 .5	0 .3
30′	∞	5′35″.7	2′47″.5	54 .4	30 .9	20. 2	13 .6	2 .4

The table also shows that only for a 5′ inclination of the mirror or less is it possible to ignore reading errors for angles exceeding 5°.

To check perpendicularity of index mirror to the plane of the limb, do as follows. Remove telescope from sextant, and put it horizontally; set the index arm at about 40° and remove coloured shades from

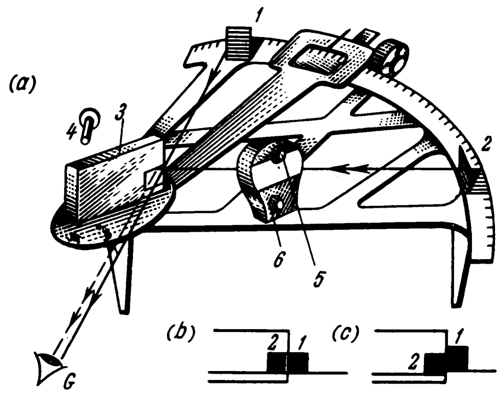


Fig. 95

index mirror. Turn the index mirror of the sextant towards yourself (Fig. 95*a*), place sighting vanes near 0° and 130° and view from a distance of 30 to 40 cm past the inner edge of the index mirror onto vane *1* standing at close to 0°; move vane *2* along the limb until its reflection is seen in the index mirror. The tops of the direct and reflected vanes should be in coincidence without any displacement (Fig. 95*b*). If there is a displacement (Fig. 95*c*), the index mirror will not be perpendicular to the plane of the limb. *To correct its position*, turn adjusting screw *3* on top of the index mirror with key *4* until the tops of the sighting vanes coincide.

By means of sighting vanes, the index mirror can be set to within $\pm 2'$ to $5'$, for which the angular error (at $C > 5°$) will be less than $0'.1$.

In the absence of sighting vanes or in a superficial check of the index mirror, the inner edge of the limb at about 0° is brought into coincidence with its reflection at about 130° in the mirror. The check is made without support, by hand. The accuracy is not great in this method of determining the inclination of the mirror and should be considered as approximate: as a preliminary and superficial check on the index mirror.

(3) **Nonperpendicularity of horizon-glass mirror to the plane of the limb.** Investigations of this error yield the following relationship:

$$\Delta C'' = -0.035\,(m')^2 \cdot \sin^2 \alpha \cdot \cot C \qquad (11.19)$$

where $\Delta C''$ is the error in angle reading (in seconds)
m' is the angle of inclination of the normal to the horizon glass to the plane of the limb (in minutes)
C is the measured angle.

From formula (11.19) it is evident that the inclination of the horizon glass causes a reading error of roughly the same numerical order as for the index mirror, with the exception of large angles for which the error changes sign. At an angle of 0°, this error equals infinity, which means that it is impossible to bring into coincidence direct and reflected images when determining the index correction on the basis of a distant object.

The perpendicularity of the horizon-glass mirror is checked after verifying that the telescope and index mirror are properly set, since we actually set the horizon-glass mirror in a position that is parallel to the index mirror.

Select a distant and clear-cut object; a faint star at night, the sun or a distant object in the daytime. Star verification will be the most exact. We shall make the test with the sun. Turn coloured shades into position in front of both mirrors (different colours is best), set the index arm at about 0°, and direct the telescope at the sun. Two images will be seen in the field: the direct image s and a twice reflected image s_1 (Fig. 96*a*); if by a slight movement of the index arm the image s_1 covers s exactly, merging with it (see ss_1 in Fig. 96*a*) then the mirrors are parallel and, consequently, the horizon-glass mirror is properly aligned. But if the images of the sun do not overlap exactly (s_1 and s in Fig. 96*b*) or pass by one another at a distance apart (s_2 and s, Fig. 96*b*), then the horizon glass is not perpendicular to the plane of the limb; and the greater its inclination, the farther the reflected image is away (it may even leave the field of view).

To correct nonperpendicularity of the horizon glass, arrange both images in the closest position (see s_1-s-s_2, Fig. 96*b*) and, while turning

the upper adjusting screw of the horizon glass by means of the key (see *5* in Fig. 95*a*), change the inclination of the horizon glass until the images coincide. In the process, the reflected image s_1 may be above or below the direct image, and the index correction (equal

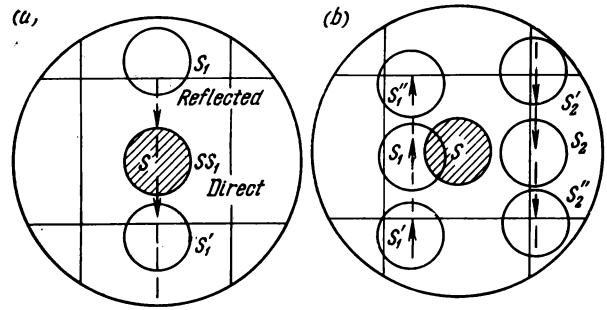


Fig. 96

to the vertical distance between the images) will change. For this reason, determine the index correction once again after proper setting of the horizon glass.

After checking and carefully adjusting the mirrors and telescope, residual errors in readings will not exceed $\pm 0'.2$ and they may be ignored. The telescope setting is checked periodically in port and roadstead, while the mirrors are checked prior to each observation, particularly if they are not frequent.

III. FINDING THE INDEX CORRECTION

Earlier, we found that each reading of the sextant must be corrected by the index correction (i). The magnitude of i varies with the conditions of observation, depending on the temperature, light jarring of the frame, the distance of the object, etc. The rule, therefore, should be *to determine the index correction for every observation.*

(1) **Finding the index correction in the daytime from the sun** The images of the sun observed in the telescope are discs of rather large diameter, and it is impossible to attain a sufficiently precise coincidence of their centres, as is required for a determination of i. Therefore, when determining i by the sun, the opposite limbs of the disc are successively brought into coincidence in the following order.

(a) Set index arm at about 0° ready for day observations with shade glasses of different colour turned into position in front of both mirrors.

(b) Direct the telescope at the sun and, turning the micrometer drum, bring the twice reflected image into contact with the direct

image first with one limb (s_1, Fig. 97), and then, moving s_1 through s, with the other limb (s_2, Fig. 97). A reading is taken for each coincidence: oi_1 for the first, and oi_2 for the second. Obviously, the average reading $\frac{oi_1+oi_2}{2}=oi_{av}$ will correspond to the reading in the case of coincidence of the centres of images s_1 and s, that is, it will be the reading of the index correction oi. Subtracting this reading from 360° (0°), we get the *index correction* i:

$$i=360°-\frac{oi_1+oi_2}{2}=360°-oi_{av} \tag{11.20}$$

The sign of the correction is obtained from the same formula. When bringing the images into coincidence, in both cases rotate the micrometer drum in the same direction. Bear in mind that in the case of micrometer-drum corrections (see above), each oi reading should be corrected by this correction s_1. In this case it is best to determine i by a star or the horizon.

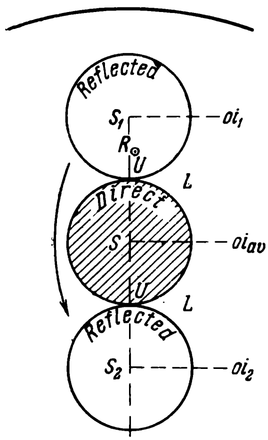


Fig. 97

An observational *check* is possible when determining i by the sun. Indeed, from Fig. 97 it is evident that the *difference of the readings* $oi_2 - oi_1$ is $4R_\odot$, which is four apparent angular radii of the sun.

It will be recalled that $R_\odot$ is equal, on the average, to 16′.0, and the exact value of R is taken from an Almanac. When comparing the reading difference $oi_2 - oi_1$ with the quadrupled angular radius of the sun from the MAE, discrepancies* are allowed that do not exceed ±0′.4, which is a check on the observational correctness. This check is valid for altitudes of the sun that exceed 15° to 17°; for observational checks at smaller altitudes, observations of oi should be made via *horizontal* coincidence of the sun's images (with the sextant in the horizontal position), because changes in refraction will affect the difference of readings for a vertical position of the sextant. Determining i by the sun is a more precise method, yielding accuracies of up to ±0′.1-0′.2.

Examples. 1. On 15 May, 1962, oi_1=359°32′.4; oi_2=360°35′.8. Find i with a check.

Solution.

$$oi_{av}=\frac{359°32'.4+360°35'.8}{2}=360°4'.1$$

$$i=360°-360°4'.1=-4'.1$$

* Due to random errors with irradiation disregarded.

Check: $oi_2 - oi_1 = 63'.4$; $R_\odot$ from MAE $= 15'.9$; $4R = 63'.6$. Observations were good.

2. On 26 November $oi_1 = 359°25'.5$; $oi_2 = 360°30'.5$. Find i with a check.

Solution.

$$i = 360° - \frac{359°25'.5 + 360°30'.5}{2} = 360° - 359°58'.0 = +2'.0$$

Check: $oi_2 - oi_1 = 65'.0$; $R_\odot = 16'.2$; $4R = 64'.8$. Observations were good.

To facilitate calculations of i there is a good practical rule; in each reading find an excess over 30′ or a complement up to 30′, a minus sign being affixed to the excess and a plus sign to the deficit. Half the sum of these quantities for oi_1 and oi_2 yields the magnitude of i with its sign. Thus, for the first example,

$$i = \frac{-2.4 - 5.8}{2} = \frac{-8'.2}{2} = -4'.1$$

for the second:

$$i = \frac{+4.5 - 0.5}{2} = +2'.0$$

(2) **Determining the index correction by a star.** In night observations, the index correction is determined by a star. For this purpose, a rather faint star is selected not too high above the horizon (up to 15°). Bright stars yield hazy images in the telescope of the sextant. The sextant is readied for night observations, the index arm is set at about 0° and the telescope is directed at the star. There will be two images of the star in the field of view: a direct image and a doubly reflected image. Bringing them into coincidence in the centre of the field, we get the reading of the index correction oi, after which

$$i = 360° - oi \tag{11.21}$$

It is particularly convenient to bring the images into coincidence in the case of very small residual nonperpendicularity of the horizon glass, as a result of which the star images pass nearby practically merging. A stellar determination of i is somewhat less accurate than a solar determination, and what is more there is no check.

Example 3. $oi = 359°57'.2$; $i = 360° - 359°57'.2 = +2'.8$.

(3) **Determining the index correction by the horizon or some object.** If the distance to the object is less than one mile, the index correction, as we know, is determined by the *direct object.* When determining i by the horizon, bring the doubly reflected image and the direct image into coincidence in the field of view of the telescope and take a reading of oi. When determining i by an object, choose a prominent feature; if it is vertical, hold the sextant in a horizontal

position, and vice versa. The reading of the sextant, when the direct and doubly reflected images of the object are brought into coincidence, yield *oi*, after which *i* is obtained from the formula (11.21). In this case, the index correction is less accurately determined than by star.

(4) **Reducing the index correction.** In astronomical observations, the magnitude of the index correction, whether +20′ or −1′, does not play any role theoretically; it needs only be known and taken into account when correcting the reading. However, in navigational determinations it is most convenient to have a small index correction, less than 0°.1. By this reasoning, if the correction exceeds 6′, it should be reduced.

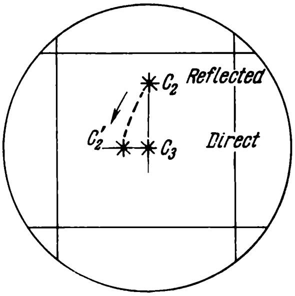


Fig. 98

The index correction is reduced by turning the horizon glass to a position parallel to the index mirror when bringing the index of the index arm into coincidence with the zero mark of the limb. To do this, after checking perpendicularity of both mirrors, set the index arm at exactly 0° and the drum at 0′.0, and direct the telescope at a star, the sun or some distant object. There will be two images in the field of view: a direct image C_3 and a doubly reflected image C_2' that are not in vertical coincidence (Fig. 98). Using the key to turn the lateral screw (see *6* in Fig. 95*a*) of the horizon glass, bring these images into coincidence or, more precisely, to a single horizontal line ($C_2' - C_3$, Fig. 98). As is evident, there may be a lack of perpendicularity in the horizon-glass mirror. For this reason, after reducing the index correction, *be sure to eliminate the nonperpendicularity* of the horizon glass (using screw *5*, Fig. 95*a*). Then find the new value of the index correction. Do not strive to reduce *i* to zero because extra turning may loosen the screws of the horizon glass.

C H A P T E R 12

MEASURING ALTITUDES OF CELESTIAL BODIES WITH A MARINE SEXTANT

SEC. 63. METHODS OF MEASURING ALTITUDE

The measured or observed altitude h' is the vertical angle between directions to the centre or limb of a body and the visible horizon, instrument corrections being taken into account. To measure altitude with a sextant, first of all bring the images of the horizon and the body into the field of view of the telescope (preliminary operation), then place the sextant vertically and bring the images into exact coincidence (basic operation). That, essentially, is what measuring altitudes at sea consist in.

The first operation is called *bringing the body down to the horizon.* It is done in a variety of ways, depending on conditions, that is, the visibility of the body and the horizon. We shall consider these methods below for the sun and for stars. After bringing the body to the horizon, the sextant reading should be approximately (to within 2°-3°) equal to the altitude and both images should be visible in the field of view of the telescope.

The chief difficulty in performing the basic operation (*bringing the images into coincidence*) is to achieve a correct vertical position of the sextant. Slight deviations from the vertical will increase the altitude. Fig. 99 shows the proper position of the image of body S_1 when brought into coincidence with the visible horizon. Angle h' is the altitude of the limb of the body above the visible horizon. When the sextant is inclined to the vertical at an angle j, the image of S_1' will coincide with the horizon outside the vertical, in L_1, and the altitude h_1' will be greater than h'.

The error in altitude $\Delta h = h_1' - h'$ will be defined as the difference between the hypotenuse and a leg of an elementary right-angled spherical triangle SLL_1 by the formula*

$$\Delta h = \frac{j'^2}{4} \text{ arc } 1' \sin 2h' \qquad (12.1)$$

For a small angle j, not noticeable to the eye, for instance, $j = 2°$ and for altitudes equal to 5°, 10,° 45°, 60°, 88°, Δh will be equal,

* See Appendix III, Item 5

respectively, to: +0′.18, 0′.36, 1′.05, 0′.91, 0′.07. Thus, for ordinary average altitudes the error is greatest, the sextant for this reason

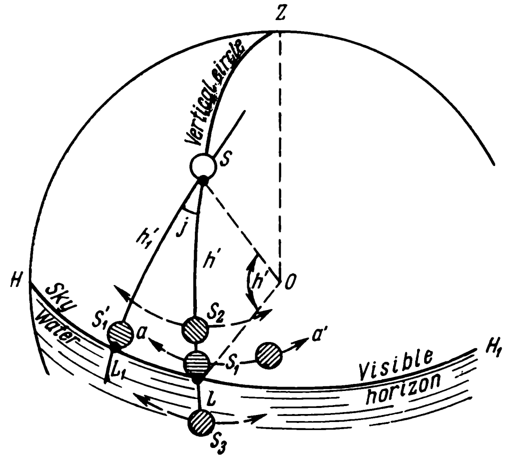


Fig. 99

should be strictly vertical when measuring altitude. To achieve this, a special method of measuring is applied that consists in *swinging the sextant* (*the arc*) so that the body image describes an arc *aa′* (Fig. 100) in the field of view relative to the horizon. In Fig. 100, the image is inverted, as seen in the telescope. The image of the body arrives at the point of the arc closest to the horizon (L) when the limb of the sextant is exactly in the plane of the vertical. If the sextant is set to a reading slightly greater than the altitude, the image of the body will (when swung) pass partially or completely "through the water" (see S_3 in Fig. 99 and 100); if the reading of the sextant is less than the altitude, the image of the body will pass completely "through the sky" (see S_2); if the reading is equal to the altitude, the image will contact the horizon.

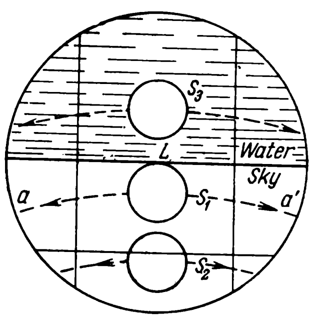


Fig. 100

There are basically three ways of making a body move in an arc: by swinging the sextant about the plumb line OZ (Fig. 99), by swinging it around the axis of the telescope OL, and around the ray SO incident on the index mirror. Combinations of these rotations are also possible.

(1) *Swinging about the plumb line OZ* is achieved by moving the telescope of the sextant in azimuth through small angles, while holding the limb of the sextant in the vertical plane, that is, by the observer turning round the plumb line. Then the image of the body will describe in the field of view a curve (parabola), the point of tangency of which with the horizon lies precisely in the vertical circle of the body.

In this method, the image of the body moves very slightly from the horizon for altitudes up to 30° and for small oscillations; if the rotation is 3° or more it will leave the field of view. It is therefore difficult to establish the point of tangency, and the first method is not suitable for small and medium altitudes; it is applicable only for very large altitudes (for the sun in the tropics).

(2) *Swinging the sextant round the axis OL of the telescope* is one of the oldest methods and is executed by moving the sextant by hand about the axis of the telescope with a simultaneous slight movement of the telescope along the horizon about the vertical circle of the body so as to bring the images of the body and the horizon into coincidence in the middle of the field of view. In this case, the doubly reflected image of the body also describes a parabola in the field of view, but for ordinary altitudes ($< 60°$) with greater curvature than in the first method. For small rotations of the sextant, the image will leave the horizon rapidly, and so the second method is the most exact one. However, it is difficult to execute, and is therefore used only by experienced observers, because when the sextant is turned, the reflected image of the body easily leaves the field of view and is lost, particularly for large (60° and more) altitudes.

In the second procedure, the horizon remains in the centre of the field in an unchanged position, while the body describes a parabolic arc about it (Fig. 100).

(3) *Swinging the sextant around a ray incident on the index mirror* is performed by rotating the sextant round the direction *OS* to the body, with the telescope moving in azimuth to right and left. That is, the sextant moves in a circle with a radius of h' and with centre in the star. Then the image of the body will all the time be in the middle of the field (Fig. 101). This is done as follows: bringing the body to the horizon and holding the sextant stationary relative to your face, swing the upper part of your body so that the image of the body is always in the centre of the field of the telescope, while the horizon tilts and recedes from the body.

It is much easier to measure altitude by this procedure than by the preceding one, because the body is kept constantly in the field of view, while the horizon leaves the field only for very large inclinations of the sextant (of the order of 15°-20°) which are not needed in practice. The curve described by the body has smaller curvature

than in the second procedure, and diminishes with increasing altitude. For this reason, the third procedure is less accurate than the second. For very large altitudes this procedure is not at all suitable, since it is impossible to establish the point of tangency and, hence,

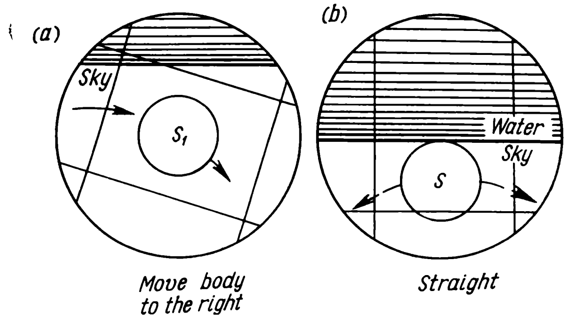


Fig. 101

the vertical circle of the body; it is recommended for inexperienced observers when observing the sun with a telescope having a small field of view.

The most common is the second procedure of swinging the sextant round the telescope axis with movement along the horizon.

Let us now consider *methods of bringing into coincidence* a body and the horizon, which is done while swinging the arc. Two procedures may be utilized to achieve this: (a) by preliminarily setting the instrument and then waiting for tangency, and (b) by making the images tangent by micrometer movement of the index arm.

The former procedure is more exact and more convenient than the latter, but it is more difficult since it requires more experience and greater skill to determine the intervals at which the index is set. This procedure is not suitable for a body located close to the meridian, in which case the latter procedure is better.

SEC. 64. MEASURING ALTITUDES OF CELESTIAL BODIES ABOVE THE VISIBLE HORIZON

I. CHOOSING A POINT OF OBSERVATION

To measure altitudes, first select a point of observation that will satisfy the following general requirements:

(a) It should be protected from the wind, water spray and, if possible, from the vibration of the ship.

(b) It should afford a good view of the chosen stars and horizon. No streams of warm air (from stacks, the machine or galley wells,

etc.) should be allowed to pass through the vertical circles of the bodies. In night-time observations, eliminate any illumination that might interfere with the visibility of the bodies and horizon.

(c) Under ordinary conditions, and in clear weather, observations are best carried out from a high spot: the upper bridge deck. This ensures more even horizon and better visibility.

(d) In case of poor visibility of the horizon (due to fog, mist, rain, or snowfall) it is better to bring the site of observation down to the lower open deck. The horizon will then be closer and visibility better.

(e) If the ship is rolling and pitching, choose a site closer to the centre-line plane of the ship, where it is more convenient to take observations. When the observation of several bodies is involved, one site may not be enough; then several convenient sites of observation will have to be found.

II. MEASURING THE ALTITUDE OF THE SUN

Preparing the sextant for observations. In cold or hot weather, the sextant is taken out to the site of observation 10 to 20 minutes beforehand so that it will take on the temperature of the ambient air.

Before observing the sun, put on the telescope (magnification from 6 to 8×), adjust it to your eye; then check the positions of the mirrors, and determine the index correction. When determining i, turn the shade glasses into position in front of the index mirror and horizon glass. For the index mirror use the same shade glasses that will be needed in the main observations. After i is determined, remove shade glasses from horizon glass. A weak shade glass is used if the surface of the sea in the direction of the sun is reflecting brilliantly or if the horizon is very bright.

Bringing the sun's image to the horizon. To bring the sun's image to the horizon, that is, to bring both images into the field of view, use one of the following methods:

1. Direct the telescope at the horizon under the sun, maintaining the sextant as vertical as possible. Then, without losing the horizon line from the field of view, use your left hand to move the index arm from small to large readings, swinging the sextant about the axis of the telescope so as to get a view of a larger belt of the sky. The movements are kept up until a reflected colour image of the sun is seen in the field of view.

For inexperienced observers and for a small field of view of the telescope, we advise turning a weaker shade glass into position in front of the index mirror during the "search" (when the sun's image passes the field of the telescope, a glowing redness will be seen).

The foregoing procedure for bringing the sun to the horizon is the most common one.

2. The shade glasses are left in the same positions as for determining i from the sun (that is, in front of both mirrors). The index arm is set at 0° and the telescope is directed at the sun; two images of the sun will be seen in the field of view. Slowly lower the telescope of the sextant to the horizon and at the same time move the index arm forward so that the reflected image remains in the field of view. Continue the movement to a horizontal position of the telescope, then take out the shade glasses of the horizon glass and bring the sun's image to the horizon. This is a more universal procedure and is sufficiently simple.

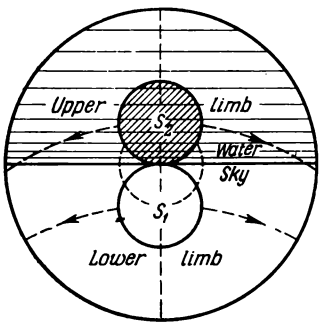


Fig. 102

3. On the sextant, set a reading roughly equal to the altitude of the sun (determined "by eye"), direct the telescope at the horizon under the sun, and, swinging the sextant and slightly moving the telescope, "catch" the image of the sun.

Measuring altitudes of the sun far from the meridian. At sea, the usual procedure is to measure the altitude of the lower limb of the sun, since in this way the image of the sun in the field of view of the telescope is projected onto the "sky" and we can more easily see the limb of the sun brought tangent to the horizon (S_1, Fig. 102); it is comparatively rare that the altitude of the upper limb (S_2, Fig. 102) is measured.

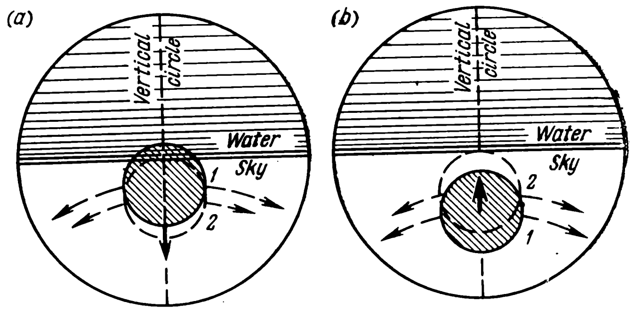
When the edges of the sun's disc are not seen clearly, for instance due to clouds or fog, it is best to bring the centre of the disc to the horizon, that is, to make the horizon split the image of the sun into two halves. When working with modern sextants in which the field of view of the telescope is large, it is best to use the second procedure when swinging the sextant: swinging it about the axis of the telescope with a slight movement along the horizon in a so-called swinging under the sun.

Bringing the limb of the sun's image tangent to the horizon is done either by constant rotation of the drum until the images are coincident, or by preliminarily set readings.

The procedure "at a set reading" is used for measuring altitudes far from meridian (when they are varying with sufficient rapidity) and is done as follows.

In the forenoon, the altitude of the sun is increasing, and so moving the drum brings the image of the sun slightly onto the "water",

sinking it so to speak (Fig. 103*a*); in the afternoon the altitude of the sun is diminishing, and so the sun's image comes a bit away from the horizon (Fig. 103*b*). The amount of overlap or divergence of images depends on the rate of variation of the sun's altitude and upon the skill of the observer; the more experienced the observer, the less "extra" overlap and the less waiting for tangency. The highest rate of variation of altitude is found near the prime vertical and image overlap here is greatest (up to 10′-15′). As the sun recedes from the prime vertical, the rate of change of altitude diminishes, and thus



Coincidence in forenoon *Coincidence in afternoon*

Fig. 103

also there is a decline in image overlap. It is best to choose sextant readings that are multiples of a small number of minutes of altitude (2′, 5′, 10′, etc.). After the index arm has been set, do not touch the drum any more; swing the sextant and wait until the image of the sun (as it comes out of the water or descends into it) makes contact with the edge of the horizon: at that instant, read the watch or chronometer.

In all these cases, rotate the drum in one direction when setting the reading—in the same direction as when determining the index correction. This is done to eliminate errors due to backlash of the tangent screw.

Since measuring altitude also involves recording the time by a watch or chronometer, it is best to work with an assistant: one observes with the sextant, the other records the time. The first observer, noting the instant when the sun (as the sextant is swung) is a slight distance from the horizon, gives the preliminary command "stand-by", and at the instant of contact, the executive command "mark". At the command of "stand-by", the second observer begins to watch the second hand; at the command "mark", he notes the reading (up to 0s.5) and records first the seconds and then the minutes

and hours. The reading of the sextant, noted by the first observer to within $0'.1$, is also recorded. When observing the sun, it is best to take 3 or 5 altitudes and compute the average.

If the observer does not have an assistant, he should give the commands to himself, and at the instant of tangency he should begin a count of seconds (half, one, half, two, etc.) to have time to switch to the timepiece (see Sec. 56).

A less precise procedure is possible with a stop watch. Start the watch at the instant of contact of the images (command "mark") and, without reading the limb, go up to the chronometer and stop the watch at some convenient reading of the second hand of the chronometer. Record this reading and put the reading of the stop watch alongside.

Measuring circummeridian and meridian altitudes of the sun. When measuring the altitude near the meridian, remember that the altitude of the body in this position changes slowly and the "waiting" method is not suitable. For this reason, circummeridian altitudes are measured like angles, that is, by using the micrometer screw to bring the arc described by the sun's image into contact with the line of the horizon, and noting the instant. All other operations are performed in the same way as for common measurements.

As we shall see later on (Sec. 124), at sea, the meridian altitude of a celestial body does not coincide with maximum altitude (due to the ship's motion and to changing declination of the sun or moon); it is thus necessary to distinguish between measuring meridian and maximum altitudes.

(a) To measure the *meridian* altitude, compute the instant of transit (meridian passage) of the sun for the meridian on which the ship will be at noon. At about this time, begin to measure circummeridian altitudes as indicated above, and take for the meridian altitude the one which will correspond to the computed instant of transit (for instance, at the watch assistant's command).

However, at sea it is more common to measure the maximum altitude and disregard the difference between it and the meridian altitude, or to recognize this difference in the form of a correction.

(b) There are two ways of measuring the *maximum altitude.*

First procedure. Begin measuring the circummeridian altitudes and note the instants and readings for some time prior to transit of the sun (5 to 10 minutes) and perform the measurements without interruption and rather quickly until the altitudes begin to decrease systematically (or increase, for lower transit). Of the recorded readings, take the greatest, which is regarded as the maximum (or meridian) altitude. In the process, see that the motion of the micrometer drum is in the same direction and coincides with that used in determining i.

Second procedure. Begin observations a few minutes prior to transit; bring the sun's image into contact with the line of the horizon and (by rotating the micrometer drum in one direction) maintain this contact while swinging the sextant as usual until the sun is seen to be moving in the opposite direction. Stop measuring, and without touching the setting of the drum, take a reading: it will be that of the maximum altitude. It is not possible to record the exact instant, but this is not required for meridian altitudes.

The first procedure is in all respects superior to the second; it is less tiring and permits computation on the basis of measured circum-meridian altitudes if for some reason the maximum altitude has not been obtained. The second procedure is used only in low latitudes when measuring very large altitudes of the sun ($>$80°-85°).

III. MEASURING ALTITUDES OF THE MOON

The best time for observing the moon is in the daytime or during twilight. As a rule, the moon is not observed at night because the horizon under it is not reliable due to dark and bright bands on the sea under the moon that may be taken for the horizon. What is more, the observations themselves are more complicated than those of the stars and planets.

Near the first and last quarters, the moon may be observed together with the sun in the daytime or just before setting (after sunset). Joint observations of the moon and sun are possible for only 4 to 6 days a month, but one should strive to utilize these observations in place of observations of the sun at different times.

In general, taking the altitude of the moon in the daytime does not differ from taking the altitude of the sun, only the shade glasses are not needed and measurements are made of the altitude of the limb that is visible in the given phase of the moon.

When measuring the altitude of the moon after sunset, use the same telescope as in the daytime, but for deteriorated visibility of the horizon, use the star telescope. Sometimes weak shade glasses are turned into position in front of the index mirror and the horizon glass so that a poorly visible horizon should not be lost in the rays of the moon's bright image. Altitude measurements of the moon are less accurate than those of the sun.

IV. MEASURING THE ALTITUDE OF STARS AND PLANETS

A star telescope (with larger field of view) is used on the sextant for observing the stars and planets. The index correction is found from some faint star in the morning prior to observations and in the evening following observations.

Measuring the altitude of stars is complicated by two circumstances: (a) insufficiently clear outline of the visible horizon (occasionally, it is not visible at all), while in daylight the stars themselves are hardly visible; (b) large numbers of stars make identification in the field of view of the telescope difficult.

Conditions of visibility of stars and horizon are improved by proper choice of the time of observation. The altitudes of stars and planets are usually measured *in twilight* when the horizon is still clear enough, while the atmosphere (then less illuminated by the

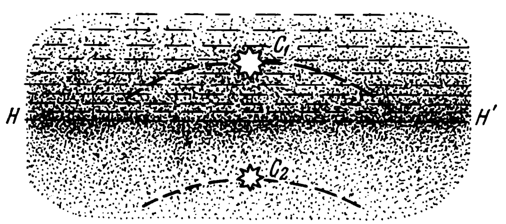


Fig. 104

sun) makes the stars and planets visible. Stellar observations are possible also on moonlight nights if the horizon is bright enough. Avoid measurements in the vertical circle of the moon.

To improve visibility of the horizon, use a *star telescope*, which has a field of view from 1.5 to 2 and more times greater than the day telescope, thus making it possible to see more of the horizon. When the illumination is reduced, a long line of the horizon is seen better and for a longer time than a short line. What is more, a star telescope gives a brighter image of the horizon (the brightness depends on the ratio of the diameters of objective lens and eyepiece lens and magnification of the telescope). A larger field of view also takes in a larger area of the sky, and this simplifies locating a desired star.

When observing on dark but clear nights, one sees only a dark band (in place of the line of the horizon), above which it is almost impossible to observe. For such cases, many foreign sextants are provided with a special *Wollaston prism* in front of the index mirror, which simplifies observations somewhat. This prism is made of quartz and has the property of splitting a ray from a star entering at an angle of 10′ so that we have two images in place of one (Fig. 104). The dark band H-H' is brought by the index arm to the middle of the interval C_1C_2 between the images of the star when swinging the arc. The reading is the altitude of star C above the axis of the dark band. The index correction is likewise determined with the Wollaston prism by alternate alignment of the direct image and both

reflections: C_1 and C_2. Computations from there on are similar to those for the sun. However, observations with this prism are not always reliable.

The measurement proper of the altitude of a star (like that of the sun) consists in bringing the image of the star to the horizon (preliminary operation) and bringing star and horizon into coincidence in the vertical circle of the star (basic operation). Bringing the image of the star to the horizon is done as described in the second and third procedures for the sun, but modified for fainter objects (stars).

First procedure. Set index arm at exactly 0°0′, direct telescope at star and move the index arm forward, at the same time slowly dropping the telescope towards the horizon and holding the reflected image of the star in the centre of the field of view until the image of the horizon appears there as well.

If the star is faint and is easily lost among adjacent stars, the sextant is turned limb upwards, the telescope is directed at the star, and the horizon is brought as close as possible to the star by moving the index arm; after this the sextant is returned to its original position and the measurement is made in the usual fashion.

Second procedure. The sextant is set to an approximate altitude of the star as obtained from a globe. The sextant is then turned to the azimuth of the star (obtained in the same way), and the image of the star is sought near the horizon. To do so, slightly move the telescope in azimuth and swing the sextant.

The second procedure is the only one suitable for day time observations of Venus and for observations of the stars and planets immediately after sunset or before sunrise when the stars are not visible with the naked eye. Since twilight is brief, especially in low latitudes, observations sometimes have to be continued in sunlight. In that case, first compute the time of sunset (sunrise) and twilight and then obtain for this time the altitudes and azimuths of the chosen bright stars by means of a star globe.

After bringing the star to the horizon, the images of the star and horizon are brought into contact by swinging the sextant about the axis of the telescope accompanied by a slight movement along the horizon. This procedure is best here because the poorly visible horizon will always be in the centre of the field of view of the telescope. When measuring the altitude of a planet, the centre of the visible disc is brought into coincidence with the horizon. As a rule, from three to five altitudes of each celestial body are taken.

The time, by watch or chronometer, is recorded by the assistant observer, as described for the sun. For this purpose and also for the recording of readings, one requires a special faint illumination (from the compass, for instance). In the absence of any special light, the assistant also reads the sextant.

In modern sextants CHO, readings can be made in the dark. The frame of the magnifier lens and the scale divisions are coated with a luminescent material that glows if the sextant has been held under a strong light prior to observations.

SEC. 65. SPECIAL CASES IN MEASURING THE ALTITUDES OF CELESTIAL BODIES

I. MEASURING ALTITUDE ABOVE THE SHORE LINE

In the daytime or in twilight, the altitudes of the sun, moon or some bright planet, like Venus, may be measured above the shore line (more precisely, above the water level) if the horizon below is blocked by the shore line. In certain cases, the water line of another ship can play the part of the shore line, if the ship lies in the vertical circle of the celestial body and is not far from the observer.

In this case, the altitude is measured in exactly the same way as above the visible horizon, but after observations a determination is made of the distance to the "shore line" by means of a range finder, or in some other way.

The index correction must be found from the "shore line", with the exception of cases when the shore lies beyond the visible horizon. Observations are taken in the same way as above the ordinary visible horizon.

II. MEASURING ALTITUDES "VIA THE ZENITH" (BACK SIGHT)

In certain cases the horizon under a body may have poor visibility (due to fog, haze, etc.), while the opposite part is clearly visible. If the body is situated high enough, its altitude may be measured above the opposite part of the horizon, "via the zenith" (Fig. 105).

The altitude measured in terms of the zenith, h_z, will, as apparent from Fig. 105, be approximately equal to $h_z \approx 180° - h$. Obviously, h_z should not be greater than the limiting angle measured by the sextant, or about 140°. Consequently, the altitude of a body h should be greater than 40°. When measuring altitude via the zenith, bear in mind that altitude equal to the arc SS_1 is the *greatest* recession of the body from the horizon, in contrast to ordinary altitude. For this reason, when swinging the sextant, the arc described by the body in the field of view of the telescope (Fig. 106) will be upside down (convex down).

When measuring the upper limb U of the sun "via the zenith", the image of its disk (S_1) is projected on the "sky"; that is, it occupies the same position as in conventional measuring of the lower limb

of the sun. So we can say that the upper limb U in ordinary altitude measurement will be the lower limb when measuring via the zenith. Now when measuring the lower limb L of the sun via the zenith, the image of its disc (S_2) will be projected completely on the "water", and in this case the picture will be similar to ordinary measurement of the upper limb of the sun. As a rule, one should use the back-sight method for measuring the altitude of the upper limb U of the sun (S_1). It is more convenient in this case to bring the images of the horizon and the star into coincidence.

Due to difficulties in measuring altitude via the zenith, it is advisable to put on a star telescope even in daytime observations. This

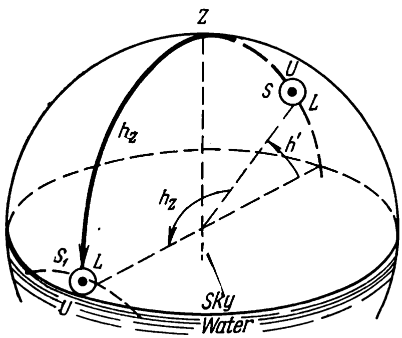


Fig. 105

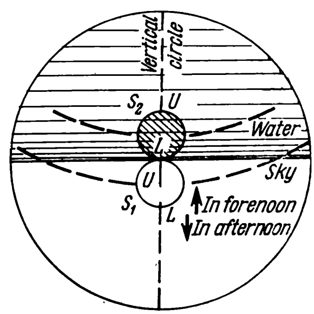


Fig. 106

simplifies observations. And the number of altitudes taken via the zenith should be increased to 5-7, so that their accuracy will approach that of ordinary measurements.

Back-sight observations are done as follows: measure the approximate altitude h' of the limb of the sun in the usual way and then compute $180° - h' \approx h_z$. The index arm of the sextant is set at this reading of h_z. The observer then turns his back to the sun (in the direction of his shadow) and directs the telescope to the horizon; then by moving the telescope and swinging the sextant, he finds the image of the sun. To make the images coincide exactly, the sextant is swung round the axis of the telescope and the telescope is moved in a direction opposite to usual. It is best to measure from set readings (see above). Back-sight observations are much more difficult than the ordinary kind. A good deal of practice has to be put in to learn the proper movements. Besides the foregoing cases, altitude sights via the zenith are used in a procedure for eliminating systematic errors in the altitude of the sun.

III. MEASURING ALTITUDE WITH AN ARTIFICIAL HORIZON

In certain cases, in the absence of a visible horizon, an **artificial horizon** may be used. It is rarely used at the present time (mostly for measuring the altitude of the sun from ice when sailing in ice-fields). However, the artificial horizon is useful for training purposes in taking altitude.

Also, the artificial horizon may be used in calm shore situations for checking human systematic errors and errors both of ordinary

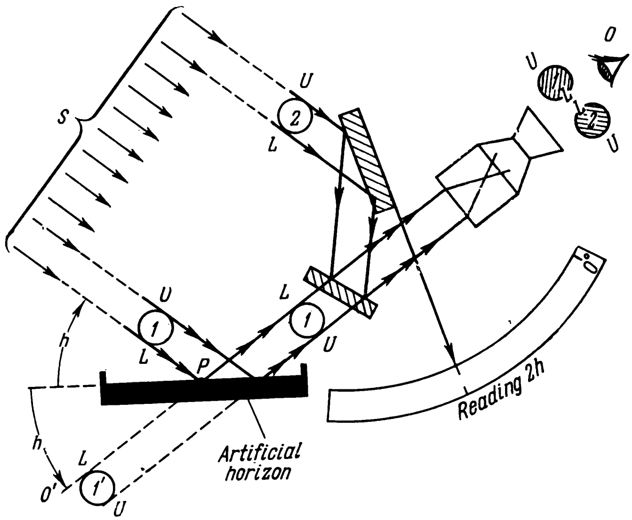


Fig. 107

sextants and sextants with artificial horizon. The artificial horizon is more often used to measure the altitude of the sun, and less often that of the stars.

For these purposes, one should use a *liquid artificial horizon*, which is a shallow flat vessel filled with mercury (Fig. 107) or some other viscous liquid. For example, any dish, broad-bottomed jar or pail filled with cylinder oil or fuel oil can be used. Put up vertical shields for protection against the wind. Mercury horizons are much more accurate than oil horizons, but they are very sensitive to vibrations and the wind.

Due to gravity, the surface of the liquid is perpendicular to the plumb line, that is, it is in the plane of the true horizon, except at the edges of the vessel.

If the sextant telescope is pointed to the sun's image in the artificial horizon, and the index mirror directly to the rays from the

sun, the angle between the directions SP and PO' will be $2h$, as is evident from Fig. 107. Hence, when the images of the sun are made to coincide, we get *twice the altitude*, $2h$.

In actual practice, the measurements are made not by coincidence but by external tangency, the reflected image *2* (Fig. 108) moves relative to the direct image *1* so that their lower (L) or upper (U) limbs are made tangent as the sextant is swung about the axis of the telescope. The path of the rays in this case is shown in Fig. 107. If the direct image (*1*) is on top, we measure the altitude of the lower limb of the sun. The direct image may be found by covering the index mirror with your hand.

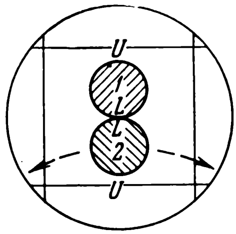


Fig. 108

To bring the image of the body *2* to the direct image *1*, set the index arm to approximately $2h$; to find image *2*, turn a weak shade glass into position as described above (Item II, Sec. 64). The images are allowed to move into external tangency (at a set reading); to achieve this, in the forenoon (when measuring the altitude of $\underline{\odot}$) the discs are made to overlap slightly, in the afternoon they are separated. Image *1* of the body should be reflected in the middle of the horizon in order to avoid distortions due to the inclination of the surface layer of the liquid near the edges of the vessel.

When measuring altitude, hold the sextant as close to the horizon as possible, and rest your elbows on a support.

Altitude measurements in an artificial horizon are much more accurate than above the visible horizon.

SEXTANTS WITH ARTIFICIAL HORIZON

SEC. 66. ESSENTIAL THEORY OF SEXTANTS WITH ARTIFICIAL HORIZON AND INTEGRATOR OR AVERAGER (TYPE ИАС)

Special sextants with artificial horizon are now used for measuring altitudes of celestial bodies at sea in the absence of a visible horizon. These sextants were originally used by aircraft and found application

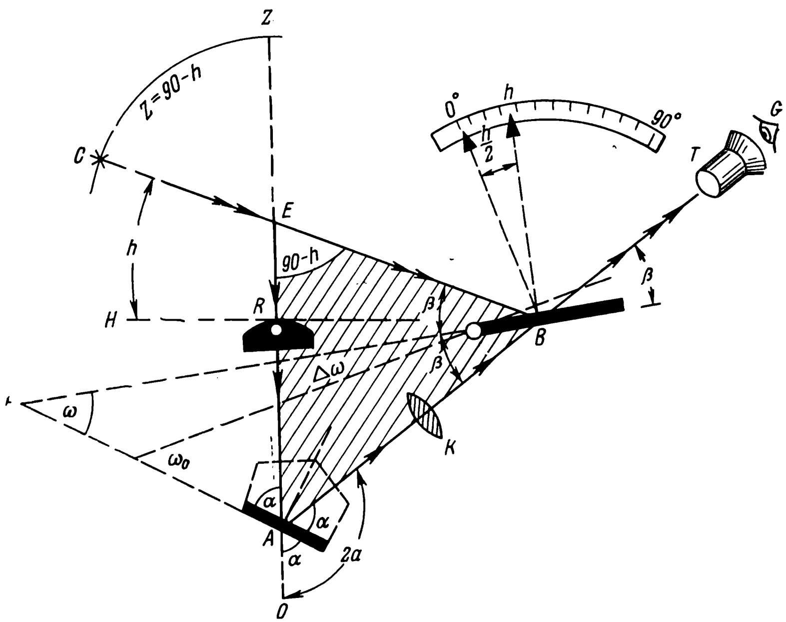


Fig. 109

at sea only after World War II. For this reason, most Soviet models have the aircraft designation (integrating aviation sextant, or ИАС) and only recently have special marine sextants of similar design appeared (ИМС).

The ИАС sextants utilize the same principle of reflection as the ordinary marine sextant, only the visible horizon is replaced by a level (or gyroscope) that yields the plumb-line direction.

By introducing mirrors, we measure the angle between the mirrors instead of measuring the angle between directions in space.

Suppose (Fig. 109) ZO is the plumb line, CB is the direction of a ray from a celestial body, BH is the horizontal plane. Then the angle CBH will be equal to the altitude of the body h, and angle $ZEC = z = 90^\circ - h$.

At B, place a rotatable half-silvered mirror called the main or index mirror; at A, a stationary mirror (in the form of a pentagonal prism). Denote the angle between the planes of these mirrors by ω. The plumb-line direction is indicated by the bubble R, the light from which is reflected from mirror A at an angle α, passes through the main mirror B and enters the eye of the observer G. The beam of light from star C impinges on the surface of the mirror B and is reflected from it at an angle β. By turning the index mirror B round the axis perpendicular to the plane of the drawing, that is, by varying the angle ω, it is possible to bring to coincidence the rays from the star C and from the bubble R; their images will coincide in the field of the telescope (see Fig. 110). The coincidence will occur for a definite relationship between the angle ω and the angle being measured $(90^\circ - h)$. From the triangle ABE (cross-hatched in Fig. 109) and on the basis of the relations mentioned in Sec. 59, we have

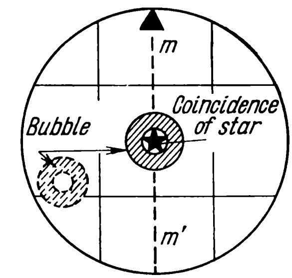


Fig. 110

$$(90^\circ - h) + 2\beta = 2\alpha$$

or

$$90^\circ - h = 2(\alpha - \beta) \tag{13.1}$$

Similarly, from the triangle ABF we have

$$\omega + \beta = \alpha$$

or

$$\omega = \alpha - \beta \tag{13.2}$$

Comparing formulas (13.1) and (13.2), we see that

$$90^\circ - h = 2\omega \tag{13.3}$$

or

$$\omega = \frac{90^\circ - h}{2} \tag{13.4}$$

that is, the *angle being measured between the plumb line and the direction of the star is equal to twice the angle between the mirrors when the images of the bubble and the star are brought into coincidence in the field of view of the telescope.*

The relation obtained, (13.3), is similar to that derived above for the marine sextant.

Differentiating formula (13.3) with respect to ω and h and passing to finite increments, we get

$$-\Delta h = 2\Delta\omega$$

or

$$\frac{\Delta h}{2} = -\Delta\omega \tag{13.5}$$

which means that an increase in altitude by the angle Δh will cause a turn of the mirror in the reverse direction by one half of this angle. Thus, the angle of rotation of the mirror relative to some initial position may serve as a measure of the altitude of a star. The angle of rotation of the mirror is measured, in the ИАС sextant, by means of a special angle-measuring drum and an averaging device.

Let us find out how the mirror B will be positioned relative to A for altitudes of a body $h = 0°$ and $90°$.

From expression (13.4) for $h = 0°$ we get $\omega_0 = 45°$, and for $h = 90°$ we have $\omega = 0°$; in other words, the angle between the mirrors varies between $45°$ and $0°$, and it would be more correct to call this instrument an octant instead of a sextant. Thus, if for an angle $\omega_0 = 45°$ we put zero on the index of the micrometer drum (see Fig. 109), and mark the half-degree divisions of the drum as whole degrees, then the angle of rotation of the drum (which is proportional to the angle of rotation of the mirror B) will be equal to the altitude of the celestial body.

The relations obtained hold only for a position of the sextant as indicated in Fig. 109; that is, when the main plane (the plane perpendicular to the planes of the mirrors A and B and passing through the centre of the spherical surface of R) coincides with the plane of the vertical circle of the star. Any deviations from this correct position of the sextant will obviously cause an error in the angle being measured. If the sextant is tilted a small angle in the main plane, it will not give rise to an error in the angle being measured. This important condition is attained by making the bubble-assembly lens R spherical and introducing a collimator lens K. The principal focal length of the lens K is taken equal to the radius of curvature of the spherical lens of the level.

As a result of this design, when the sextant is inclined in the main plane (say, by an angle γ), the bubble of the level will also move over the spherical surface of the level through an angle γ, and will

again occupy a position on the plumb line, while remaining in the focus of the lens K. The bubble image in the field of view of the telescope will move by 2γ. On the basis of (13.5), the image of the star will also move this much so that both images will remain coincident and the angle being measured will not change.

But if the principal plane of the sextant is inclined perpendicular to the plane of the vertical circle of the star and the star image is brought into coincidence with the bubble image (which in this case will be to the right or left of the centre of the field of view), the angle being measured will have an error dependent on the angle of inclination of the sextant and the altitude of the star*. An analysis of this phenomenon shows that the lateral tilt of the sextant should not exceed $1°$ so that the altitude error does not exceed $1'$.

In the field of view of the telescope of the sextant (Fig. 110) is a grid on the lens of the level; there is an angular distance of $2°.8$ between the sides of the central square of this grid; it is thus possible to judge the angle of lateral tilt of the sextant from the position of the bubble relative to the grid. If, for example, the centre of the bubble has shifted to a side line, the inclination is $1°.4$, which is impermissible.

It is thus possible to bring about image coincidence near the longitudinal line mm' without deviating to one side more than $1°$, which is twice the angular diameter of the sun or moon.

Let us now consider faulty coincidences of images.

The images of the bubble and star may *diverge* in the field of view along the longitudinal line mm' and also at right angles to this direction. In the former case, divergence is due to insufficient rotation of the principal mirror and causes an error in the measured altitude. These errors are eliminated by turning the angle-measuring drum until the images coincide.

Divergence of images in a lateral direction is caused by a transverse inclination of the sextant or by taking it out of the vertical circle. If this divergence is exactly transverse and within the limits of the grid square of the level, there will be no altitude error because the principal mirror remains in accord with expression (13.4). On this basis, one might make the images *tangent* in the transverse direction (instead of *coincident*); but this is harder to do, and ordinarily the images are brought into coincidence.

From the foregoing it follows that measuring altitude with an ИАС sextant is done by bringing three points into coincidence in the field of view of the telescope: the centre of the celestial body, the

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* By the formula $\sin\frac{z'}{2}=\sin\frac{z}{2}\sec\gamma$ where z' and z are the measured and true zenith distances, γ is the tilt of the sextant.

centre of the bubble, and the centre of the field of view. This is rather hard to achieve on a moving support, and to obtain reliable results one has to develop definite skills through long practice.

SEC. 67. DESIGN FEATURES OF THE ИАС SEXTANT

The ИАС sextant consists of the following basic components:
(1) optical system and level,
(2) angle-measuring device,
(3) averaging device,
(4) illumination system,
(5) frame of instrument and auxiliary parts.

(1) The **optical system** of the sextant is shown in Fig. 111, where the double arrowhead indicates the path of rays from a body, while the single arrowhead shows the path of rays illuminating the bubble. The eye of the observer is shown in two positions: G_1, usual position

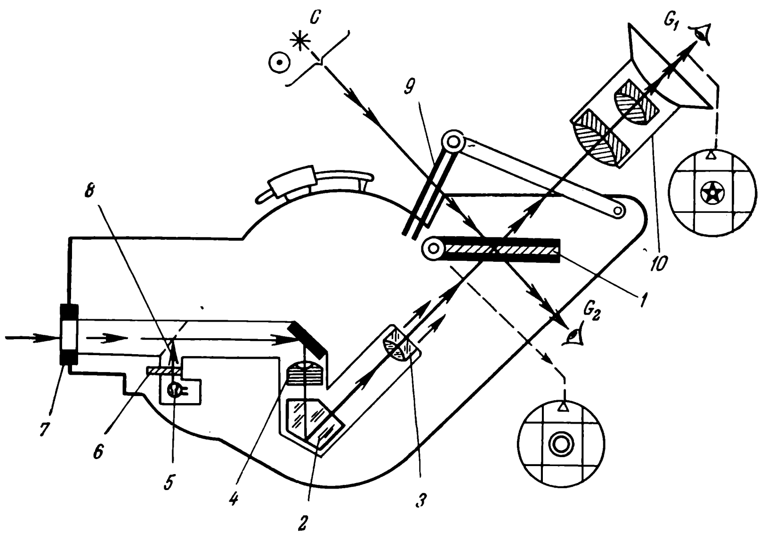

Fig. 111

for observations through the telescope, and G_2, the position for observation through the index mirror, in which the body images are brought to the horizon (i.e., the bubble).

The principal mirror *1* is a plane parallel glass covered with a special lacquer so that part of the rays passes through the glass, while the other part is reflected. The small mirror *2* is in the shape of a pentagonal prism for convenience in mounting and to eliminate prismatic errors.

The main purpose of the collimator lens *3* consists in the following. The bubble, for reasons already indicated, always lies in the principal focus of the collimator lens; therefore, light rays coming from the bubble emerge from this lens as a parallel beam and the bubble image appears removed to infinity, just like the image of the celestial body. As a result, both images may be regarded jointly for an eye identically adapted to "infinity" and in any point of the field of view.

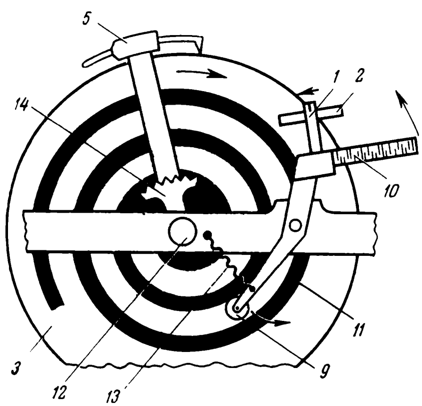

Fig. 112

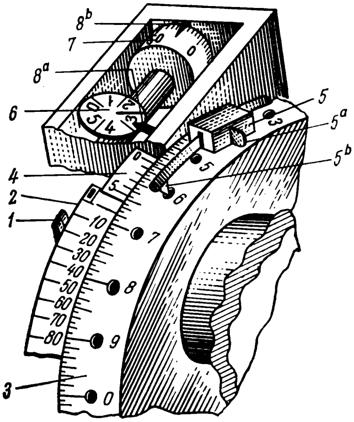

Fig. 113

The telescope of the sextant serves to increase the accuracy of alignment of images (magnification, 2.2×, field of view 7°.5). The relatively large field simplifies identification of the required star among its neighbours.

Lamp *5* and shade glass *6* are used in night observations of stars and planets for illumination of the bubble and for setting up a dark-red background about it. White and yellow stars are particularly clear against such a background. In night-time observations, the frosted glass *7* is replaced by a reflecting mirror *8*, and the solar shade glasses *9* are thrown out of position. The bubble assembly *4* consists of an upper spherical lens and a lower plane glass, the space between them being filled with ethyl alcohol that contains an air bubble. This assembly is mounted in a special body of the level equipped with an expansion chamber and an adjusting screw at the bottom of the instrument (Fig. 115).

(2) The **angle-measuring** drum (see Figs. 112 and 113) is designed for measuring the angle of inclination of the principal mirror relative to the initial position marked 0°0′. Fixed rigidly to the principal mirror is an index *1* (also see Fig. 109) that indicates tens of degrees

of rotation of the mirror on the dial *2*. The angle-measuring drum *3* (Fig. 112) is held by a handle and is connected with the principal mirror *10* via a roller *9* that rolls along the spiral *11* cut in the inner surface of the drum. As the drum rotates about the axis *12*, the roller, pressed to the spiral by spring *13*, rolls along the spiral, receding from the axis (or approaching the axis), as a result of which the angle of inclination of the mirror varies as indicated by the index *1*. A rotation of the drum through 360° makes the principal mirror tilt only 30°. This device permits increasing 12-fold the linear magnitude of a degree on the rim of the angle-measuring drum, thus considerably enhancing the accuracy with which the rotation angle of the mirror is measured. On this basis, the circumference of the drum is divided into 30°, three intervals from 0° to 9°, and into smaller divisions with values of 5′ each. Tens of degrees are indicated, as already mentioned, directly by the index *1*. Single measurements (minutes) are read by the minute vernier *4* (Fig. 113), but at sea single measurements are not taken.

The result of a series of measurements is averaged by an *integrator*, the angle-measuring scales (*6*, *7*) of which are shown in Fig. 113.

The integrator is switched in by stop *5* after the drum has been rotated to approximate alignment of the images of bubble and celestial body in the field of view of the telescope. By pressing catch 5*a*, insert pin 5*b* into aperture cut in each degree of the drum. Now, each motion of the drum and, hence, rotation of the mirror is transmitted via the stop and the gear sector *14* (Fig. 112) to the integrator, which records it on the degree scale *6* (Fig. 113) and the minute drum *7* by means of indexes 8*a* and 8*b*. The readings of these scales are increments ($\pm\Delta h$) to the reading on pin 5*b* of the stop and the ten-degree scale; hence, the total reading of the measured altitude is obtained as the sum of four readings (Fig. 113):

10-degree scale	10°
reading of stop on angle-measuring drum	6°
degree scale of integrator	3°
minute drum of integrator	55′
Reading of measured altitude	19°55′

The scales *6* and *7* of the integrator do not indicate negative increments of altitude. This is achieved by setting the zero position of the degree scale of the integrator at +3°. To compensate, the scale of the angle-measuring drum is displaced by —3° relative to its actual position. The ИАС sextant can measure altitudes approximately from —1°15′ to +81°15′, which is a range of 82°30′.

(3) The **averaging device** (integrator), a schematic diagram of which is shown in Fig. 114, consists of a carriage *1*, inside of which is a shaft *2* with the minute drum *7* and a worm *5* that engages the

gear of the degree scale *6*. The shaft can turn about its longitudinal axis, but due to friction against roller *3*, the shaft will turn only when the angle-measuring drum turns. This rotation is transmitted to roller *3* via a transmission mechanism and fork *4*. When the carriage is stationary, these rotations do not cause any consequences. But the carriage can move along guide strips *9* by means of gear *10* of the clock mechanism that acts as an engine. When the carriage and shaft are in motion in the direction of the arrow, the roller will roll along the shaft.

As the angle-measuring drum turns through some angle (say, a diminishing angle), the fork and roller *3* will turn in the direction

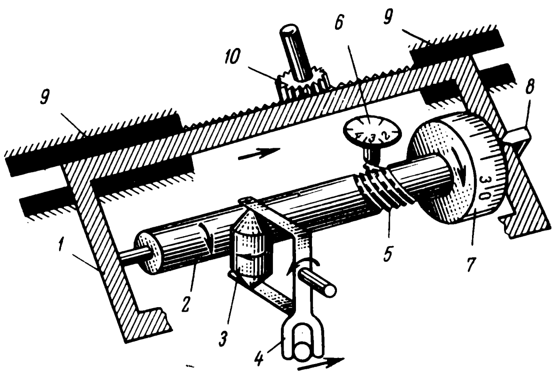

Fig. 114

of the arrow and, due to friction, the shaft *2* will begin to roll onto the roller. The readings of the drum *7* by index *8* will then be decreasing, thus reflecting the diminishing. angle of rotation of the principal mirror.

If the drum executes more than one rotation, the readings of the degree scale *6* of the integrator will change via the worm gear *5*. The rate of motion of the carriage depends on the work time of the time mechanism, which in the ИАС sextant is set equal to 40s, 120s, and 200s. The short period of work, ΔT, of the mechanism will obviously yield a small number of increments of altitude. That is why the 40s period is used in good conditions. If conditions deteriorate, the period should be increased.

The angle through which the minute drum turns is the sum of the positive and negative rotations of the angle-measuring drum during the period of operation of the clock mechanism. The drum is turned by the observer in order to keep the images of bubble and celestial body aligned; its rotation reflects the variation of altitude and fluctuations of the bubble relative to the plumb line due to rolling of

the ship. The sum of the bubble fluctuations for a long time of coincidence approaches zero, and the integrator reading will be equal to the increment of altitude for the period of operation of the mechanism.

The total sum of scale readings yields the altitude reading (*sr*), which corresponds to the mean instant of observations or the starting time (T_{st}) of the mechanism plus half the interval of operation: $T_{ch} = T_{st} + \frac{\Delta T}{2}$.

(4) The **illumination system** of the ИАС sextant is designed to illuminate the bubble and the instrument scales for night-time observations and consists of three 2.5-volt lamps, wiring, rheostat, switch and cord with plug for connecting to 24-volt ship mains. Storage or dry batteries (3 to 4 volts) can also serve as a power supply.

(5) All the foregoing components are mounted in the **frame of the instrument**, which consists of two plates, a handle for holding the instrument and a number of auxiliary parts. A general view of the sextant is shown in Fig. 115.

Design features of the ИМС sextant. The ИМС marine sextant (Fig. 116) has a supplementary device that permits determining the

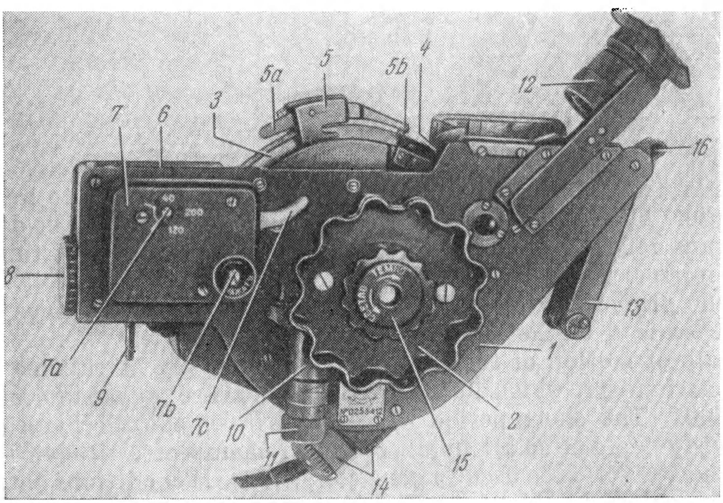

Fig. 115

1—frame, 2—handle, 3—micrometer drum, 4—main mirror, 5—lock of connecting device, 5a—tail of lock, 5b—catch of locking device, 6—integrator, 7—clockwork of integrator, 7a—head of interval switch, 7b—winding head, 7c—release lever, 8—mirror for night observations (or glass for day observations), 9—lever of night colour shade, 10—level, 11—level regulating-head, 12—telescope, 13—daytime colour shades, 14—socket and holder for electric input, 15—light regulator, 16—support for observations through index mirror.

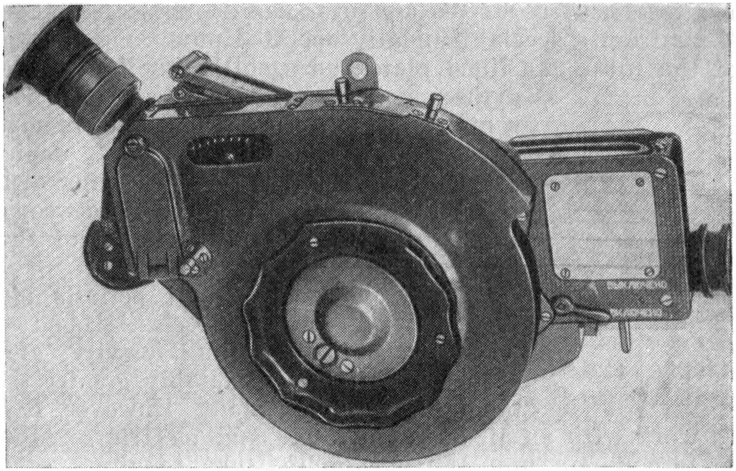

Fig. 116

zero-point correction by the bodies. For this purpose, it has inside a telescopic system with an objective lens in place of the frosted glass (*8* in Fig. 115 in the ИАС sextant). In Fig. 116, the objective lens of this system is covered with shade glasses for sun sights. When determining the correction, point the objective lens at a celestial body to obtain a direct image; then rotate the angle-measuring drum (with a special wheel) to bring the reflected image of the body into coincidence with the direct image in the field of view, and then read a. The zero-point correction will be found from the formula $i = 10° - (a + 3°)$.

In addition, the ИМС sextant is supplied with a certificate that contains instrument corrections of readings s so that one can obtain a total correction $\Delta = i + s$. This increases the reliability of measured altitudes.

The angle-measuring drum of the ИМС sextant is protected from spray by a jacket on top of which are two pins for switching the integrator on and off.

Design features of sextants with gyrohorizon. Besides bubble sextants, gyrohorizon sextants have been manufactured. This type of sextant is constructed in the same way as the ИАС sextant, with the exception that in place of a bubble assembly there is a gyrohorizon chamber. The gyroscope rotor is a massive bronze disc, the centre of gravity of which is located below the point of support. The rotor is actuated either by air or, in the latest models, by electricity.

The lateral surface of the air rotor has depressions that receive jets of air from a special manually operated pump. On the top surface of the rotor is a lined plate and a collimator lens. When the gyroscope is in rotation, its surface is horizontal and the lines on the plate, removed to infinity by the collimator lens, depict an artificial horizon. When measuring altitude, the image of the star should fit between close-lying lines (Fig. 117), the image of the sun, between more distant lines.

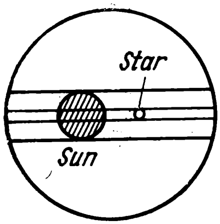

Fig. 117

The gyrohorizon sextant permits measuring the altitude of a celestial body even when the ship is rolling heavily, and the accuracy will be considerably greater than in the ИАС bubble sextant. However, it is not easy to work with an air-operated rotor, and electric sextants are not yet in common use. What is more, they require a source of electricity, which is a handicap.

SEC. 68. USING THE ИАС SEXTANT

I. CARE OF THE SEXTANT

As is clear from the foregoing, the ИАС sextant is a far more complex mechanical instrument than the ordinary sextant. For this reason, the following additional rules should be observed:

(1) do not subject the sextant to sharp variations of temperature and humidity in storage and protect it from shock;

(2) use a brush to remove dust from the optical system, and use clean rags to remove moisture; take special care when cleaning the principal mirror so as not to disturb it;

(3) when in storage, the clock of the sextant should not be wound; the bubble-adjustment lever should be turned to the stop in a direction opposite to that of the index pointer; the angle-measuring drum should be turned to maximum reading, and the stop released;

(4) during operation, periodically check performance of the sextant; check proper functioning of the basic components, bubble adjustment, the clock and integrator, and the stability of the correction obtained.

The clock is checked by comparing periods of operation (40s, 120s, 200s) with simultaneous readings of an accurate stop watch. Discrepancies should not exceed $\pm$2s, $\pm$6s, and $\pm$10s, respectively, otherwise the sextant should be repaired.

To check the integrator, set the clock mechanism at 40s and proceed as follows:

(a) start the clock without switching in the stop; when it stops, the integrator drum should read 0°0′;

(b) turn angle-measuring drum exactly ±2° by the vernier and start clock. When stopped, the integrator scales should read 1°0′ or 5°0′. Deviations of the order of ±2′ are permissible.

II. DETERMINING THE CORRECTION OF THE ИАС SEXTANT

Like any other instrument, the ИАС sextant has instrument errors. The large number of mechanical and optical components and the complicated kinematic scheme of the ИАС sextant lead to constant and random errors in altitude readings being many times greater than those of a marine sextant. The constant instrument errors can sum up to 5′ to 10′. These errors must therefore be taken into account in the form of an instrument correction to the reading. However, the certificate of the ИАС sextant does not include tables of instrument corrections of readings (the ИМС sextant includes such tables), and so these corrections must be obtained together with the zero-point correction in the form of a **total correction** Δ, called the correction of the ИАС sextant. In the course of time, the correction of the sextant may change under considerably changed environmental conditions. It therefore has to be determined from observations periodically (as frequently as possible).

To obtain the total correction Δ, the altitude of the celestial body measured by the ИАС sextant (h_{meas}) is compared with a more precise value of the altitude of the body (h_{tr}) obtained at that moment. This work is done in one of the following ways.

(a) Determining the ИАС Sextant Correction on a Test Instrument

Under laboratory conditions, a special test instrument can yield the exact (true) values of altitude (h_{tr}). At the same time, these angles are measured by the ИАС sextant every 5°-10° twice for each value of altitude; then the sextant corrections are computed from the formula

$$\Delta = h_{tr} - h_{meas} \tag{13.6}$$

and tabulated. This procedure is the most exact and speedy; however, test instruments are not yet widely available.

(b) Determining the Sextant Correction by Comparing with the Computed Altitude

This method may be used on shore, in a roadstead, and at sea. The sole requirement is that the coordinates φ and λ of the position be known to within ±0′.5 and the external conditions (visibility of celestial body, rolling and pitching of ship) be favourable.

by turning the switch in the direction 40-200-120s or 120-200-40s to coincidence of the index (point) and the desired number. The switch-over should be made only when the *clock is operating*.

V. OBSERVING CELESTIAL BODIES WITH THE ИАС SEXTANT

The ИАС sextant may be used to measure the altitude of the sun and moon in the daytime and the stars, planets and the moon at night. It is first necessary to bring the celestial body to the bubble, that is, to bring the image of the body into the field of view together with the bubble. This is done in two ways:

(a) by setting the approximate altitude of the body on the angle-measuring device;

(b) by observing the body through the index mirror from position G_2 (see Fig. 111); by rotating the drum, bring the image of the body in line with the bubble.

The first procedure is used in daytime observations of the sun or moon and in night-time observations of very bright stars and planets. However, the latter procedure may be used also for the sun if for some reason its image cannot be seen.

The actual measuring of the altitude of a celestial body is done by *protracted coincidence* of the centres of the images of bubble and body in the centre of the field of view of the telescope during the entire clock interval.

The bubble image is brought to the centre of the field in the following manner: holding the sextant with both hands, tilt it in the longitudinal and transverse planes, which is best done by the observer bending his body. The image of the celestial body is brought to the centre of the field of view by moving the sextant along the horizon (to a precise position in the vertical circle of the body) and by rotating the angle-measuring drum until the image of the celestial body coincides with the bubble.

If the ship is rolling (pitching), take up a stable position when working with the ИАС sextant (for instance, with your back to a support of some kind). Note the watch or chronometer time when the clock mechanism is started, because the stopping is not always clear-cut. The total reading obtained (*sr*) will refer to the mean instant, that is, $T_{ch} = T_{st} + \frac{\Delta T}{2}$, where ΔT is the period of operation of the clock.

Measuring altitude with the ИАС sextant requires a good deal of practice under shoreline conditions and then at sea. The accuracy obtained with these instruments (especially when the ship is rolling) is considerably lower than that provided by the marine sextant.

SEC. 69. FUNDAMENTALS OF THE RADIO SEXTANT

The earth's atmosphere lets in from outer space not only visible light, but also radio waves of a specific wavelength. Figuratively speaking, the atmosphere has two "windows": the optical window and the radio window through which we can see and study the universe. From Fig. 118*a*, in which on the y-axis we have the amount of energy transmitted by the atmosphere (in percentage), it is obvious that the terrestrial atmosphere is more transparent to radio emissions (wavelengths from 0.7 mm to 40 metres) than to light. However, the radio-emission energy is exceedingly small and requires very sensitive and intricate instruments for reception; that was why the construction of radio telescopes became technologically feasible only in the 1940's.

Ship radio-sextants are small-size radio telescopes on a movable basis. Technically, they are more complicated than stationary land-based radio telescopes, which explains why the first operating ship instrument appeared only in 1952.

The objects of extra-terrestrial radio emission include the sun, moon and several thousand discrete sources of emission mainly in the regions of the Galaxy, for example in the constellation Cassiopeia ("Cassiopeia A"), Cygnus, Orion, and elsewhere.

Of all these sources, only the sun and moon are of practical use at the present stage of technology. The other sources are considerably less intensive; what is more, their emission maxima lie in the longer wavelengths, as may be seen from the curves in Fig. 118*b*, where the vertical axis indicates the density of a monochromatic flux of radiation in watts per square metre per cycle per second. For the sun and moon, emission maxima come at the same wavelengths (about 0.8 to 2 cm). Waves shorter than 7 mm are strongly absorbed by the oxygen and water vapour of the air, whereas metre waves are distorted and absorbed by the ionosphere. Now the maximum emission of "Cassiopeia A" (the shortest-wave source) lies at about 10 metres. It is thus difficult to construct a ship instrument suitable both for sun and moon and also for stellar sources of emission, so modern radio sextants are designed only for observing the sun and moon.

The radio sextant is a complex instrument consisting of the following basic components (Fig. 119):

(a) antenna system,
(b) receiving system,
(c) tracking system,
(d) stabilized platform in gimbals,
(e) panels containing recording, averaging and control apparatus.

For receiving weak cosmic radio emission, the antenna has a metal parabolic reflector (from 30 cm to 1.5 metres in diameter), the purpose of which is to concentrate the radiation energy to a focus where

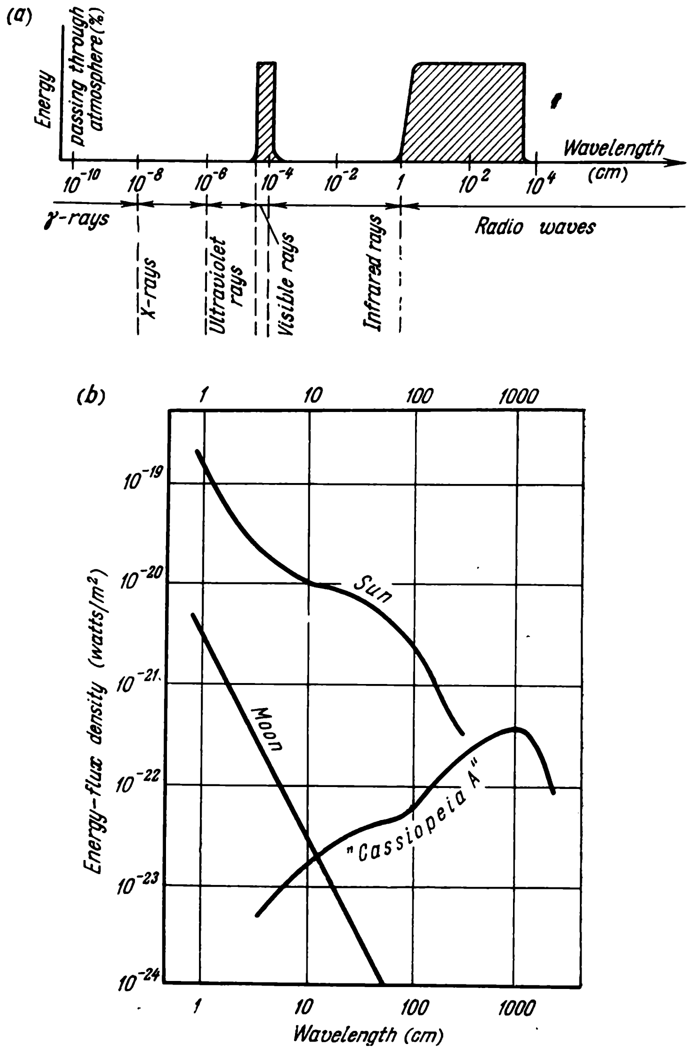

Fig. 118

the antenna is located in the form of a half-wave dipole (see Fig. 120). However, it is difficult to separate the weak cosmic signals from

the receiver noise, and a special method of modulation has to be used that consists in the following. If a flux of energy of some extraterrestrial radio source is periodically interrupted or varied with a definite frequency, then the signals being received acquire the same additional frequency in the receiver. The modulated signals can then be separated from the nonmodulated noises and amplified.

Radio Sextant (block diagram)

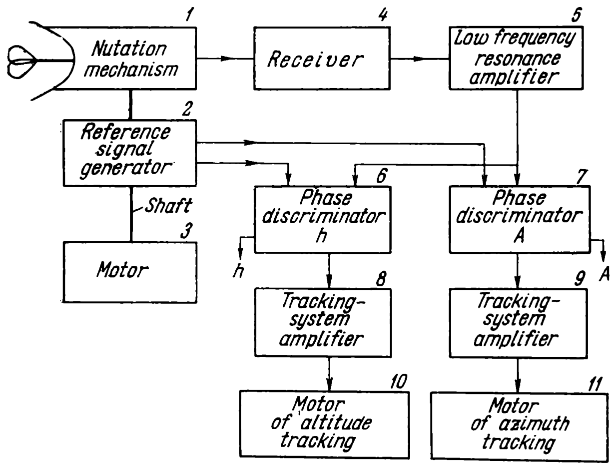

Fig. 119

For this purpose, a special mechanism and motor (*1* and *3* in Fig. 119) feed to the antenna of the radio sextant small "nutation" oscillations with a frequency of 20 to 30 cycles per second.

Due to nutation, the axis of the antenna moves relative to the maximum of emission (the sun, let us say), and this results in modulation of the incoming signal.

The incoming signals are fed, via a waveguide, to receiver *4* of the radio sextant, which is essentially the same as a radar receiver. Here, the radiation impulses are transformed from their frequencies (λ = 0.8-1.9 cm) to an intermediate frequency that allows for amplification. After amplification and rectification, the signal acquires the form of an envelope, that is, it is modulated depending on the nutation oscillations, external interference, etc. The output signal of the receiver is fed to a low-frequency resonance amplifier *5* that passes and amplifies only frequency bands close to the nutation frequency of the antenna. The output signals of the amplifier are fed to two phase discriminators *6* and *7*, whose functions will be discussed below.

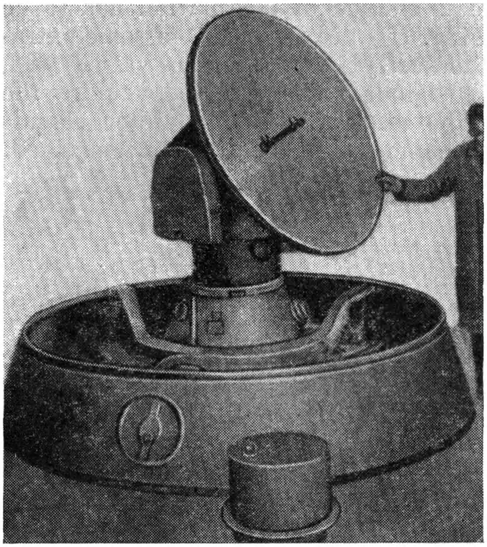

Fig. 120

To track the sun, or follow it in the radio sextant, a tracking system is provided (separately for altitude and azimuth) that operates as follows. When the antenna occupies some specific position relative to the stabilized platform and a diametric plane (the direction of the vertical circle is set by a gyrocompass), the reference-signal generator *2* delivers two signals for altitude h and azimuth A that have a phase difference of 90°; the frequency of the reference signals is equal to the nutation frequency. These signals are fed to a different lead-in of the phase discriminators *6* and *7*; here also are fed the cosmic-radiation signals picked up by the antenna after they have passed through the receiver and amplifier. The discriminators determine the difference between the principal and reference signals, and the output signals of the phase discriminators express deviations of the antenna in h and A from the centre of emission on the sun. These signals are amplified and transformed into commands to the motors *10* and *11* of the tracking system, which correct the position of the antenna. Simultaneously these signals are fed to special averaging systems, and thence to the selsyn indicators that give the antenna position in h and A; these data may be further fed to a computer to obtain φ and λ. All systems of the radio sextant are powered by direct and alternating currents of a variety of voltages, thus requiring transformers, rectifiers and motor-generator sets.

Much attention in recent models of radio sextants is given to the problem of stabilization of the antenna system on a rolling ship. The use of inertial stabilization by means of gyrointegrators has been reported. A gyrointegrator is capable of maintaining a given direction to within 0.5% of the angular velocity of the earth and with a systematic error in the drift of the gyroscope axis of about 3′ per hour. The radio sextant introduces corrections for inertial drift of the antenna axis due to the gyrointegrators, while the values of h and A are obtained directly from the gyrointegrators. The entire

antenna system of this type of radio sextant is hung in gimbals made up of several rings and is balanced to a state of equilibrium in order to reduce the effect of accelerations due to rolling of the ship and to wind. These radio sextants have greater precision than earlier models.

In addition to errors in the radiometric part and stabilization errors, radio-sextant readings are greatly affected by atmospheric radiation that causes a drift of the antenna axis and has a maximum at the same frequencies that the radio sextant operates on, and also by radio-refraction. These two effects must be taken into account in the form of corrections that depend (according to recent studies) on the temperature, pressure and humidity of the air and the altitude of the celestial body. Radio-refraction is similar to ordinary optical refraction, and the correction is introduced in a similar manner, though it is somewhat less stable.

The first ship radio-sextant (model AN/SAN of Collins Co., U.S.A.) was designed for sun sights only, and operates on a wavelength of 1.9 cm. It has a 91-cm-diameter antenna reflector. There are also radio sextants with small reflectors (30 to 60 cm) operating on 8.7 mm. They are, however, less stable and more dependent on weather conditions, particularly for small altitudes of the sun. The Collins AN/SRN-4, put out in 1959, has a 1.5-metre reflector and operates on a wavelength of about 2 cm. This radio sextant is for sighting the sun and moon (see Fig. 120).

Studies of the first models of radio sextants have demonstrated that these instruments can function in any weather, and only strong thunder storms reduce the accuracy of their readings. Marine investigations into determining location by radio sextant have yielded a probable (50%) error in altitude of about $\pm 2'$ or $\varepsilon_h \approx \pm 2'.8$ in 222 measurements. Subsequent studies carried out in the Antarctic (aboard the "Arneb") yielded, from 528 observations, a mean deviation of $\pm 2'.7$ from altitudes measured by the conventional sextant, with maximum discrepancies reaching $8'.2$. The chief source of errors was unreliable stabilization of the unit.

The merits of this new navigational instrument are: the possibility of observing the sun and moon under a cloud cover, automatic and continual operation, thus permitting a link-up with computer for continuous determination of φ and λ of position and of compass correction. The drawbacks of the radio sextant are that it is extremely complex and cumbersome, depends on sources of power, is applicable only to a limited number of celestial bodies, and is still rather expensive.

CHAPTER 14

CORRECTING SEXTANT-MEASURED ALTITUDES OF CELESTIAL BODIES

SEC. 70. THE NECESSITY FOR CORRECTING MEASURED ALTITUDES

The astronomical triangle, whose solution is a part of almost every astronomical problem, includes the altitude of the celestial body.

From the definition of altitude given at the beginning of this text, it follows that the arc h, equal to the altitude, lies between the celestial horizon and the place of the celestial body, which is given as a geometric point. But at sea, altitudes are measured above the *visible* horizon, and the direction to the star is also *apparent*; for bodies with a visible disc, the altitude of the limb is measured. It is obvious that before the measured altitude can be used in the astronomical triangle, a number of corrections must be applied. *Correction of altitude is the transition from measured altitudes to so-called true geocentric altitudes.*

Let us now consider in more detail the causes that deflect light rays and the corrections applied to measured altitudes.

SEC. 71. ASTRONOMICAL REFRACTION

When a ray of light coming from a celestial body moves through the earth's atmosphere, it changes its original direction. This is called **astronomical refraction.** Observer A on the surface of the earth will not see the body in the actual. "true" direction AC_1 (Fig. 121) but along AC_2, called the *apparent* direction. *The angle* C_1AC_2 *between the apparent and true directions of the body is called the altitude correction for astronomical refraction* (ρ), *or simply the astronomical refraction* (in Fig. 121, angle ρ is greatly exaggerated).

For the sake of simplicity, let us assume that the earth's atmosphere consists of a series of infinitely thin concentric layers, the optical density of which diminishes with altitude. We determine the path of a ray of light on the basis of the laws of refraction of light (Fig. 122), namely:

(1) the incident ray and refracted ray lie in the same plane as the normal to the line of junction of the media;

(2) when a ray of light passes into a denser medium, it approaches the normal;

(3) if the index of refraction of the medium A is denoted by μ_a, of medium B by μ_b, and the angles of incidence and refraction are

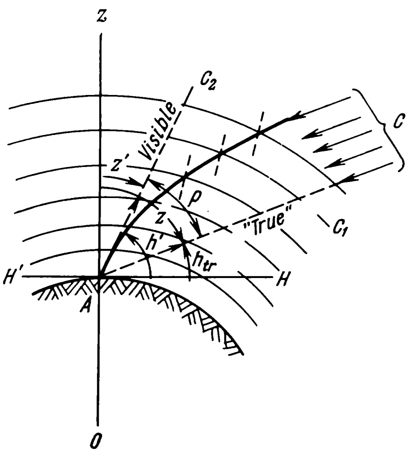

Fig. 121

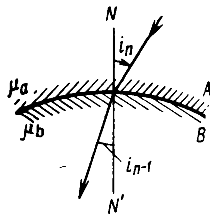

Fig. 122

i_n and i_{n-1}, respectively, then

$$\frac{\sin i_n}{\sin i_{n-1}} = \frac{\mu_b}{\mu_a} \tag{14.1}$$

Applying these laws, it is possible to obtain the value of astronomical refraction for each infinitely small layer.

As a result of refraction, a ray will occasionally describe a curve in the plane of the vertical circle of the body and the observer will see the body closer to the zenith, in the direction of the line AC_2, tangent to the ray. From Fig. 121 it is clear that

$$z = z' + \rho \quad \text{or} \quad h_{tr} = h' - \rho \tag{14.2}$$

that is, the true altitude of the body will be less than the measured altitude, and the astronomical refraction is *always subtracted* from the measured altitude of the body.

For purposes of nautical astronomy, we can derive an approximate formula obtained on the assumption that the layers of air are parallel. This formula disregards the law of change of density with height of layer. As is seen from Fig. 123, the angle of incidence i_n of a ray on the outer layer is equal to the true zenith distance z, while the

angle of refraction i_1 at the ground layer is equal to the apparent zenith distance z', hence

$$z = z' + \rho = i_n \tag{14.3}$$

Denoting the refractive index at the ground layer by μ_1, and at the boundary of the atmosphere by $\mu_n = 1$, we can write (on the

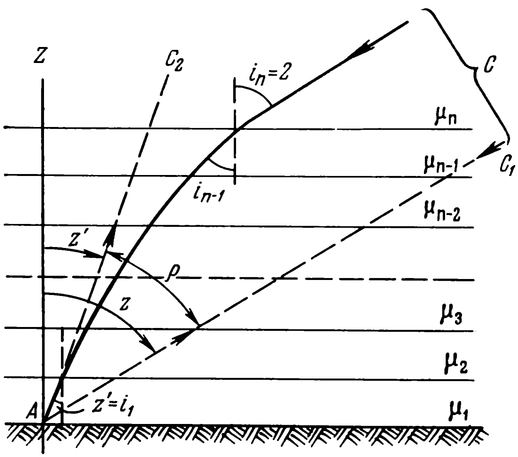

Fig. 123

basis of the third law)

$$\frac{\sin i_n}{\sin i_{n-1}} = \frac{\mu_{n-1}}{\mu_n}; \quad \frac{\sin i_{n-1}}{\sin i_{n-2}} = \frac{\mu_{n-2}}{\mu_{n-1}}; \ \ldots; \quad \frac{\sin i_2}{\sin i_1} = \frac{\mu_1}{\mu_2}$$

Multiplying together the right and left sides of these equations and cancelling, we get

$$\frac{\sin i_n}{\sin i_1} = \mu_1$$

or, taking into account (14.3), we have

$$\frac{\sin(z' + \rho)}{\sin z'} = \mu_1$$

Removing brackets and substituting $\cos\rho \approx 1$, $\sin\rho \approx \rho' \operatorname{arc} 1'$, we get

$$1 + \cot z' \rho' \cdot \operatorname{arc} 1' = \mu_1$$

whence

$$\rho' = \frac{\mu_1 - 1}{\operatorname{arc} 1'} \tan z' \tag{14.4}$$

At a temperature of $t = +10°C$ and a pressure of $B = 760$ mm, the refractive index μ_0 obtained on the basis of observations is $\mu_0 = 1.0002916$.

The refraction for all these mean conditions is called the **mean astronomical refraction** ρ_0 and after substitution of μ_0 into (14.4), we have

$$\rho_0' = 1'.0026 \tan z'$$

or finally

$$\rho_0' = 1'.00 \cot h' \tag{14.5}$$

Despite the fact that formula (14.5) was obtained on very rough assumptions concerning the structure of the atmosphere, the values of ρ_0' will be sufficiently precise for our purposes for altitudes in excess of 15°; for smaller altitudes, this formula is totally unsuitable. Thus, for $h = 0°$, ρ_0 is equal to infinity from (14.5), whereas the refraction on the horizon is actually about 35′.

In the U.S.S.R. refractions are calculated from tables published by the Pulkovo Astronomical Observatory. These tables are compiled from more complicated and precise formulas based on Gülden's theory.

More exact formulas of refraction are obtained if for each infinitesimal concentric layer of the atmosphere we take its value $d\rho$ expressed by the differential equation $d\rho = \tan z' \cdot f(\mu)\, d\mu$, where z' is the zenith distance in the given layer, $f(\mu)$ is the law of variation of the refractive index.

Integrating this equation over the entire totality of layers from ground (refractive index μ_1) to the boundary of the atmosphere (refractive index μ_n), we get the refraction for an observer A:

$$\rho = \int_{\mu_1}^{\mu_n} \tan z' \cdot f(\mu)\, d\mu$$

It has been established experimentally that $\mu - 1 = K\delta$, where δ is the optical density of the air, and K is an empirical coefficient. Therefore, the refraction will depend on the laws of variation of air density with height of layer and with changes in the environment. All refraction formulas have been obtained on the basis of various hypotheses concerning changes in air density; however, due to the complexity of this phenomenon, the best results are given by tables compiled from semi-empirical formulas.

For changes in the temperature and pressure of the air relative to the values given above, corrections are introduced into the mean astronomical refraction for temperature and pressure, $\Delta\rho_t$ and $\Delta\rho_B$. These corrections take into account air density changes in accord

with the formula

$$\mu - 1 = K\delta_0 \frac{B}{760} \times \frac{273^\circ}{273^\circ + t}$$

where δ_0 is the density of the air at $t = 0^\circ$ and $B = 760$ mm; physical determinations yield $\delta_0 = 0.0012928$.

After applying these corrections to the mean refraction we get a quantity closer to the real value of the refraction, the so-called **true refraction**, ρ, that is

$$\rho = \rho_0 + \Delta\rho_t + \Delta\rho_B \tag{14.6}$$

In nautical tables, the mean astronomical refraction is given in Table 12, MT-63, and the corrections for temperature and pressure in Tables 14a and 14б, MT-63.

From these tables it is seen that for small altitudes of the body, the refraction varies very rapidly, and the values are very considerable. One has to take into account these peculiarities of astronomical refraction when studying the rising and setting of celestial bodies, when correcting small altitudes of bodies and in other cases. The compression of the disc of the sun and moon in the vertical direction at about rising and setting time is also due to a rapid change in refraction with altitude.

For a small altitude, the rays from a body pass through a maximum thickness of atmosphere, a considerable portion of which is directly above the earth's surface. The density and temperature of the air here are subjected to perceptible fluctuations, and so the actual values of refraction for altitudes less than 2° may differ somewhat from the tabular values. In addition, an important factor when correcting small altitudes is the dip of the horizon, which is usually not accurately known.

From the foregoing it follows that corrections of altitudes of celestial bodies that are less than 2°-3° are not always reliable; one should be cautious about these altitudes.

SEC. 72. DIP SHORT OF THE HORIZON, THE DIP OF THE HORIZON

Terrestrial refraction. Light rays coming from objects on the earth's surface or near it are refracted just like rays entering the atmosphere from celestial bodies. *This displacement in the atmosphere of a ray of light coming from a terrestrial object is called terrestrial refraction.*

Observer A (Fig. 124) does not see an object B along the line AB, but along the curved ray AaB, or along the tangent AB' raised above AB by an angle ρ_1. This angle ρ_1 is what is usually called the terrestrial refraction.

Similarly, observer B will see object A along the line BA' raised through an angle ρ_2, which is not, in the general case, equal to the angle ρ_1. However, for an approximate solution (which is permissible in navigational problems) we assume the curvature of the ray AaB to be constant. Then $\rho_1 = \rho_2 = \rho$ and the path of the ray will be the arc AB of radius R_1 which is considerably greater (6- to 7-fold) than the earth's radius. The angle 2ρ, equal to the central angle O_1. will be (in radians):

$$2\rho = \frac{\text{arc } AB}{R_1} \qquad (*)$$

The arc $A_1B_1 = D$ (on the earth's surface) is proportional to the central angle C, that is,

$$D = R_E \cdot C \qquad (**)$$

where R_E is the earth's radius.

Due to the small difference between the arcs AB and D, we can put

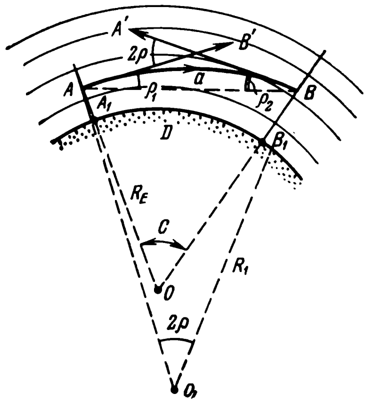

Fig. 124

arc $D \approx$ arc AB and after substitution of (**) in (*), we have

$$\rho = \frac{1}{2} C \cdot \frac{R_E}{R_1}$$

The ratio of radii $\frac{R_E}{R_1}$ is called the coefficient of terrestrial refraction K.

We then finally have

$$\rho = \frac{1}{2} K \cdot C \qquad (14.7)$$

or the magnitude of the angle ρ in minutes of arc is proportional to the distance C between the objects expressed in nautical miles.

The coefficient K depends on the optical density of the air in the lower layers of the atmosphere. Since the conditions of air masses are constantly changing at sea level, the air density and, hence, the curvature of ray AB is constantly changing, and this changes the coefficient K. With the exception of fluctuations due to local conditions and temperature, the coefficient K has rather regular diurnal and annual variations, which have been studied mainly on land. During a 24-hour period, K varies from maximum at sunrise to a mean value of 0.16 at noon, and then again increases, reaching a maximum at sunset. For this reason, the visibility of low objects is better

at sunrise and sunset, and this is utilized by navigators when approaching low shores. The variability of K under sea conditions has not yet been studied very thoroughly. K has been observed to fluctuate from 0.01 to 0.36. For coastal conditions the mean value of K is usually taken at 0.14, for sea conditions, 0.16.

Dip short of the horizon and dip of the horizon. At sea, the altitudes of bodies are measured above the *apparent* (*visible*) horizon, that is, above the line along which the surface of the sea is projected on the celestial sphere. This line is a small circle on the surface of

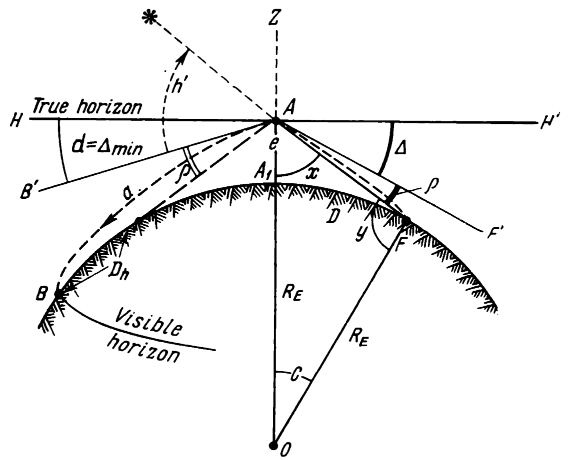

Fig. 125

the sea described by the line of sight AaB (Fig. 125) by which observer A at height of eye e sees the most distant point B of the sea surface. The arc A_1B is roughly equal to the arc AaB and is the distance of the visible horizon D_h, while the *vertical angle* HAB' *between the plane of the true horizon* HH' *and the tangent* AB' *to the ray to the visible horizon is called the dip of the sea horizon* d, or simply *the dip of the horizon* (or the dip). To bring the observed altitude to the true horizon it is obviously necessary to subtract angle d from it. Due to the action of terrestrial refraction, the visible horizon will be farther away than the "geometric" horizon, or the horizon formed by the tangent to the earth's surface, and its dip d will be less than the "geometric" dip. From the figure it is seen that the higher the height e of the observer's eye, the greater the dip, but if $e = 0$, then $d = 0$ and the visible horizon coincides with the true horizon.

Now if the line of sight is not directed at the visible horizon, but at a closer point of the sea, say F, then the angle Δ *between the tangent* AF' *to the ray directed at a close point of the sea, and the plane*

of the true horizon is called the dip short of the horizon. Obviously, inclination of the line of sight is the more general case, a particular instance of which is the dip, when point F is at limiting distance. Then the dip short of the horizon is minimal and equal to the dip d of the horizon, thus $\Delta_{min} = d$.

Deriving the formula for the dip short of the horizon. Let us derive the formula for the dip short of the horizon Δ as the more general one, from which we readily obtain formulas for dip and horizon distance. Besides, the formula for Δ will itself be needed when correcting altitudes measured above the shoreline.

Let us obtain the quantity Δ as a function of the height of eye, distance of point F, and refraction.

From the plane triangle AOF (Fig. 125), in which the angles A and F are denoted by x and y, we can write on the basis of the tangent theorem:

$$\frac{\tan\frac{y-x}{2}}{\tan\frac{y+x}{2}}=\frac{\frac{1}{2}(AO-FO)}{\frac{1}{2}(AO+FO)}=\frac{\frac{1}{2}(e+R_E-R_E)}{\frac{1}{2}(e+R_E+R_E)}=\frac{\frac{e}{2}}{R_E+\frac{e}{2}}$$

whence

$$\tan\frac{1}{2}(y-x)=\tan\frac{1}{2}(y+x)\frac{\frac{e}{2}}{R_E+\frac{e}{2}} \tag{14.8}$$

From this expression we determine the angle x, which subsequently is easily related to the desired Δ.

Replace angle $x+y$; to do this, we find from the triangle AOF

$$x+y+C=180^\circ$$

or

$$\frac{x+y}{2}=90^\circ-\frac{C}{2} \tag{14.9}$$

Substituting (14.9) in (14.8) and neglecting $\frac{e}{2}$ in the denominator (it is less than the accuracy with which R_E is computed) we get

$$\tan\frac{1}{2}(y-x)=\cot\frac{C}{2}\times\frac{\frac{e}{2}}{R_E}=\frac{1}{\tan\frac{C}{2}}\times\frac{e}{2R_E} \tag{14.10}$$

Due to the smallness of the angles $\frac{1}{2}(y-x)$ and $\frac{1}{2}C$, we replace the tangents by the first terms of their expansion in a Taylor's

series (in radian measure):

$$\tan\frac{1}{2}(y-x)=\frac{1}{2}(y-x)+\ldots;\quad \tan\frac{C}{2}=\frac{C}{2}+\ldots$$

Putting these expressions into (14.10), we have

$$\frac{1}{2}(y-x)=\frac{1}{\frac{C}{2}}\times\frac{e}{2R_E}=\frac{e}{CR_E} \tag{14.11}$$

Subtracting expression (14.11) termwise from (14.9), we determine the value of the angle x:

$$x=90^\circ-\frac{C}{2}-\frac{e}{CR_E} \tag{14.12}$$

Considering the angle OAH', we can write

$$x+\rho+\Delta=90^\circ$$

or

$$x=90^\circ-\Delta-\frac{1}{2}K\cdot C \tag{14.13}$$

where ρ is substituted by the expression (14.7).

Equating the right-hand sides of expressions (14.12) and (14.13), we have

$$90^\circ-\frac{C}{2}-\frac{e}{C\cdot R_E}=90^\circ-\Delta-\frac{1}{2}K\cdot C$$

whence

$$\Delta=\frac{C}{2}(1-K)+\frac{e}{C\cdot R_E} \tag{14.14}$$

This equation is the theoretical formula for dip short of the horizon.

Expressing Δ and C in minutes of arc, e in metres and taking the earth radius in metres to be 1,852: arc 1′,* we get

$$\Delta'\text{ arc }1'=\frac{C'\text{ arc }1'}{2}(1-K)+\frac{e_{\text{metres}}}{C'\cdot 1{,}852}$$

or, after substituting the numerical values: $\text{arc }1'=\frac{1}{3{,}438}$ and the mean value $K=0.16$, we finally obtain the *formula for the inclination of line of sight*:

$$\Delta'=0'.42C+1'.856\frac{e_{\text{metres}}}{C} \tag{14.15}$$

---

* Here, the earth radius is taken to be that of a sphere, the length of one minute of the meridian of which σ is equal to the standard nautical mile of 1,852 metres. Suppose angle C at the centre of the earth is expressed in terms of the arc σ on the surface in radians (fractions of radius), or $C=\frac{\sigma}{R_E}$; taking $\sigma=$ one nautical mile, $C=\text{arc }1'$, we get $R_E=\frac{1{,}852\text{ metres}}{\text{arc }1'}$.

where C is the distance to the object (the shoreline, for example) in nautical miles.

Formulas for dip of the horizon and its distance. It will be recalled that the minimum dip short of horizon is equal to the dip of the horizon, that is, $d = \Delta_{min}$. We find the minimum of expression (14.14). To do this, equate the derivative $\frac{d\Delta}{dC}$ to zero:

$$\frac{d\Delta}{dC} = \frac{1}{2}(1-K) - \frac{e}{C_{max}^2 R_E} = 0 \tag{14.16}$$

Then the angle C will take on its greatest value, equal to the *distance of the visible horizon* D_h. Thus, from expression (14.16) we obtain

$$C_{max} = D_h = \sqrt{\frac{1}{1-K}} \times \sqrt{\frac{2e}{E_E}} \tag{14.17}$$

Substituting D_h for C in (14.14) and noting that $\Delta_{min} = d$, we obtain

$$\Delta_{min} = d = \frac{D_h}{2}(1-K) + \frac{e}{R_E D_h} = D_h \left(\frac{1-K}{2} + \frac{e}{R_E D_h^2}\right)$$

or after substituting the values of D_h from (14.17) and some simple manipulation we have

$$d = \sqrt{1-K} \times \sqrt{\frac{2e}{R_E}} \tag{14.18}$$

Expressing d in minutes of arc: d' arc $1'$, e in metres and taking the earth radius as before at 1,852 m : arc $1'$, we get

$$d' \quad \frac{\sqrt{1-K}}{\text{arc } 1'} \times \sqrt{\frac{2e_{\text{metres}} \text{ arc } 1'}{1{,}852 \text{ metres}}} = 1'.927 \sqrt{1-K} \times \sqrt{e_{\text{metres}}} \tag{14.19}$$

or the dip depends on the height of the observer's eye and the value of the coefficient of terrestrial refraction K.

If we take the mean value of K, which corresponds to the mean atmospheric conditions, $K = 0.16$, we finally get

$$d' = 1'.766 \sqrt{e_{\text{metres}}} \tag{14.20}$$

In nautical tables (MT-63), formulas (14.15) and (14.20) are used to compute special tables: 11a and 11б.

If in (14.17) we express D_h in minutes (nautical miles, that is) and take the above-indicated values of K and R_E, we will get a familiar navigational formula for the distance of the visible horizon:

$$D_h = 2.102 \sqrt{e_{\text{metres}}}$$

The indicated tabular value of dip corresponds to the mean state of the atmosphere. Actually, the state of the atmosphere and, consequently, the coefficient of terrestrial refraction K, may differ considerably from the mean at the time of observation; for this reason, the actual values of dip may differ appreciably from tabulated values; for instance, deviations of $1'$ to $2'$ are rather frequent. Occasionally, they attain greater values.

Investigations have shown that the dip depends on the temperature difference of air and water, the humidity, air pressure, wind, local conditions, and others. Numerous attempts to establish an empirical formula that would take into account the principal cause (temperature difference between air and water) have not yielded satisfactory results, and the formulas obtained are sometimes not even as good as the tabulated values.

Extensive experience has established that:

(1) The dip in open sea for stable hydrometeorological conditions is approximately in accord with the tabulated value.

(2) In enclosed seas and coastal regions the dip is often (and sometimes rather substantially) different from the tabulated value. For instance, appreciable deviations of d are observed in the Red Sea, along the western coast of America where the Andes come close to the shore, and in other places.

(3) The dip differs considerably (up to $15'$) from the tabulated value after the passage of a squall.

(4) At points of encounter of large ocean currents with different temperatures or where currents come to the surface, we constantly find d deviating from the tabulated value. The following characteristic regions have been defined:

(a) south of Newfoundland on the boundary of the Gulf Stream;

(b) near the western coast of Africa from the Cape of Good Hope up to the Congo River and from Cape Blanco to the town of Mogador;

(c) near the islands of Lofoten.

In some regions there are seasonal deviations of d from the tabulated values, for example:

(a) in summer in polar seas, especially on the ice fringe;

(b) in spring in the northern seas (White, Barents, and others);

(c) in spring and at the beginning of summer in moderate ocean areas.

Thus, the tabulated value of the dip can very often give errors in corrected altitude. To avoid such errors, it is best to determine the actual value of the dip at a given instant by means of a special angle-measuring instrument called a dipmeter. In certain cases, the error due to the dip may be eliminated from the final result together with other systematic errors by special observational procedures.

SEC. 73. ESSENTIALS OF THE DIPMETER

Many specialized instruments have been proposed for measuring the actual magnitude of the dip of the horizon. Some of them are devices attached to the sextant for measuring the dip, others are separate instruments called **dipmeters.** Only dipmeters have gained popularity as being more convenient and accurate than sextant attachments.

Essentially, all these instruments are based on measuring the vertical angle H_1AH_1 (Fig. 126) between directions to opposite sides H_1 and H_1 of the visible sea horizon.

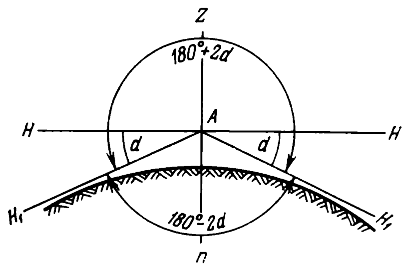

Fig. 126

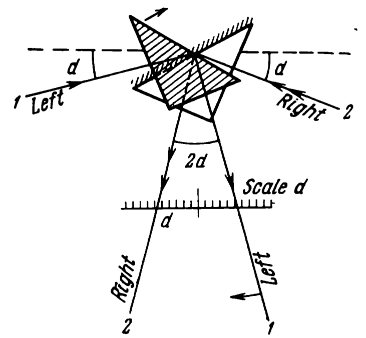

Fig. 127

When measured through the zenith, this angle will be equal to $180° + 2d$; when measured through the nadir it is $180° - 2d$. Whence we can obtain d. It is assumed here that the dip in opposite directions is the same. As experience shows, this assumption is not always true, but in the open sea detected variations of dip in azimuth have been within the limits of accuracy of nautical observations.

Dipmeters, being instruments for observations at sea, must be designed to measure by hand the above-indicated angles on a rolling ship. For this purpose, dipmeters use a reflecting optical system in which rays from opposite sides of the horizon are reflected from the faces of two prisms and then enter the field of view of the observer simultaneously at an angle of $2d$ to one another (Fig. 127). Placing a scale graduated in units of d near the movable prism and turning one prism relative to the other until rays *1* and *2* coincide, we can read the value of dip d on the scale.

The above principle is used in the Pulfrich dipmeter.

In the Soviet dipmeter (designed by Professor Kavraisky), the rays are moved not by rotating the prism, but by turning the entire instrument about the longitudinal axis, the horizon images in the

field remaining parallel to each other. As the instrument is turned, the images of the right and left horizons move different distances in the field of view due to the fact that the left and right objective lenses have different magnifications.

The optical system of Kavraisky's dipmeter (*HK*) and the path of the light rays are shown in Fig. 128. The distant (relative to

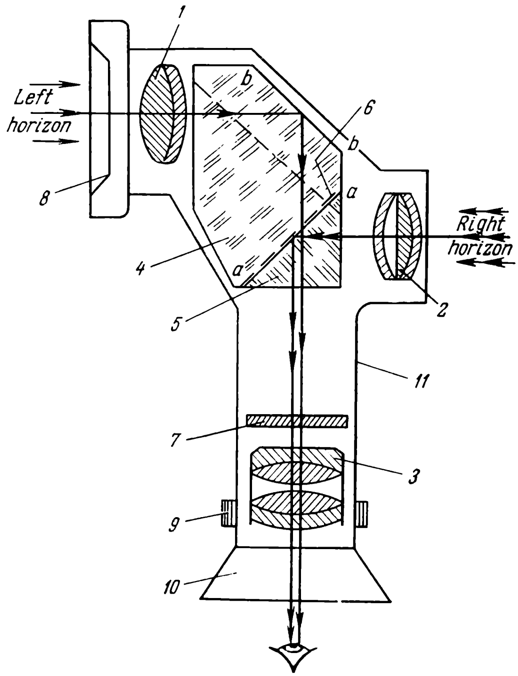

Fig. 128

eyepiece) objective lens *1* gives a magnification of 4.7×; the closer objective lens *2* magnifies 4.1×; the eyepiece *3*, which consists of two systems of lenses, is set to the eye of the observer by means of the dioptric ring *9*. The rays from opposite parts of the horizon are bent by prisms *4* and *5*, which are glued together along the face *a-a*. Prism *4* is a complex polyhedral roof-like prism in which the faces that meet along edge *b-b* intersect at 90°, which means that they are inclined ±45° to the plane of the figure. By virtue of this design of the roof-prism, rays coming from the left horizon not only change direction by 90°, but also turn through 180°; they are inverted, and as a result, the image of the left horizon is "direct" in the field of view, while the image of the right horizon reflected

from the face a-a of the ordinary rectangular prism 5 will be inverted (see Fig. 129).

The face a-a of the roof-prism 4 has an aluminium coating in the form of strips 6 that are perpendicular to the plane of the figure

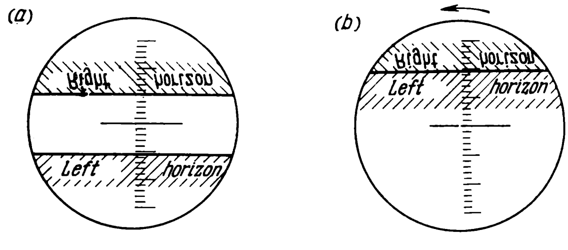

Fig. 129

and equal in magnitude to the blank spaces between them. As a result, roughly one half of the rays from the left horizon passes between the strips and enters the observer's field of view. At the same time, the other half of the rays from the right horizon is reflected from the strips and also enters the observer's field. Both right and left horizons are thus visible at the same time. Both horizons are projected onto a glass 7 that contains a scale graduated in divisions of $1'$.

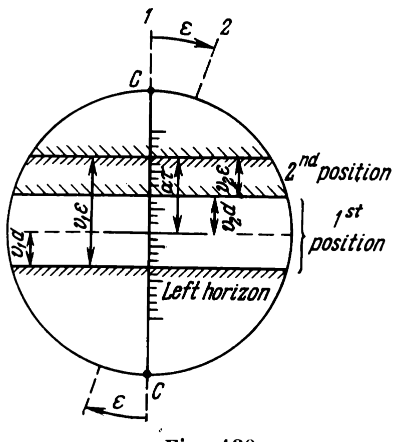

Fig. 130

With this type of optical system, the images of the horizons in the field of view remain parallel to one another and perpendicular to the base of the scale c-c (see Fig. 130) when the instrument is in a horizontal position.

The brightnesses of the two portions of the horizon may differ, the brighter one making the fainter one invisible. The more distant objective lens is supplied with a diaphragm 8 for the purpose of adjusting brightness. The eyepiece has a rubber blinker 10 to protect the eye from extraneous rays. All parts of the dipmeter are rigidly mounted in the body 11.

Fig. 129a shows the scale and images of left and right horizons for a horizontal position of the instrument, while Fig. 129b shows them after rotation of the instrument through some angle about

the longitudinal axis. The magnitude of d is given by the scale reading along the line of coincidence of the horizon images.

The dip of the horizon thus measured includes the constant position error of the zero of the scale (which we denote by x). To eliminate this error from the result, it is necessary to make two measurements: through the zenith and the nadir. To do this, invert the instrument.

The scale in the Kavraisky dipmeter is designed and positioned so that the reading in the first instance is $d_1 = d + x$, and in the second, $d_2 = d - x$; the zero error of the instrument is eliminated in the half sum of the readings:

$$d = \frac{d_1 + d_2}{2} \tag{14.21}$$

where d is the value of the dip without the zero error.

Operating the Kavraisky dipmeter. The dip may be measured either before or after observations of celestial objects, depending on the visibility of the horizon. Before taking observations, focus the dipmeter to your eye. Select a spot at the same height as that used to measure the altitude of the body, and one from which opposite parts of the horizon are visible. If visibility of the horizon is poor, lower the height of eye; this will improve visibility. The dipmeter is held at the eye of the observer horizontally with the far objective lens towards the brighter part of the horizon (let us take it on the left) and visibility of horizons is balanced with the aid of the diaphragm. Turn the dipmeter round the longitudinal axis and align the images of the horizons; at the line of coincidence, read d_1 on the scale. The dipmeter must be horizontal; if its axis is tilted, the images of the horizons will not be perpendicular to the base of the scale. To take the second observation, invert the dipmeter with the far objective lens to the right, then turn through 180° so that the far objective lens is again directed at the bright part of the horizon. The second reading (d_2) is obtained like the first. After taking the measurement, compute the magnitude without the zero error from (14.21).

The sign of d is determined from the instrument scale: if horizon alignment is above the mean division of the scale, the dip has a normal value and is subtractive from the measured altitude, but if alignment occurs below the mean division (which happens very rarely) then the dip is additive to the measured altitude.*

The accuracy with which dip is measured with the Kavraisky instrument depends on the conditions of observation and the skill of the observer. Under good conditions, in the absence of rolling,

* The visible horizon is higher than the true horizon.

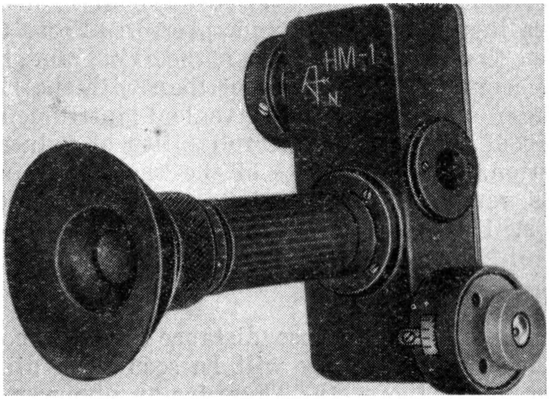

Fig. 131

wind, vibration, and with good visibility the mean square error of one complete measurement will be of the order of $\pm 0'.2$-$0'.3$. For a very rough-to-high sea (magnitude 5-6), the error increases to $\pm 0'.5$. For a relatively inexperienced observer in average (that is, real) sea conditions, the error will be of the order of $\pm 0'.8$.

A drawback of Kavraisky's dipmeter is the difficulty involved in aligning the horizons with a simultaneous reading of the scale when the ship is rolling and vibrating and there is a wind, all of which makes the instrument shake in the hand producing all kinds of tilting. It is extremely difficult in such conditions to hold the horizons in coincidence for the time necessary to take a reading, and this naturally reduces the accuracy of the measurement appreciably. In this connection, Soviet manufacturers began the production of a new dipmeter (HM-1, Fig. 131), which is an intermediate type between the Pulfrich and Kavraisky instruments and the new type HM-5. In type HM-1, the prism block closest to the micrometer drum is rotatable, as a result of which the horizons are aligned by rotating the micrometer drum, which gives the dip reading, and so there is no scale in the field of view. Otherwise, the operating principle and diagram of the instrument are the same as those of the Kavraisky dipmeter.

The HM-1 dipmeter is operated in exactly the same way as the HK instrument (with the exception of rotation of the instrument).

In this dipmeter, the images of the horizons after alignment by means of the micrometer will not diverge, thus simplifying observations in a rolling ship. Besides, the horizons are aligned in the centre of the field of view (in the HK instrument they are

aligned out of the centre), and evaluation of divisions and reading of the dip are less subjected to human errors. From the foregoing it will be seen that observations with the HM-1 and HM-5 dipmeters are more accurate and convenient than with the HK dipmeter.

Despite these comparatively convenient instruments for measuring the dip of the horizon, the reliability of values obtained is open to question. Further studies of the variability of dip in azimuth and its variability in different areas for different times of the year are still needed.

SEC. 74. DIURNAL PARALLAX

When observing a body whose distance away is comparable to the size of the earth, the body will be seen from different angles at different points of the earth. Thus, for an observer A (Fig. 132) moving with the diurnal rotation of the earth to positions A_1, A_2,

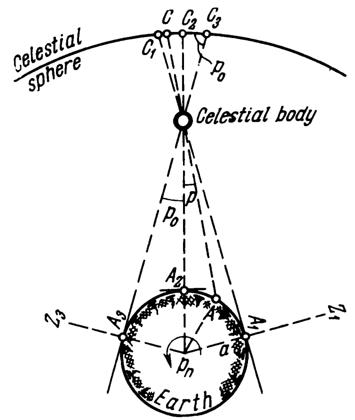

Fig. 132

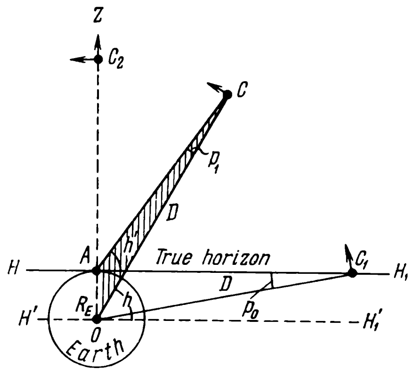

Fig. 133

A_3, the celestial body will be projected on the celestial sphere at the points C_1, C_2, C_3; in other words, we have a parallactic shift, and the coordinates of the body will therefore change.

The diurnal parallax p is the angle at which the radius of the earth (corresponding to the site of observation) is seen from the centre of the celestial body, or, more precisely, the arc CC_2 on the celestial sphere between the geocentric position of the body C_2 and the place of the body C as seen from the earth's surface.

From Fig. 132 it is seen that for an observer at the point A_1 the star will be on the horizon and the diurnal parallax will be greatest. For an observer at A_2, the star will be in the zenith and the diurnal parallax will be zero.

Only bodies of the solar system whose distances away are relatively small exhibit a diurnal parallax. This phenomenon is similar to the annual parallax of stars, the only difference being that the observer moves a distance of one earth radius and the period is diurnal. If all coordinates are referred to the centre of the earth, that is, if we obtain geocentric coordinates, then the diurnal parallax will have no effect on them. Almanacs give the geocentric coordinates for bodies of the solar system. But altitudes measured from the surface of the earth are subjected to the effects of the diurnal parallax and must therefore be referred to the centre of the earth before they can be introduced into computations.

In Fig. 133, let the angle $CAH_1 = h'$, the altitude of the body for an observer at the earth's surface, and the angle COH_1' be the geocentric altitude of the same celestial body; this is the altitude referred to the centre of earth. We then have

$$h = h' + p \tag{14.22}$$

The value of p for a body on the horizon is called the horizontal parallax. Since the shape of the earth is much like an ellipsoid of revolution, the greatest horizontal parallax will be found for an observer on the equator, a distance away from the centre of the earth equal to the semimajor axis a. This maximum parallax is called the equatorial horizontal parallax p_0, where the subscript 0 is replaced by the symbol of the celestial body. But since the difference between the equatorial and polar parallaxes even for the moon does not exceed $0'.2$, tables give the equatorial horizontal parallax p_0 for all observers.

When the body has an intermediate altitude, its parallax p will obviously be greater than zero but less than the horizontal parallax p_0. From the triangle CAO (Fig. 133) we obtain

$$\frac{\sin p}{R} = \frac{\sin(90° + h')}{D}$$

whence

$$\sin p = \frac{R}{D}\cos h' \tag{14.23}$$

where R is the radius of the earth;
D is the distance of the body.

But from the triangle C_1AO we have

$$\frac{R}{D} = \sin p_0$$

Substituting this value in the preceding formula, we have

$$\sin p = \sin p_0 \cdot \cos h'$$

Due to the smallness of the angles p and p_0, put the first terms of a series expansion in place of the sines, we then finally get

$$p = p_0 \cos h' \qquad (14.24)$$

The angle p is called the diurnal parallax of the body at an altitude, and it is applied to correct altitudes on the basis of formula (14.22).

A more precise formula is obtained if in the expression (14.23) in place of the earth's radius R we take the value of the radius of the vector ρ of a given point of the earth as derived in navigation: $\rho = a(1 - \alpha \sin^2 \varphi)$, where a is the semimajor axis, α is the oblateness of the earth, and φ is the latitude of the place.

Then $\sin p = \frac{a}{D}(1 - \alpha \sin^2 \varphi) \cos h'$ and, substituting $\frac{a}{D}$ by $\sin p_0$, we get

$$\sin p = \sin p_0 (1 - \alpha \sin^2 \varphi) \cdot \cos h' \qquad (14.25)$$

The closer a body is to the earth, the greater its diurnal parallax. Thus, the value of p_0 for the sun is an average of 0′.15; for the moon, about 57′ (from 61′.5 to 53′.9), for Venus, from 33″ to 5″; for Mars, from 24″ to 4″, and so forth. The values of p_0 for the sun, planets and the moon are given in the daily tables of the MAE for 0h Greenwich time (for the moon, an additional value is given for 12h).

SEC. 75. SEMIDIAMETERS OF CELESTIAL BODIES

When measuring the altitude of the sun and moon, the horizon is brought to coincidence with the limb of the body and not its centre, so that we get the altitude of the limb of the body. But the MAE and other almanacs give the coordinates of the centre of the

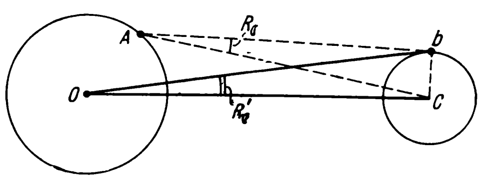

Fig. 134

body. Obviously, the measured altitude of the limb of the body must be referred to its centre too. To do this, add the apparent angular radius R of the body to the measured altitude of the lower limb, and subtract R from the altitude of the upper limb.

The apparent angular radius R, or *the semidiameter*, *of the body is the angle* bAc (Fig. 134) *at which the radius of the body is seen from*

the earth's surface. But the MAE gives the central true angular radii (semidiameters) R', which are the angles bOc referred to the centre of the earth. For the sun and the planets, the difference $R' - R$ is negligibly small, for the moon however it reaches 0'.3; for this reason, tables for correcting the altitude of the moon include the computed apparent radius of the moon as a function of its parallax and its altitude. The MAE does not take into account the effect of refraction.

The refraction effect here lies in the fact that rays coming from the centre and from the upper and lower limbs of the body are refracted differently, as a result of which the apparent radii of the sun and moon differ slightly from those given in the MAE. This difference, for altitudes less than 15°, is sensible. Thus, for small altitudes of the sun, one should not take the measured vertical radius as equal to that given in the MAE, and when correcting the altitude one should introduce only R from the MAE or tables.

The apparent angular radius of the sun varies from a maximum 16'.3 on about 3 January to a minimum 15'.8 at about 3 July, yielding an average $R_{\odot} = 16'.0$. For the moon, the angular radius varies from 16'.8 (at about new moon) to 14'.7 (at about full moon); the average angular radius of the moon is 15'.5.

The semidiameters of the planets are neglected in nautical astronomy due to their smallness and to the fact that the centre of the apparent disc of the planet is aligned with the horizon.

ERRORS DUE TO IRRADIATION

Experiment has shown that a bright body on a dark background appears to the observer's eye to be greater than its actual dimensions. This is due to an optical illusion called *irradiation.* When the bright sun is viewed on a comparatively dark sky background, the disc of the sun appears exaggerated (about 0'.6 on either side, according to certain authors). Similarly, a bright sky appears to "encroach" somewhat on the dark background of the sea (also about 0'.6). Due to irradiation, the stars do not appear as points, but rather as bright spots. In observations of the lower limb of the sun, errors due to irradiation of the solar disc and the horizon just about balance, and no corrections are needed. But if the upper limb of the sun is observed, the corrected altitude may have a total error due to increased disc (0'.6) and "depression" of the horizon (0'.6). In British and American tables and the MAE, this error is taken care of by a special correction for irradiation for the upper limb of the sun equal to —1'.2. This correction is not introduced for other bodies. Since this phenomenon has not been sufficiently verified, it is best not to take sights of the upper limb of the sun.

SEC. 76. CORRECTING THE ALTITUDES OF BODIES MEASURED ABOVE THE VISIBLE HORIZON

"Correcting altitude", as we have noted, is the transition from a sextant reading to the true geocentric, or observed, altitude. In each case we take into account the corrections necessary for the given specific measurement. The general sequence for making corrections and the appropriate terminology are as follows:

(1) the sextant reading (*sr*) or (*sext. alt.*) corrected by the index correction (i) and the instrument correction (s) taken from the cer-

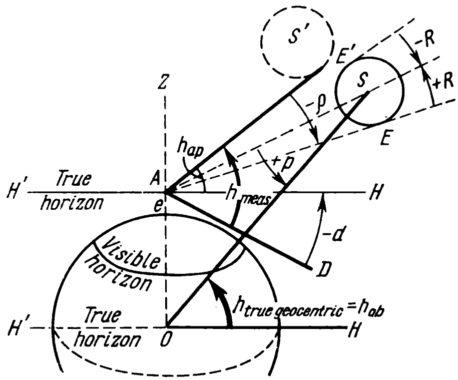

Fig. 135

tificate of the sextant is called the *measured altitude* h' of a celestial body or of its limb ($h' = h_{meas}$ = angle $E'AD$ in Fig. 135);

(2) the measured altitude corrected by the dip d is called the *apparent altitude* (h_{ap}) of the body or of its limb (h_{ap} = angle HAE');

(3) the apparent altitude corrected by the astronomical refraction ρ is called the *true altitude* of the body or of its limb (angle HAE);

(4) the true altitude of the limb of the sun or the moon corrected by the angular radius R is called the *true altitude of the centre* of the body (angle HAS);

(5) the true altitude of the centre of a body (the sun or moon) corrected for parallax p is called the *true geocentric altitude of the centre* of the body (angle SOH) or the *observed altitude*, and is denoted by h or h_{ob}. The signs of all corrections are shown in Fig. 135 for the ordinary case.

Bringing all corrections together, we get the following general formula for altitude correction:

$$h = sr + i + s + (-d) + (-\rho_0) + p + (\pm R) + (\pm \Delta\rho_t) + (\pm \Delta\rho_B) \quad (14.26)$$

where $\Delta\rho_{t,\,B}$ are corrections to altitude due to variation of the mean refraction ρ_0 with temperature and pressure (for $h < 30°$)

$+R$ is for correcting the altitude of the lower limb of the body

$-R$ is for correcting the altitude of the upper limb.

I. CORRECTING ALTITUDES OF THE SUN

To correct the altitude of the sun on the basis of the basic formula (14.26) (disregarding $\Delta\rho_{B,\,t}$), three arguments are needed: height of eye (for d), altitude of celestial body (for ρ_0 and p) and date (for $R_\odot$). To simplify the work, a single table of "general corrections" is compiled (on the basis of two arguments) and a supplementary table of small corrections for the date. To do this, transform (14.26) separately for the lower (14.27) and the upper (14.28) limbs of the sun:

$$h = sr + i + s + (-d - \rho_0 + p + R_{av}) + [\Delta R] \qquad (14.27)$$

$$h = sr + i + s + (-d - \rho_0 + p + R_{av}) - [2R_{av} + \Delta R] \qquad (14.28)$$

where $p = p_\odot \cos h$

$p_\odot = 0'.15$

R_{av} is the average (mean) angular radius of the sun equal to $16'.0$

ΔR is the correction to it for the given date.

The bracketed quantities are combined into a total correction, Δ_{tot}, included in a single table 8a, MT-63, based on the arguments: height of eye in metres and measured altitude of body (from 3° to 90°). The quantities in square brackets are given as additional corrections Δ_{ad} in the tables: 8б, MT-63, for the lower limb; 8в, MT-63, for the upper limb. The date is the argument for entering these tables. The quantities $\Delta\rho_B$ and $\Delta\rho_t$, given in Tables 14a and 14б, MT-63, are considered only for $h < 30°$.

Examples 1 and 2 indicate the outline of computations and designations of corrections.

Example 1. On 21.10 a measurement was made of the altitude of the lower limb of the sun: $sr_\odot = 32°18'.6$, height of eye of observer $e = 8.5$ metres; $oi_1 = 359°31'.2$; $oi_2 = 360°35'.8$; certificate yields $s = 0'.4$. Find h.

	sr	$32°18'.6$
	$i+s$	$-\ 3\ .1$
	h'	$32°15'.5$
Table 8a	Δ_{tot}	$+\ 9\ .4$
Table 8б	Δ_{ad}	$+\ 0\ .1$
	h_{ob} or $h_\odot$	$32°25'.0$

$$i = 360° - \frac{oi_2 + oi_1}{2} = -3'.5$$

Check: $oi_2 - oi_1 = 64'.6$

$R_\odot \times 4 = 16'.1 \times 4 = 64'.4$

Example 2. On 8.11, $sr_{\underline{\odot}} = 7°27'.4$; $e = 9.4$ metres; $oi_1 = 360°32'.2$; $oi_2 = 359°27'.0$; $s = -0'.2$; $t = +1°$; $B = 775$ mm. Find h.

	sr	$7°27'.4$
	$i+s$	$+\ 0\ .2$
	h'	$7°27'.6$
Table 8a	Δ_{tot}	$+\ 3\ .7$
Table 8в	Δ_{ad}	$-\ 32\ .2$
Tables 14a, б	$\Delta\rho_{t,B}$	$-\ 0\ .5$
	$h_{\odot}$	$6°58'.6$

$$i = \frac{-2'.2 + 3'.0}{2} = +0'.4$$

Check: $oi_1 - oi_2 = 65'.2$
$R_{\odot} \times 4 = 64'.8$

Note. For altitudes less than 5°, it is best to enter Table 8 (Δ_{tot}) with $h_{ap} = h' - d$, where d is taken from Table 11б. In this case, enter Table 8 with $e = 0$ metres. When correcting the altitude of the upper limb of the sun, bear in mind the possible correction for irradiation ($-1'.2$).

II. CORRECTING ALTITUDES OF STARS

For the altitudes of stars there are no corrections for semidiameter and diurnal parallax, and so the general formula (14.26) will have the form

$$h_{*} = sr + i + s + (-d - \rho_0) \qquad (14.29)$$

The quantities d and ρ_0 are combined into the total correction Δ_{tot} and given in a single Table 9a, MT-63, the arguments for entering this table are height of eye in metres and the measured altitude of the star. The correction from Table 9a will always be negative.

Example 3. $sr_{*} = 31°43'.4$; $oi = 359°57'.5$; $e = 11$ metres; $s = +0'.4$. Find h_{*}.

	sr	$31°43'.4$
	$i+s$	$+\ 2\ .9$
	h'	$31°46'.3$
Table 9a	Δ_{tot}	$-\ 7\ .5$
	h_{*}	$31°38'.8$

III. CORRECTING ALTITUDES OF PLANETS

The altitudes of planets are corrected by the same table as those of stars, but for Mars and Venus, and sometimes Jupiter, it is necessary to introduce an additional correction for parallax from Table

9б, MT-63, using the arguments: equatorial horizontal parallax taken from the MAE for the date of observation and the altitude of the planet.

Example 4. Date: 13.11.68. Morning measurement of altitude of Venus: mean $sr = 12°33'.4$; $i = +1.8$; $s = -0'.5$; $e = 9.8$ metres; $t = -10°$; B 770 mm. Find $h_♀$.

	sr	$12°33'.4$	
	$i+s$	$+\ \ 1\ .3$	From MAE $p_♀ = 0'.1$
Table 9a	h'	$12°34'.7$	
Table 9б	Δ_{tot}	$-\ \ 9\ .9$	
Tables 14a, б	Δ_{ad}	$+\ \ 0\ .1$	
	$\Delta\rho_{t,B}$	$-\ \ 0\ .4$	
	$h_♀$	$12°24'.5$	

IV. CORRECTING ALTITUDES OF THE MOON

When correcting the altitude of the moon, formula (14.26) takes the form

$$h_☾ = sr + i + s - d - \rho_0 + p_h \pm R_☾ \pm \Delta\rho_{t,B} \tag{14.30}$$

The diurnal parallax of the moon and its angular radius vary rapidly; therefore, they are taken from the MAE for a given date

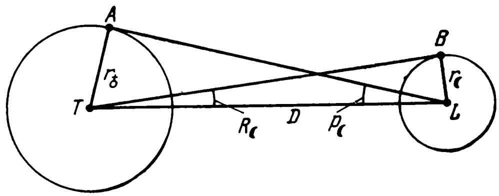

Fig. 136

and are interpolated for Greenwich time, T_{gr}. The result is that (14.30) acquires many arguments, and the construction of a general table becomes complicated. But a relationship can be established between $p_☾$ and $R_☾$ that simplifies the formula for altitude correction.

Take the earth T for a sphere of radius $r_♁$ (Fig. 136), denote the distance of the moon (L) by D, and the linear radius of the moon by $r_☾$. Then from the triangles TAL and TBL we have

$$\sin p_☾ = \frac{r_♀}{D} \text{ and } \sin R_☾ = \frac{R_☾}{D}$$

whence we get

$$\sin R_{\leftmoon} = \frac{r_{\leftmoon}}{r_{\earth}} \sin p_{\leftmoon} = K \sin p_{\leftmoon}$$

The ratio of the linear radii of the moon and the earth $\frac{r_{\leftmoon}}{r_{\earth}} = K$, which is a constant equal to 0.2725.

Due to the smallness of angles $R_{\leftmoon}$ and $p_{\leftmoon}$, we can substitute the first terms of the series for the sines of the angles. We then obtain the following approximate equation:

$$R_{\leftmoon} = 0.2725 p_{\leftmoon} \tag{14.31}$$

The general formula (14.30) in MT-63 is transformed as follows:

$$h_{\leftmoon} = sr + i + s + (-d) + \Delta h_{\leftmoon} + \Delta h_{\underline{\leftmoon}\ (\text{or}\ \overline{\leftmoon})} + \Delta h_{t,\,B} \tag{14.32}$$

Here, $\Delta h_{\leftmoon} = 54'.0 \cos h_{ap} + 0.2725 \times 54' + 0'.26 \sin h_{ap} - \rho_0 - 2'$

where $54'.0$ is the least value of the equatorial horizontal parallax of the moon

$0.26 \sin h_{ap}$ is a correction for the increase of the geocentric angular radius of the moon when passing to the surface of the earth.

This correction is given in Table 10a, MT-63. The corrections $\Delta h_{\underline{\leftmoon}}$ and $\Delta h_{\overline{\leftmoon}}$ are given in Table 10б (lower limb) and in Table 10в (upper limb). Enter these tables with altitude and parallax taken from the MAE for the given hour of T_{gr}. Enter the main table (Table 10a) with altitude only.

In Table MT-53 and also TBA-57, BAC-58, the tables for moon altitudes are based on other formulas:

$$\left.\begin{aligned} \underline{\leftmoon}\ h_{\leftmoon} &= sr + i + s + (-d) + [-\rho_0 + p_{\leftmoon}(\cos h' + 0.2725)] \\ \overline{\leftmoon}\ h_{\leftmoon} &= sr + i + s + (-d) + [-\rho_0 + p_{\leftmoon}(\cos h' - 0.2725)] \end{aligned}\right\} \tag{14.33}$$

The corrections in the square brackets are functions of only two arguments and are given in Table 10a, MT-53, for the lower limb of the moon and in Table 10б for the upper limb.

In formulas (14.32) and (14.33) the correction d is taken from Table 11б, MT-63 or MT-53, for eye height. First introduce this correction into the measured altitude and then with the *apparent* altitude obtained enter Table 10a and Table 10б.

V. CORRECTING ALTITUDE OF BODIES MEASURED "ABOVE THE SHORE LINE"

When correcting altitude measured above the shore line, introduce the correction for "dip short of the horizon" derived above (Sec. 72) in place of the dip correction.

Example 5. **Date:** 17.05.68. $T_{gr} \approx 6^h\,30^m$; $sr_{☾} = 16°21'.8$; $i =$ [illegible]; $s =$ [illegible]; $e = 12.4$ metres; $t =$ [illegible]; $B = 750$ mm. Find $h_{☾}$.

MT-53

	sr	$16°21'.8$
	$i+s$	$-\ 0\ .4$
	$h'_{☾}$	$16°21'.4$
Table 11б	d	$-\ 6\ .2$
	h_{ap}	$16°15'.2$
Table 10б	Δ_{tot}	$+\ 37\ .0$
Tables 14a, б	Δ_{ad}	$+\ 0\ .1$
	$h_{☾}$	$16°52'.3$

MT-63

	sr	$16°21'.8$
	$i+s$	$-\ 0\ .4$
	h'	$16°21'.4$
Table 11б		$-\ 6\ .2$
	$☾\ h_{ap}$	$16°15'.2$
Table 10a		$+\ 61\ .3$
Table 10в		$-\ 24\ .2$
Tables 14a, б		$+\ 0\ .1$
	$h_{☾}$	$16°52'.4$

From MAE at $T_{gr} = 0^h$	$p_{☾}$	$58'.9\ (-0'.9)$
Correction at $T_{gr} = 6^h.5$	$\Delta p_{☾}$	$-0'.2$
	$p_{☾}$	$58'.7$

From formula (14.15) for the inclination of the line of sight

$$\Delta' = 0.42C_{\text{miles}} + 1.856\,\frac{e_{\text{metres}}}{C_{\text{miles}}}$$

reduced to the form

$$\Delta' = 0.042C_{\text{cab leng}} + 18.56\,\frac{e_{\text{metres}}}{C_{\text{cab leng}}} \tag{14.34}$$

we have the Table 11a, MT-63. The arguments for entering this table are: distance to shore line C in cable lengths and eye height in metres. The formula for correcting the altitude of a body will have the form

$$h = sr + i + s - \Delta - \rho + p \pm R \tag{14.35}$$

After applying the correction Δ to the measured altitude, we get the *apparent altitude*; therefore, when subsequently entering the tables for general corrections of the sun, stars and planets, it is necessary to choose the values of Δ_{tot} for zero height of eye, i.e., Δ_0, given in the first column under the argument 'altitude of body'.

When correcting the altitude of the moon, use the correction from Table 11a (in place of Table 11б), otherwise there are no changes.

Example 6. On 10.08.68 altitude of sun measured above "shore line", $sr_{\odot} = 49°27'.5$; $i = +1'.2$; $s = -6''$, $e = 12.5$ metres; radar-determined distance to shore line, $D = 27$ cable lengths. Find $h_{\odot}$.

	sr	49°27′.5
	$i+s$	1 .1
	h'_{shore}	49°28′.6
Table 11a	Δ	— 9 .7
	h_{ap}	49°18′.9
Table 8₀	$(\Delta_{tot})_0$	+ 15 .3
Table 8б	Δ_{ad}	— 0 .2
	$h_{\odot}$	49°34′.0

VI. CORRECTING ALTITUDES OF BODIES MEASURED "VIA THE ZENITH"

Above it was mentioned that via the zenith (or back-sight observations) is a method used in measuring mainly the altitude of the sun, sometimes also the moon. We shall therefore consider correcting the altitude of the sun. If altitude corrections for other celestial bodies are needed, the procedure is fully analogous.

In this method, it is more convenient to measure the altitude of the upper limb of the sun, that is, $h_{\overline{\odot} z}$; it will be the lower limb when measuring via the zenith (Fig. 137). An altitude measured via the zenith is reduced to the visible horizon by subtracting the dip (Table 11б), after which the result is subtracted from 180° and we obtain the apparent altitude measured in the ordinary way. Total corrections are then applied, but without the dip of the horizon. In this case, the correction for irradiation is not needed. The formula for the upper limb of the sun measured via the zenith will be

$$h_{\overline{\odot}} = [180° - (sr + i + s - d)] + (-\rho + p + R_{av}) - (2R_{av} + \Delta R) \quad (14.36)$$

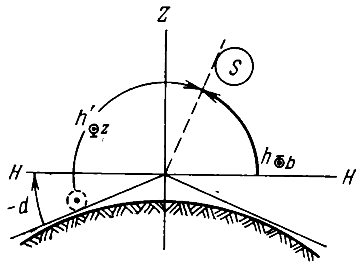

Fig. 137

The quantity in the second parentheses is taken from Table 8a, MT-63, for height of eye equal to zero (8^{a}_{0}); that in the final parentheses, from Table 8в. Similarly, for stars use Table 9^{a}_{0}, and for the moon, Tables 10a, 10б, 10в without change.

Example 7. On 3.07.68, altitude of ordinary upper limb of the sun measured via the zenith: $sr_{\overline{\odot} z} = 122°43'.6$; $i = +1'.5$; $s = -73''$; $e = 10.4$ metres.

	sr	122°43'.6
	$i+s$	+ 0 .3
	$h'_{\overline{\odot} z}$	122°43'.9
Table 10б	d	− 5 .7
	$h_{ap\overline{\odot} z}$	122°38'.2
	$h_{ap\overline{\odot}}$	57 21 .8
Table 8^{a}_{0}		+ 15 .4
Table 8в		− 31 .8
	$h_{\overline{\odot}}$	57° 5'.4

VII. CORRECTING SMALL (LESS THAN 3°-5°) ALTITUDES OF THE SUN

It has already been pointed out that the tabular refraction for small altitudes is not quite reliable, particularly if the dip is not known exactly. However, when sailing in high latitudes one finds it necessary to measure small altitudes of the sun. In such cases,

altitude corrections have certain peculiarities. Due to a rapid and nonuniform variation of refraction about the horizon, even slight changes in the argument of the altitude cause appreciable changes in refraction. For this reason,

(a) enter tables of refraction with the apparent altitude, that is, as corrected by the dip of the horizon;

(b) it is best to correct each sextant reading separately, and not the mean of 3 to 5 readings, as usual, otherwise discrepancies will appear, especially when deriving the root-mean-square error of observation. Under ordinary conditions, one can correct the mean sextant reading as usual.

Table 8 does not contain solar altitudes less than 3°, and so separate corrections are applied from different tables according to the formula

$$h_{\odot}=sr+i+s+(-d)+(-\rho_0+p)+(\pm R_{\odot})+(\pm \Delta_{\rho t})+(\pm \Delta_{\rho B}) \tag{14.37}$$

Here, the quantities taken from separate tables are given in brackets; $R_{\odot}$ (Table 13a) is taken with the "+" sign for the lower limb of the sun and with the "—" sign for the upper limb.

Example 8. On 13.10.68, three altitudes of $\underline{\odot}$ were measured: $sr_1=1°15'.5$; $sr_2=1°17'.2$; $sr_3=1°19'.4$; $i=+1'.4$; $s=+0'.2$; $e=11.2$ metres; $t=-5°$; $B=774$ mm. Find $h_{\odot\,av}$.

		I	II	III
	sr	1°15′.5	1°17′.2	1°19′.4
	$i+s$	+ 1 .6	+ 1 .6	+ 1 .6
	h'	1°17′.1	1°18′.8	1°21′.1
Table 11б	d	— 5 .9	— 5 .9	— 5 .9
	h_{ap}	1°11′.2	1°12′.9	1°15′.1
Table 13a	$\rho+p$	— 22 .8	— 22 .6	— 22 .4
	$\Delta\rho_t$	— 2 .0	— 2 .0	— 1 .9
	$\Delta\rho_B$	— 0 .5	— 0 .5	— 0 .5
	h_{tr}	0°45′.9	0°47′.8	0°50′.3
Table 13б	$R_{\odot}$	— 16 .1	— 16 .1	— 16 .1
	$h_{\odot}$	0°29′.8	0°31′.7	0°34′.2

$$h_{\odot\,av}=0°31'.9$$

VIII. CORRECTING ALTITUDE OF BODIES WITH DIP MEASURED BY A DIPMETER

If d is obtained from observations, it should obviously be excluded in all the foregoing total corrections. To do this, enter tables for the sun and stars with height of eye equal to zero. The dip will then be eliminated and the other corrections will remain unchanged. Dip is not included in the total correction of the moon in MT-63. The only change therefore is that the measured value d_{meas} is used instead of Table 11б. The working formulas will then be

$$\left.\begin{aligned} h_{\odot} &= sr + i + s + (-d_{meas}) + \text{Table } 8^{\text{a}}_{0} + \text{Table } 8б \text{ (or } 8\text{в)} \\ h_{*} &= sr + i + s + (-d_{meas}) + \text{Table } 9^{\text{a}}_{0}) \\ h_{☾} &= sr + i + s + (-d_{meas}) + \text{Tables } 10\text{a} + 10б \text{ (or } 10\text{в)} \end{aligned}\right\} \quad (14.38)$$

For altitudes less than 30° and temperature and pressure different from mean values, also add to these values the corrections from Tables 14a and 14б.

Example 9. On 4.08 68, $sr_{\odot} = 14°27'.5$; $i = +3'.6$; $s = +0'.4$; $e = 10.3$ metres; measured $d = -6'.4$; $t = +24°$; $B = 745$ mm. Find $h_{\odot}$ and the dip error.

	sr	14°27'.5
	$i+s$	+ 4 .0
	h'	14°31'.5
	d_{meas}	− 6 .4
	h_{ap}	14°25'.1
Table 8^a_0		+ 12 .5
Table 8в		− 31 .8
Tables 14a, б		+ 0 .3
	$h_{\odot}$	14° 6'.1

d_{table}	−5'.7
− d_{meas}	−6 .4
error	+0'.7

SEC. 77. CORRECTING ALTITUDES MEASURED WITH ARTIFICIAL HORIZON AND BUBBLE SEXTANT ИАС

1. Correcting altitudes measured with artificial horizon. When observing celestial bodies in an artificial horizon, we really measure $2h'$, so before correcting the measured altitude, divide it by two. No dip correction is needed, and the remaining correction is similar to the above-considered case with measured dip. The formulas for

this correction will be of the following form:

$$\left.\begin{aligned} &\underline{\odot}\ h_{\odot} = \frac{sr+i+s}{2} + (-\rho + p + R_{av}) + (\Delta R) \\ &\ast\ h_{\ast} = \frac{sr+i+s}{2} = \rho \\ &\underline{☾}\ h_{☾} = \frac{sr+i+s}{2} + [-\rho + p_{☾}\ (\cos h' + 0.2725)] \end{aligned}\right\} \quad (14.39)$$

In these formulas, s is taken for sr and not $\frac{1}{2}\ sr$.

For the sun, use Table 8_0^a (for zero height of eye) and an additional correction from Table 8б or 8в; for a star, use Table 9_0^a or, which is the same thing, Table 12, and for the moon, Tables 10a, 10б, 10в without any changes.

Example 10. On 20.07.62, altitude of lower limb of sun was measured in artificial horizon, $sr_{\underline{\odot}} = 74°28'.6$; $i = -1'.8$; $s = -0'.8$.

	sr	74°28′.6
	$i+s$	— 2 .6
	$2h'$	74°26′.0
	h'	37 13 .0
Table 8_0^a		+ 14 .9
Table 8б		— 0 .2
	$h_{\odot}$	37°27′.7

II. **Correcting altitudes measured by ИАС sextant.** Unlike the artificial horizon, sextants with artificial horizon in the form of a bubble or gyroscope measure the altitude of the centre of the body and not the limb; accordingly, the formulas for correcting altitude take the form

$$\left.\begin{aligned} &\odot\ h_{\odot} = sr + \Delta - \rho + p \\ &\ast\ h_{\ast} = sr + \Delta - \rho \\ &☾\ h_{☾} = sr + \Delta - \rho + p_{☾} \cdot \cos h' \end{aligned}\right\} \quad (14.40)$$

where Δ is the correction of the ИАС sextant.

For the sun, use Table 13a, which gives the sum $(-\rho + p)$; for stars, use Table 12 or 9_0^a, for planets a parallax correction is added from Table 9б. The altitude of the moon is corrected by Tables 10a and 10б; from these corrections subtract $R_{☾}$ taken from the MAE.

Example 11. On 18.09.68, at $T_{gr}=14h$, the altitudes of the sun and moon were measured with the ИАС sextant: $sr_{\odot}=12°40'$; $sr_{\text{☾}}=19°28'$; $i=-8'.2\approx 8'$.

		$\odot$
	sr	12°40′
	$\Delta\approx$	− 8
	h'	12°32′
Table 13a	$\rho+p$	− 4
	$h_{\odot}$	12°28′

		☾
	sr	19°28′
	Δ	− 8
	h'	19°20′
Table 10a		+ 61
Table 10б		+ 4
	$R_{\text{☾}}\approx$	− 15
	$h_{\text{☾}}$	20°10′

$0h\ p'_{\text{☾}}$	55′.6 (+0′.7)
$14h\ \Delta p$	+0 .4
p	56′.0

$R_{\text{☾}}=15'.3$

When correcting altitudes measured by radio sextant, the formulas will be similar in structure to those considered above, but the values of refraction of extraterrestrial radio emission will be different, requiring special tables.

SEC. 78. ERRORS INVOLVED IN CORRECTING ALTITUDE

Systematic and random errors may creep into altitude corrections; besides, the calculator may make blunders.

Blunders occur most often in the numbers of tables and in signs of the corrections. The only way to avoid blunders is by neat and careful work. One check is to compare the corrected altitude with the measured altitude. For stars and planets, the corrected altitude is *always less* than the measured altitude, for the lower limb of the sun the corrected altitude (for $h > 10°$) is always *greater* than the measured altitude, and for the upper limb, it is *always less*. The corrected altitude of the moon, irrespective of the limb, will *always exceed* the measured altitude.

Systematic errors are due mainly to the fact that the actual conditions of observation differ from the mean conditions on which tables are based.

Practically the only source of systematic errors in tables of total corrections for the sun and stars is the dip error (Δd). This error, it will be recalled, can be eliminated by taking the value of dip measured with a dipmeter instead of the tabulated dip. If this is not done, the error Δd will enter the altitude and consequently the line of position in toto.

To make the tabulated value of refraction closer to the actual value, introduce corrections from Tables 14a and 14б for altitudes less than 30°.

Slight residual errors due to the above-mentioned causes may be considered random errors.

Total-correction tables for the moon utilize formulas that neglect:

(a) variation of parallax with latitude;

(b) the effect of azimuth on parallax.

The errors generated by these simplifications are not great, of the order of 0'.1 to 0'.2 in each case. Since appropriate corrections are not introduced, these errors completely go into the altitude, which means that the altitude of the moon is corrected with a slight additional systematic error.

Random errors of altitude correction occur mainly due to interpolation. Experience has shown that the greatest error due to all these causes for tables of total corrections of the altitude of the sun and stars does not exceed $\pm 0'.3$, for the moon, $\pm 0'.5$. Errors of interpolation obey the law of equal probability, and so the root-mean-square error of interpolation for stars and the sun will be

$$\varepsilon_1 = \pm \frac{0'.3}{\sqrt{3}} = \pm 0'.17$$

and for the moon,

$$\varepsilon_1 = \pm \frac{0'.5}{\sqrt{3}} = \pm 0'.29$$

These values may be taken as the mean tabulated accuracy of altitude correction of celestial bodies.

CHAPTER 15

ERRORS IN OBSERVING ALTITUDES AT SEA, METHODS OF DETECTING AND REDUCING THEM

SEC. 79. ERRORS IN MEASURED ALTITUDES AND WAYS OF DETERMINING THEM

From theory we know that all errors are divided, as to the way they affect a measured quantity, into **systematic** (including recurrent) and **random** (see Appendix V). The true, geocentric, altitude obtained from observations and called the *observed altitude* h or h_{ob} inevitably contains both systematic and random errors of measurement and correction of altitude that are independent of the will of the observer. These errors differ with different observers, conditions and instruments.

Besides, an observer can always commit a **blunder** when measuring or correcting altitude; this will also enter into the observed altitude. Thus, the observed altitude h_{ob} differs from the true altitude (h_{tr}) for a given observer by the sum of the systematic (Δ) and random ($\pm\delta$) errors; in certain cases, h_{ob} also contains a blunder of some magnitude. Consequently,

$$h_{ob} = h_{tr} + \Delta + (\pm\,\delta) \tag{15.1}$$

which means that h_{ob} contains a total error, the magnitude and sign of which are usually unknown to us.

Problems in detecting blunders in the measured altitude will be considered together with random errors, while blunders in computations will be examined in the method of altitude lines of position*. In this chapter we shall dwell only on systematic and, particularly, random errors in observed altitude.

(1) **Systematic errors in observed altitude.** We shall consider only such systematic errors as have the same sign and magnitude (recurring errors) in the process of a single measurement.

The main sources of such altitude errors are:

(a) the personal error of the observer due to a property of his eye when aligning images: he does not achieve complete tangency or permits overlapping;

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* Chapter 20, Section 113.

(b) errors due to inaccurate adjustment of sextant before observations;

(c) "instrument" errors: errors in the instrument correction s taken from the certificate; prismatic errors of shade glasses, and errors due to the tangent screw and its backlash;

(d) error in index correction;

(e) error in the tabulated value of dip;

(f) error in value of refraction (for small altitudes);

(g) error in chronometer correction, which may be converted to an altitude error by the formula (3.16) in the form

$$\Delta h' = -0.25 \cos \varphi \sin A \Delta T^{\mathrm{sec}} \tag{15.2}$$

where ΔT is a systematic error in the chronometer correction.

The greatest of these errors is that of dip of the horizon, the others are usually small, given proper handling of the sextant and chronometer. Systematic errors may be eliminated either by introducing corrections or by special methods of observation. Both procedures are used in nautical astronomy and will be considered specifically for each method of determining the position of a ship and its coordinates. Finding personal systematic errors is shown in Sec. 84, Item 4.

(2) **Random errors.** As we know, random errors cause the results of measurements of one and the same quantity to differ; they have a certain spread; the smaller the spread, the more accurate the measurements. Accuracy of measurement is characterized by the so-called root-mean-square error or standard error (see Appendix V).

Let us try to find out what mean-square error may be expected in individual operations in measuring and correcting altitudes and let us determine their total magnitude from present data.

(1) The error due to deviation of the plane of the limb from the plane of the vertical circle when measuring (see Sec. 63). Since the inclination of the sextant is unknown in each measurement, the error expressed by formula (12.1) is of a random nature. For the average observer, this error may be taken at about $\varepsilon_1 = \pm 0'.2$.

(2) Sighting error, or error of "alignment". This error is due to inaccurate alignment of the images of celestial body and horizon in the field of view of the telescope. Under average conditions, the resolving power of the eye is $c = 45''$ (0'.75), which means that two points are still distinguishable by the eye at this angle. This quantity is taken as the sighting accuracy of the naked eye. A telescope increases this quantity in proportion to the magnification of the telescope, w,

$$\varepsilon_s = \frac{c}{w}$$

Magnification of telescopes of present-day sextants ranges from 3.5 × for star telescopes to 6 × and 7 × for day telescopes. Depending on this factor, the sighting error will be $\varepsilon = \pm 0'.21$ for a star telescope and $\varepsilon_2 = \pm 0'.12$ for a day telescope, respectively.

(3) The error in micrometer-drum reading due to interpolation "by eye" may be taken* at $\varepsilon_3 = \pm 0'.12$.

(4) We take the error in determining the index correction at $\varepsilon_4 = \pm 0'.1$.

(5) We take the random errors in correction of altitude (see Sec. 78) at $\varepsilon_5 = \pm 0'.17$.

(6) The random error in chronometer time can also be referred to errors of altitude. To do this, we rewrite formula (15.2) for the mean-square error ε_h

$$\varepsilon_h = -0'.25 \cos \varphi \sin A \varepsilon_T^{\sec} \tag{15.3}$$

where $\varepsilon_T^{\sec}$ is the error in the chronometer time and chronometer correction. ε_T does not usually exceed 1s; for this reason, under average conditions, $\varepsilon_6 = \varepsilon_h \approx \pm 0'.1$.

It may be taken that all causes that give rise to errors operate independently, then from the formula of error theory we get the expected random error in the corrected altitude:

$$\varepsilon_h = \pm \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2 + \varepsilon_6^2} = \pm 0'.35$$

This is the possible theoretical precision of the observed altitude of the sun or a star obtained under average conditions.

Experience has shown that the principal factors determining the accuracy of altitude measurement are: brightness of horizon, skill of the observer, quality of the instrument, and care taken in making the observation. Due to poor visibility of the horizon, the sighting error is almost always greater than the theoretical value, so that even in the case of a good observer the mean-square error in the observed altitude is always greater than the possible theoretical value obtained above.

Studies have shown that for an observer of average skill, the errors ε_h fluctuate as follows under various conditions:

in daytime for the sun: from $\pm 0'.3$ to $\pm 1'.0$ (average $\pm 0'.7$);

at night for stars: from $\pm 0'.5$ to $\pm 1'.5$ (average $\pm 1'.0$).

For the ИАС-1 sextant, the magnitude of a random observational error is greater than for an ordinary sextant. Under very good conditions (in a roadstead or at a wharf), the error of measurement ε_h may be of the order of $\pm 1'.0$-$1'.5$. Errors increase drastically in the case of rolling or pitching of the ship. For rolling or pitching

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* According to Professor N. Matusevich.

up to 2° (moderate-to-rough sea), $\varepsilon_h \approx \pm 3'$; for rolling or pitching from 5° to 8°, the error $\varepsilon_h = \pm 5'$-$7'$. For greater values do not use an ИАС sextant for measuring altitude.

The highest accuracy in altitude measurements is obtained with a mercury artificial horizon (under coastal conditions); here, $\varepsilon_h = \pm(0'.21$-$0'.26)$ and is close to the highest theoretically possible precision in sextant measurements of the altitude of a celestial body.

(3) **Finding the mean-square error of measured altitude.** Error theory has worked out methods for determining the mean-square error ε in the measurement of a quantity when the true value of this quantity is not known. In this case, a procedure is applied called "by inner agreement of measurements" in which are found the deviations v_i of individual measurements from their arithmetic mean, after which the error of one randomly taken measurement in a given series is determined from the formula

$$\varepsilon = \pm \sqrt{\frac{\Sigma v_i^2}{n-1}} \tag{15.4}$$

where n is the number of observations.

This and other formulas that follow from the normal (Gaussian) law of distribution of random errors will be used to determine altitude errors.

Recent studies in the field of error theory indicate that the methods and formulas derived from the Gaussian law hold only for very large numbers of observations ($n \to \infty$). For a small number of observations (5 to 7), other laws of error distribution have to be invoked (Student's law, for example), also special tables and formulas that yield better results. However, for navigational purposes we still make use of the classical methods of determining ε, as being simpler and of more practical convenience.

To compute ε from formula (15.4) it is necessary to take *many measurements* of one and the same quantity, angle α, say. Now the altitude of a celestial body is constantly varying, and so it is not possible to take a series of measurements of the same altitude.

Measuring a series of different altitudes of the body, we get different angles each time; it is therefore impossible to appraise observational errors from the data obtained.

Change of altitude is due to the following causes:

(1) diurnal rotation of the sphere,

(2) movement of the ship over the earth's surface,

(3) change of declination of the celestial body,

(4) random errors of observation.

If we eliminate from consecutively measured altitudes the effects of the first three factors by introducing corrections, we can take it

that these "reduced" altitudes (h_{red}) are equally exact and represent the result of a multiple measurement of one and the same quantity. From a comparison of reduced altitudes we can judge the observational accuracy and compute the error ε_h.

Systematic errors do not usually vary perceptibly during the time of observation, and so we shall consider that they will have no effect on change of altitude. Due to their operation, all measured altitudes are erroneous in the same direction and by the same amount. The same is assumed in the correcting of altitude. On this basis, we can "reduce" sextant readings directly without any corrections. The altitude thus obtained is the reduced altitude, h_{red}.

SEC. 80. REDUCING ALTITUDES OF CELESTIAL BODIES TO A SINGLE INSTANT

To reduce an altitude to a given single instant, that is, in order to eliminate the effect of the diurnal rotation of the sphere, apply to each altitude the correction Δh_T, which expresses the magnitude of change of altitude for the interval of time from the instant it is taken to the instant to which the altitude is reduced.

In Sec. 11 we obtained the formula (3.27) for increment of altitude due to diurnal motion, taking into account the second terms of the series expansion of the altitude. For small intervals of time and for large distances from the meridian of the observer, we can confine ourselves to the first term of this formula, or

$$\Delta h = -\cos\varphi \cdot \sin A \cdot \Delta t$$

where Δt is the variation of hour angle or, approximately, the change in time (ΔT).

Near the meridian of the observer, however, do not neglect the second term of the series (acceleration of altitude rate). In these cases, it is best to reduce each altitude to the meridian (as an ex-meridian* altitude), and this will take account of the diurnal motion of the sphere. From the foregoing we have two ways of reducing to a single instant.

(a) Reducing to a single instant altitudes measured outside the meridian. To reduce, apply formula (3.16), in which the quantity $\Delta t^{sec} = \Delta T^{sec}$ is the increment of time:

$$\Delta h'_T = -0.25\cos\varphi \cdot \sin A \cdot \Delta T^{sec} \tag{15.5}$$

Altitudes may be reduced to any instant in the past or future, but when using the formula remember that when reducing an earlier

* See Section 127.

altitude to a subsequent instant, consider ΔT in this formula positive for west hour angles and negative for east hour angles. When reducing a subsequent altitude to a previous instant, the signs of ΔT will be reversed. Incidentally, it is hard to make a mistake in the sign of the reduction even without these hints because the very nature of the altitude change will indicate clearly the sign to be affixed to the correction Δh_T.

In order to deal with smaller numbers, it is more convenient to reduce all altitudes to T_{av}, the instant when the average altitude is measured; then ΔT in the above formula (15.5) will represent the difference between the instant of each of the observations (T_i) and this mean instant. The azimuth in (15.5) is ordinarily obtained for T_{av} by compass and is converted to true azimuth or is computed from the formula $\sin A = \cos\delta \cdot \sin t \cdot \sec h$. To compute Δh_T from (15.5), first compute the factor $K = 0.25 \cos\varphi \cdot \sin A$, which is change of altitude in one second of time, or the rate of change of altitude. It is computed from tables of logarithms or from Table 156 to the third decimal place. The quantity $\Delta T_i^{\text{sec}} = T_i - T_{av}$ is obtained in seconds, and then $\Delta h'_T = K \cdot \Delta T_i^{\text{sec}}$ is computed by slide-rule. All computations are arranged as shown in Example 1.

Example 1. On 10.09.68, $\varphi = 46°$ N, at anchor measured altitude of $\odot$ in forenoon and recorded instants; average conditions of observation; at mean instant, $A_{\odot} = 57°$ SE. Reduce altitudes to mean instant and determine ε_h.

(a)

T_{ch}	ΔT_i^{sec}	$\Delta h_T = K\Delta T_i$	sr	h_{red}	Δ	Δ^2
6h 08m 32s	141s	20′.6	33°24′.3	33°44′.9	−0′.3	0.09
09 08	105	15 .3	30 .4	45 .7	+0 .5	0.25
10 12	41	6 .0	40 .0	46 .0	+0 .8	0.64
10 53	00	00	45 .3	45 .3	+0 .1	0.01
11 26	33	4 .8	49 .5	44 .7	−0 .5	0.25
11 53	60	8 .8	53 .2	44 .4	−0 .8	0.64
12m 35s	102s	14′.9	34°00′.2	33°45′.3	+0′.1	0.01

$h_{av} = 33°45'.2$; $\Sigma\Delta^2 = 1.89$

(b) Determination of $K = \Delta h$ during 1s

0.25	log	9.3979
$\varphi = 46°$	cos	9.8418
$A_{\odot} = TB^* = 123°$	sin	9.9236
	log K	9.1633

$K = 0'.146$ in 1s

(c) $\varepsilon_h = \sqrt{\dfrac{1.89}{6}} = \pm 0'.56$

* True bearing.

(b) Reducing to a single instant altitudes measured near meridian. In this case, the measured altitudes are regarded as ex-meridian and are reduced to the meridian by adding to each a reduction r taken from Table 17б, MT-53; we then consider the altitudes free from the effect of diurnal motion. This method is analogous to determining latitude on the basis of ex-meridian altitudes (Sec. 126).

SEC. 81. REDUCING ALTITUDES OF CELESTIAL BODIES TO A SINGLE ZENITH

If an observer on board ship moves over the earth's surface, his zenith on the celestial sphere and the associated plane of the celestial horizon will also move. As a result, the altitudes of bodies will change independently of their variation during the diurnal rotation of the sphere. To compare altitudes, all consecutively measured altitudes must be referred or reduced to a single place on the earth or to a single zenith. The reduction is performed by introducing into the altitude a correction for ship movement (Δh_Z).

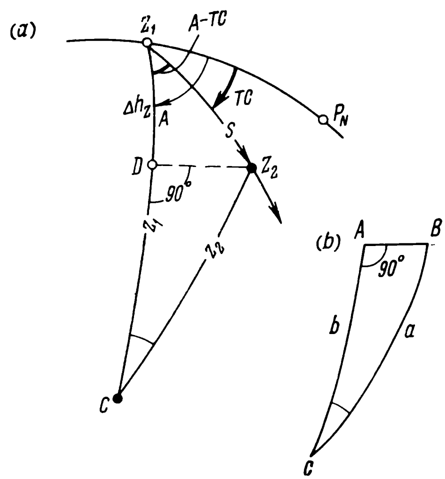

Fig. 138

Suppose that at the first measurement of the altitude of a celestial body C, the zenith of the observer on the sphere (Fig. 138a) was at Z_1 and the body had a zenith distance $z_1 = 90° - h_1$; its azimuth in circular reckoning will be A; the course of the ship (true course, TC, or K) is reckoned from the meridian of the observer $P_N Z_1$ in circular reckoning. By the time of second set of observations, the ship's zenith will have moved to Z_2; the arc Z_1Z_2 in minutes will be equal to the run S of the ship in nautical miles during this time. The zenith distance of body C will now equal $z_2 = 90° - h_2$. The difference between z_2 and z_1 will represent the change in altitude of the body due to movement of the ship. If we drop from z_2 a spherical perpendicular Z_2D on arc Z_1C, then Z_1D will to a first approximation represent the difference $z_1 - z_2 = h_2 - h_1 = \Delta h_z$, or $h_2 = h_1 + \Delta h_z$. Taking the small spherical triangle Z_1Z_2D for a plane triangle, we get

$$\Delta h_Z = S \cdot \cos (A - K) \tag{15.6}$$

where $(A - K)$ is the course angle of the celestial body. Consequently, to reduce the first altitude to the second zenith, simply introduce the small correction Δh_Z; then both altitudes may be considered as measured from a single place on the earth.

Apply the following rule of signs in formula (15.6). If the reduction is forward, that is, the earlier altitude is reduced to a subsequent zenith, the run is regarded as positive and the sign of Δh_Z is determined from the sign of $\cos(A - K)$; if the reduction is backward, the run S is negative and the signs are reversed. Now the sign of $\cos(A - K)$ depends on the course angle $CA = A - K$: if $CA < 90°$ or $CA > 270°$, then the cosine is positive, otherwise it is negative. Thus, *if the preceding altitude is reduced to subsequent zenith, the correction* Δh_Z *will be positive when the ship is sailing towards the celestial body and negative when sailing away from it.* In reduction back (subsequent altitude to preceding zenith), the signs are reversed.

This correction will attain its maximum value when the body is located ahead or astern: in this case $\Delta h_Z' = S$ in miles. But if the body is located abeam the correction Δh_Z is zero. As the observer's zenith moves, the altitude changes and so also does the azimuth of the body; but change in azimuth is slight and does not affect the magnitude of reduction Δh_Z.

It is more convenient to compute the correction Δh_Z with the aid of Table 16, MT-63. For compilation of this table, formula (15.6) is given in the form

$$\Delta h_Z = \frac{V}{60}(T_2 - T_1)^{\min} \cdot \cos(A - K) \tag{15.7}$$

where V is the speed of the ship in knots, $T_2 - T_1$ is the difference between first and second observations in minutes. If we take $T_2 - T_1 = 1$ minute, we get the formula

$$\Delta h_{1\min} = \frac{V}{60}\cos(A - K) \tag{15.8}$$

which is used to compile Table 16—"Reduction of Altitude to a Single Zenith" (for one minute). The arguments for entering this table are speed V and course angle $(A - K)$, where the azimuth of the body is computed from formula (2.3) or is obtained by compass. The sign of the correction $\Delta h_{1\min}$ is indicated at the top and bottom of the tables, depending on the course angle.

After taking out $\Delta h_{1\min}$ we get

$$\Delta h_Z = \Delta h_{1\min}(T_2 - T_1)^{\min} \tag{15.9}$$

It is required to reduce altitudes to a single zenith not only when deriving observational errors, but also in night-time observations of the stars and planets and day observations of the sun and moon.

Example 2. At $\varphi_c=51°20'$N; course 221° true; speed 15 knots; measured altitudes of two stars: av. $sr_1=27°34'$; $T_1=5$h 5m 25s; $RCB_{*}=21°$; $\Delta K=1°.5$; av. $sr_2=39°21'$; $T_2=5$h 11m 34s.

Reduce h_1 to zenith h_2.

(1)			(2)		
	A	199°.5		T_2	5h 11m 34s
	$CMG=K$	221		T_1	5 5 25
	$A-K$	338°.5=21°.5		T_2-T_1	6m 9s $\approx$ 6m.2

(3) From Table 16 with $A-K$ (on left) and $V=15$, we have $\Delta h_{1\,\text{min}}=+0'.23$.
(4) $\Delta h_Z=+0.23'/\text{min}\ 6.2\ \text{min}=+1'.4$.
(5) $h_{red}=27°34'+1'.4=27°35'.4$.

When deriving formula (15.6), the triangle Z_1Z_2D is taken to be plane, actually however it is a spherical triangle. For this reason, when working to a second approximation, one must take into account a correction for sphericity.

SEC. 82. REDUCING ALTITUDES OF CELESTIAL BODIES TO A SINGLE DECLINATION

If the declination of a body changes, the altitude of the body will also change independently of its change due to the diurnal rotation of the sphere and the motion of the observer. Therefore, to derive the error, all measured altitudes in a series must be reduced to a single declination by introducing a correction, Δh_δ. The formula for the increment of altitude with change in declination is derived in Sec. 107 and has the form

$$\Delta h'\delta=\Delta\delta'\cdot\cos q \tag{15.10}$$

where $\Delta\delta=\frac{\Delta\text{ for 1h}}{60\text{ min}}\cdot\Delta T^{\text{min}}$ is the change in declination during time ΔT between observations, and Δ is the hourly change in declination taken from the MAE.

The angle q may be obtained from the formula

$$\sin q=\sin A\cdot\cos\varphi\cdot\cos\delta$$

or from formula (3.10) of Sec. 11 in the form

$$\sin q=\frac{\Delta h'}{15\Delta T^{\text{min}}}\sec\delta \tag{15.11}$$

For the sun and the planets, the correction Δh_δ has perceptible values only for very large intervals of time ΔT amounting to 10 to 20 minutes, since the hourly change of $\Delta\delta_\odot\leqslant 1'.0$, and $\Delta\delta_{pl}\leqslant 1'.3$.

For the moon, the declination of which can change very considerably (up to $\Delta\delta_☾ = 20'$ per hour), the correction Δh_δ may come out appreciable and has to be taken into account. Reduction to a single declination is applied only for deriving observational errors for the moon.

Example 3. On 27.09.58, measured two altitudes of the moon: $h_1 = 38°29'.5$; $T_{gr1} = 4h\ 28m\ 35s$ and $h_2 = 39°4'.6$, $T_{gr2} = 4h\ 33m\ 43s$. Reduce first altitude to declination at second instant.

(1) $\delta_T^☾$	$0°22'.9N$ $(\overset{+}{19}.2)$	
$\Delta\delta$	$+\ 4\ .6$	
$\delta_☾$	$0°27'.5N$	

(2) $\Delta\delta_☾ = \frac{+19'.2}{120m}\,5m.1 = +0'.82$

(3) $\sin q = \frac{35'.1}{15'/m \times 5m.1} \times 1.00 = 0.46$

(4) $q \approx 28°$; $\cos q = 0.88$

(5) $\Delta h_\delta = +0'.7$

SEC. 83. COMPUTING THE MEAN-SQUARE ERROR OF ALTITUDE FROM OBSERVED ALTITUDES

The principal method for obtaining the mean-square error of altitude ε_h in the open sea is by computing it from deviations of separate measurements from their arithmetic mean, that is, from "the inner coincidence" of the altitudes. Here, use formula (15.4).

If there is an opportunity in some way to obtain or compute the true values of altitude at the instants of observation, then the total personal errors will be equal: $\Delta + \delta_i = h_i - h_{tr}$ and the mean error ε_h, after elimination of the systematic error Δ, is obtained from the formula

$$\varepsilon_h = \pm \sqrt{\frac{\Sigma\delta_i^2}{n}} \tag{15.12}$$

where n is the number of observations. This method is called computing from true or absolute errors.

To derive the mean error ε_h from "the inner coincidence of altitudes", measure 7 to 11 altitudes of the body and note the chronometer time. It is not advisable to measure more than 11 altitudes in succession because fatigue of eye and hand will impair the result. When measuring the mean altitude or after measurements, take the bearing of the celestial body, note T_{sh}, speed, course and log reading from which the ship's coordinates are taken. These data are put into the computation diagram, which is used to reduce to a single (averaged) instant and zenith, and (for the moon) the declination as well.

After the error has been found, the series of observations obtained must be analyzed. To do this, compute the limiting error $\varepsilon_{lim} =$

$3\varepsilon_h$ and the probable error (50%) $\varepsilon_{prob} = {}^2/_3\, \varepsilon_h$. In an analysis, an error in v that is greater than or equal to ε_{lim}, should be considered a blunder and the observation result deleted; a new value of ε_h is then computed. If a series of observations has been performed correctly, then half of the errors in v is greater than ε_{prob} in absolute value, and half is less. In addition, the mean-square error in derivation, that is, in the average altitude, h_{av}, is computed from the formula

$$\varepsilon_0 = \pm \frac{\varepsilon_h}{\sqrt{n}} = \pm \sqrt{\frac{\Sigma v^2}{n(n-1)}} \tag{15.13}$$

The quantity ε_0 gives an estimate of accuracy of the arithmetic mean (h_{av}) and depends on the number of observations whereas ε_h (for large n) is independent of the number of observations. It is advisable to compute Δh_T and Δh_Z by slide-rule.

Example 4. On 10.04.68, at about T_{sh}=10h 25m; lr=37.4, φ_c=42°35′N; on course 280° at speed 14 knots; $CB_{\odot}$=121°; ΔK=—3°; measured series of altitudes of $\underline{\odot}$. Reduce altitudes to mean instant and to zenith; compute ε_h, ε_0, ε_{prob} and ε_{lim} and analyze the given series of observations.

(1)

T_{ch}	ΔTs	ΔTm	$\Delta h_T = K_1 \Delta Ts$	$\Delta h_Z = \Delta h_1 \cdot \Delta Tm$	sr	$h_{lim} = sr + \Sigma\Delta h$	V	V^2
0h 23m 18m	104s	1m.7	+16′.9	—0′.4	36°44′.4	37°0′.9	—0′.5	0.25
23 54	68	1 .1	+11 .0	—0 .2	50 .4	1 .2	—0 .2	0.04
24 29	33	0 .6	+5 .4	—0 .1	36°56′.7	2 .0	+0 .6	0.36
25 02	00	0 .0	0 .0	0 .0	37 1 .0	1 .0	—0 .4	0.16
25 38	36	0 .6	—5 .8	+0 .1	6 .4	0 .7	—0 .7	0.49
26 18	76	1 .3	—12 .3	+0 .3	13 .8	1 .8	+0 .4	0.16
26 52	110	1 .8	—17 .8	+0 .4	19 .5	2 .1	+0 .7	0.49

h_{av}=37°1′.4 $\Sigma\Delta^2$=1.95

The sign of Δh_T is determined from the change in altitude and of Δh_Z from Table 16.

(2) Auxiliary computations:

(a) $TB_{\odot}$=118°; φ =42°.6

From Table 156 K_1=0′.162 in 1s

(b)

$TB = A$	118°
K	227
$A - K$	201°

V=14 knots

from Table 16, MT-63

Δh_1= —0′.22 in 1 min

(c) Check on computation of v.

The sum of deviations from the mean (Σv) should equal zero; in the given case, $\Sigma v_i = -0'.1$, which is permissible and is the result of rounding off.

(3) Computing mean errors.

$$\varepsilon_h = \pm \sqrt{\frac{\Sigma v_i^2}{n-1}} = \pm \sqrt{\frac{1.95}{6}} = \pm 0'.57 \approx \pm 0'.6$$

$$\varepsilon_0 = \pm \frac{\varepsilon_h}{\sqrt{n}} = \pm \frac{0'.57}{\sqrt{7}} = \pm 0'.21 \approx \pm 0'.2$$

$$\varepsilon_{lim} = 3\varepsilon_h \approx \pm 1'.7$$

$$\varepsilon_{prob} = \frac{2}{3}\varepsilon_h = \pm 0'.4$$

When analyzing a series of v, it is seen that there are no blunders ($3\varepsilon_h$) and that the series of observations h was taken rather well. The resultant ε_h corresponds to any randomly taken measurement with a probability of 68.3%.

SEC. 84. METHODS OF CHECKING MEASURED ALTITUDES

Several methods may be applied to check measured altitudes for blunders, quality of measurement and total accuracy.

(1) **Approximate check from the differences of measured altitudes.** After taking a series of altitudes, a computation is made of the differences of readings and instants between adjacent observations. These differences are then analyzed: larger Δh should correspond to a larger interval of time. If the measurements were carried out on preset readings, the differences Δh are constant (5′ to 10′) and the difference of instants ΔT should be roughly (to within several seconds) the same.

Then from Tables 15a and 15б, MT-53, with computed latitude and azimuth of the body take out the theoretical increments in altitude and write them in a column next to the actual differences. If any two adjacent differences differ radically from the theoretical value and from other differences, it is probable that the mean observation between these differences is in error and should be deleted. If the discrepancies between differences are comparatively great—of the order of 1′ to 2′—and are evident throughout the series, then all observations were apparently executed badly, with a spread. Discrepancies in differences less than 1′ are common, and the observation series may be considered good.

Example 5.

No	T_{ch}	sr	ΔT	Δh_{ob}	Δh Table 156
1	5h 3m 30s	37° 8′.2	39s	5′.3	6′.0
2	4 9	13 .5	27	4 .0	4 .2
3	4 36	17 .5	42	7 .1	6 .5
4	5 18	24 .6	27	6 .6	4 .2
5	5 45	31 .2	35	3 .7	5 .4
6	6 20	34 .9	31	4 .3	4 .8
7	5 6 51	37 39 .2			

Auxiliary computations from Table 156 based on $\varphi_c = 49°50'$N and $A_{\odot} = 47°$.
Δh for 10s—
—1′.54
20—3.08
30—4.62
40—6.16
1—0.15
5—0.72

From a comparison of Δh_{ob} and Δh from Table 15 it is seen that the differences of observations 4-5 and 5-6 do not agree with the others and with the theoretical values; obviously, the middle, fifth, observation is inaccurate. Delete it and form the difference 6-4; $\Delta h_{ob} = 10'.3$; $\Delta T = 1$m 2s; Δh Table 156 = 9′.5, which lies within the limits of random deviations. The other observations have still better agreement.

(2) **Check by comparing observations of two or three observers.** A beginning navigator will find it very helpful to compare his measurements with those of a more experienced observer. To do this, altitudes are observed on command and compared after applying instrument corrections. If there are discrepancies, find their causes. Later, two or more observers measure a series of altitudes at about the same time. Reducing the altitudes to one common instant of time and zenith, find the mean values and compare them. Derive the errors ε_h and ε_0, and analyze them too.

(3) **Check on observations and instrument by comparing observed altitude with the altitude computed at that instant from known φ and λ.** Every navigator should utilize any spare time when standing in roadstead or when sailing along the coast with an open and nicely visible horizon to determine the quality of his observations and the reliability of his sextant. To do this, take a series of altitudes of a body, and, if possible, the dip of the horizon. Taking the average altitude and time, correct the *sr* obtained. With the mean instant obtained and with known φ_{ob}, λ_{ob}, find the "computed" altitude in the ordinary way using a five-place table. This altitude will in the given case be the actual altitude. The difference $h_{ob} - h_{tr}$ will indicate the total error of observation; if the value is greater than 0′.4-0′.5, we have either a constant error of the sextant, or a personal error of the observer.

In this case, the mean random error ε_h is found after reducing the corrected altitudes to the instant (in roadstead) or to the instant and the zenith of the computed altitude. Then the differences $(h_{ob})_i - h_{tr} = \pm\delta_i + \Delta$, after eliminating Δ, will represent the true errors and ε_h is computed from the formula (15.12).

(4) **Determining a personal systematic error.** To an observer, the sun's disc against a dark background appears greater than it actually is; this is due to irradiation. For this reason, the semidiameter derived from observations may be greater than the actual value. The personal error combined with the irradiation error may be found in the following way.

If R is the actual value of the apparent angular radius of the sun (from an almanac), and R' is its value obtained from observations ($4R' = sr_2 - sr_1$), then

$$4R' - 4R = \xi \pm \delta$$

where ξ is the personal error and δ is the random error of measurement.

Taking a number of observations, we get

$$4\left(\frac{\Sigma R'}{n} - \frac{\Sigma R}{n}\right) = \frac{\Sigma \xi_i}{n} + \frac{\Sigma \delta_i}{n} = \xi$$

since $\Sigma\delta_i$ is close to zero due to the property of errors.

(5) **Checking quality of observations from the magnitudes of ε_h.** Every observer, particularly beginners, should periodically compute the mean-square error of observation of the altitude of the sun and stars to determine the spread in one's observations. These computations should be carried out under a variety of conditions with a simultaneous determination of position, or, still better, with known coordinates. The mean error under identical conditions should be roughly the same. For a beginning observer, ε_h should gradually diminish towards the normal value.

(6) **On combining observations into a single mean.** On the basis of a theoretical analysis we may draw the following conclusions about averaging a series of altitudes:

(1) In all latitudes, with altitudes of the sun less than 70° and with rapid measuring of altitude (at intervals of about 30s) it is possible to average three and five altitudes even if they are measured near the meridian.

(2) In high latitudes, observed altitudes of the sun may be averaged in all cases, since the time interval ΔT never actually exceeds 5 to 6 minutes.

(3) In low latitudes do not average a series of altitudes for very large altitudes of the sun near the meridian; in other words, treat each altitude separately.

SEC. 85. WAYS OF REDUCING THE EFFECTS OF RANDOM AND SYSTEMATIC ERRORS

By adhering to certain general rules and taking the necessary measures, an observer can reduce the magnitude of errors in altitude and increase reliability in determining the coordinates φ, λ or the position of the ship.

I. TO REDUCE THE EFFECT OF RANDOM ERRORS ON THE ALTITUDE

(1) When observing altitudes, choose the most suitable conditions, such as place of observation, a reliable sextant and appropriate telescope, the brightest celestial body, the illuminated part of the horizon, etc.

(2) Strive to make every altitude measurement as carefully as possible and record the exact time; altitudes are reckoned to within $0'.1$ and time to $0^s.5$. It is hard to make the number of observations compensate for carelessness in observing.

(3) Always measure more than one altitude (3 or 5) and as fast as possible in a series, recording the corresponding time. The number of altitudes measured in succession should not, however, be too great, because fatigue of eye and hand will gradually reduce the accuracy, and subsequent increases in number do not make the arithmetic mean more accurate. It is best to take an odd number of altitudes to obtain a rough first check; the arithmetic mean for uniform intervals between altitudes should be close to the mean of the readings. The next check is done on the basis of differences, as shown above.

Derive the arithmetic mean from the series of altitudes and instants obtained and use it in subsequent analysis.

The mean instant is usually rounded off to the least integral second, since observers are prone to delay registration of tangency of the images.

(4) Practice measuring altitudes regularly and make a periodic check of your observations by the procedures indicated above.

II. TO REDUCE SYSTEMATIC ERRORS IN ALTITUDE

(1) prior to each series of observations, make a rough check of the sextant and adjust it periodically;

(2) in each series of observations, determine the index correction;

(3) eliminate possible backlash of the tangent screw;

(4) remember that the instrument correction s changes with time; for this reason, have the sextant checked and certified at periodic intervals (every 3 years);

(5) always reduce the chronometer correction to the instant of observations by the daily rate;

(6) strive to measure the dip with an instrument rather than relaying on tabulated values;

(7) as far as possible, do not use very small altitudes of bodies. But if they are measured, obtain the dip by a dipmeter and take into account the peculiarities of correcting small altitudes;

(8) bear in mind that personal errors are possible in measured altitudes;

(9) in a heavy sea, take the height of the eye above the wave top; that is, reduce it by about 0.5 of the height of the wave.

Systematic errors in altitudes are ordinarily greater than random errors.

CHAPTER 16

THE CELESTIAL GLOBE AND AIDS THAT REPLACE IT

SEC. 86. THE CELESTIAL GLOBE, DESIGNATION AND CONSTRUCTION

Many problems of nautical astronomy may be solved approximately with the aid of a map and the celestial sphere, as in Sec. 5. But if the same pattern is made in the form of a model of the celestial sphere with stars indicated appropriately, these problems can be solved more simply and accurately.

The celestial globe is an instrument that models the celestial sphere and is designed for approximate solutions of problems in nautical astronomy. With a globe, one can give an approximate solution of nearly all problems of nautical astronomy, yet it is mostly used for star identification (particularly for cases of poor visibility of the sky) and choice of stars for determining positions at sea. Globes come in a variety of designs. One of the best designs is the Soviet "ЗГ" globe (Fig. 139).

The globe is a hollow plastic (or metallic) sphere of diameter 168 mm, containing pasted-on star maps, section by section, in projections such as to practically eliminate distortions. The maps contain the principal circles: the celestial equator, parallels (at intervals of 10°), meridians (at intervals of 15° = 1h) and the ecliptic. The celestial equator is divided into degrees (at 1° intervals) and in addition, at 15m intervals below with every hour numbered. These divisions represent a scale of right ascensions α, and since the right ascension of zenith $\alpha_Z = S_{loc} = t_{loc}^{\Upsilon}$, this same scale, when setting, yields local sidereal time. Reckoning begins with Aries and is indicated by number XXIV (360°). At this point, the ecliptic intersects the equator, that is, the sun passes from the southern hemisphere into the northern. The ecliptic and the meridians of points with $\alpha = 360° - 180°$ and $90° - 270°$ are also divided into 1° intervals.

At the poles, the sphere has depressions for the axis of a metallic ring that encompasses the sphere and indicates an arbitrary meridian. When the sphere is set in its box, the horizontal metal ring inside the box depicts the celestial horizon; the ring of the meridian

is inserted into the slots at points N and S of the horizon and will now represent the *meridian of the observer*.

On top of the globe, a crosspiece of vertical circles is mounted with movable index; the zenith of the observer is represented by a ball on top of the crosspiece.

The celestial globe portrays the celestial sphere (with centre in the eye of the observer) as if looked at from *outside*. As a result, the figures of all constellations are in positions the reverse of those seen on the celestial sphere.

So that the globe should reproduce the stellar sky as seen by the observer at a given time, set the globe for the latitude of the observer and turn the sphere to the time of observations. Since the globe is a model of the celestial sphere, remember when setting it in latitude that the name of the elevated pole always corresponds to the name of the latitude of the observer, whereas the altitude is equal to the latitude of the observer (φ). Thus, if the latitude is N, set P_N of the globe over point N of the ring; P_N is identified by the star Polaris (constellation Ursa Minor); but if the latitude is S, then set P_S of the globe (opposite P_N) over point S of the ring. *The inclination of the globe axis to the horizon must equal* φ *of the observer*. Remember that the reading on the arc of the ring will

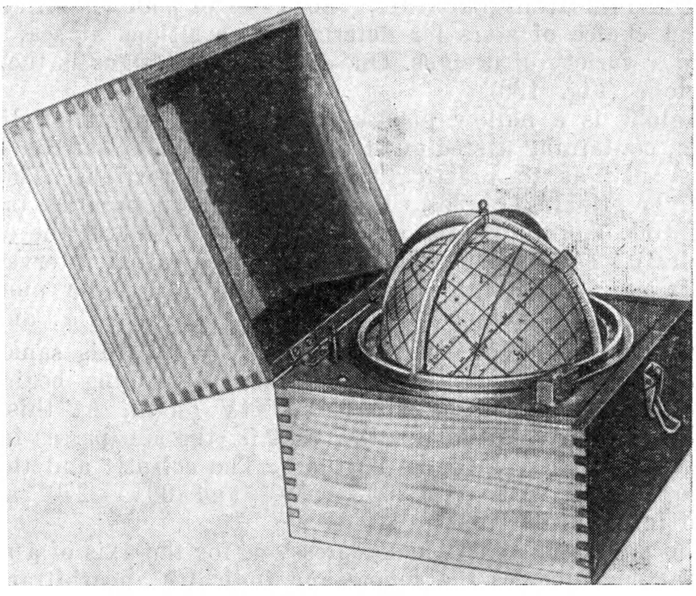

Fig. 139

be $90° - \varphi$, since the division marks on the meridian ring are meant for declinations and proceed from the equator. From the relation $\varphi = \delta_Z$, the reading on the ring of the meridian near the zenith is also equal to φ.

To set the globe to the time of observation, turn it relative to the meridian of the observer to a position corresponding *to local sidereal time* $S_{loc} = t^{\Upsilon}_{loc}$.

To do this, turn the globe so that under the arc of the observer's meridian (the upper branch) we have the equatorial reading equal to t^{Υ}_{loc} in degrees of hours. The globe will not turn on its own due to friction of the sphere against the cushion and spring inside the box.

The value of $t^{\Upsilon}_{loc} = S_{loc}$ may be computed from the MAE in the usual way (see Sec. 46, Item I) or approximately (to 4m), utilizing the time of transit of Aries (T^{Υ}_{tr}) on the Greenwich meridian as given in the daily tables of the MAE. Subtracting this time from T_{loc}, we get S_{loc} approximately. However, this procedure is no simpler than the first due to the trouble in getting T_{loc} under ordinary conditions.

A number of other problems may also be solved by turning the sphere: to bring a celestial body to the horizon, the prime vertical, the observer's meridian, to a given azimuth or altitude, and so forth.

The horizontal coordinates h and A are reproduced on the globe by means of the *crosspiece of vertical circles.* To set the crosspiece in azimuth, turn one of the vertical circles along the ring of the horizon to a reading equal to azimuth A in quadrantal reckoning; the altitude is laid off along the vertical circle and is fixed by the index.

As already mentioned, the globe indicates the *fixed positions* of 167 bright stars. Star movements on the celestial sphere are slight (see Sec. 23) and these positions will be sufficiently accurate for approximate solutions for 20 to 30 years.

Bodies that have noticeable proper motions (the sun, moon and planets) *are positioned on the globe by the observer himself* as needed or at periodic intervals. To fix these bodies on the globe, take their declinations and right ascensions from the MAE. Remember that to obtain α of the moon or sun, and also for more precise computation of planetary α, select t_{gr} of the body and t^{Υ}_{gr} for the same hour, then: $\alpha_{body} = t^{\Upsilon}_{gr} - t^{body}_{gr}$ (see Sec. 47, Item IV).

The α of the body thus obtained is reckoned from point XXIV (360°) of the globe on the scale of the equator; but the declination is laid off along the ring of the meridian to the N or S. Use a special wax pencil to mark the point on the surface of the globe and indicate the astronomical symbol of the celestial body. The planets and

the moon should be located near the ecliptic. For better orientation the planetary positions are indicated periodically: Venus every week, Mars every two weeks, Jupiter and Saturn every month.

However, when solving a specific problem, the position of a given body should be revised for the given 24-hour period, and, with respect to the moon, for the given hour. The position of the sun is always on the ecliptic, which makes it easy to indicate it on the basis of $\alpha_{\odot}$.

Example 1. On 12.09.62, indicate for evening twilight, $T_{sh} = 18\text{h}\,40\text{m}$ (ZD = 11E), the position of the moon and the planets Jupiter and Saturn which are visible at this time.

(1) Approximately, from MAE

	Jupiter	Saturn
	$\alpha = 22\text{h}\,34\text{m}$	$\alpha = 20\text{h}\,32\text{m}$
	$\delta = 10°.5\text{S}$	$\delta = 19°.5\text{S}$

(2) Exactly, from MAE.

$$\begin{array}{r|l} -\,T_{sh} & 18\text{h}\ 40\text{m}\ 12.09 \\ \text{ZD} & 11 \\ \hline T_{gr} & 7\text{h}\ 40\text{m} \approx 8\text{h} \end{array}$$

	Jupiter	Saturn	Moon
t_T^{Υ}	110°51′	110°51′	110°51′
t_T^{body}	132 22	162 56	144 53
α_{body}	338°29′	307°55′	325°58′
δ_{body}	10°32′S	19°39′S	15°3′S

Problem solving by the "ЗГ" celestial globe is done with an accuracy to within ±0°.5-1°.5. Therefore, the requisite data (α and t_{loc}^{Υ} and also α and δ of the celestial bodies) do not need to be more accurate than ±0°.3-0°.5.

Many firms in other countries manufacture globes that differ in design from the "ЗГ" globe. In some cases, the box is replaced by a support made up of two half-rings; in others there is no crosspiece of vertical circles, a mobile ring taking its place, and so forth. Yet they function in similar fashion to the "ЗГ" globe.

SEC. 87. SOLVING PROBLEMS WITH THE CELESTIAL GLOBE

The celestial globe is used to solve three basic types of problems:

(1) determining the names of observed but unidentified stars or planets;

(2) obtaining h and A of stars or planets at a given time, and varieties of this problem, such as: (a) choosing stars for observations, (b) finding ΔK;

(3) determining the time of arrival of a body at a given position; for example, the time a body rises, crosses the prime vertical, transits, etc.

I. DETERMINING THE NAME OF AN UNIDENTIFIED STAR OR PLANET

There are cases when the sky is overcast with breaks in the clouds showing separate stars. In such cases it is rather difficult to identify a star that has been observed, so the celestial globe is resorted to. Also, problems of this kind are solved in studies of the stars.

This problem is solved in the following sequence:

(1) After measuring the altitude of the star, determine its bearing by compass and note T_{sh}. From the map take φ and λ.

(2) Compute T_{gr}; take t_{gr}^{Υ} out of MAE and compute $t_{loc}^{\Upsilon} = t_{gr}^{\Upsilon} \pm \lambda_{W}^{E}$.

(3) Set globe for φ and t_{loc}^{Υ} (see preceding section).

(4) Transfer bearing to A in quadrantal reckoning. Set arc of vertical circle in azimuth and the index of the vertical circle in altitude.

(5) Under the index find the star by its *position in the constellation*, which is given in Latin, for instance, α of the constellation of Taurus (α Tauri). Using the star list in the MAE, find the Russian name of the constellation and the number of the star. Using this name (or number) take the coordinates out of the MAE. Thus, Taurus α is Телец α, No. 24 (Aldebaran).

(6) If there is no star under the index, or a bright body was observed, then a blunder has been made in solving the problem, or a planet was observed. When you are sure the solution is correct, identify the planet. This may be done by one of two procedures: approximate or exact.

(a) In the approximate procedure, use the table of "Planet Visibility" given at the beginning of the MAE; to identify by the globe, find the name of the constellation near the index of the vertical circle and using the constellation find the name of the planet in the table.

(b) In the exact procedure, take α and δ of the point under the index from the globe. With these data and the date enter the daily tables of the MAE and find the planet for which δ and α are closest to those given.

Example 2. 13 September 1962 at $\varphi_c = 56°20'$N; $\lambda_c = 20°54'$E at $T_{sh} = 18$h 35m observed unidentified body: $sr = 38°7'.5$; $RCB = 330°$; $\Delta K = +3°$. Find the name of the body.

$-\ T_{sh}$	18h 35m
ZD	1
T_{gr}	17h 35m 13.09

t_T^{Υ}	247°12′
Δt	8 46
$+\ t_{gr}^{\Upsilon}$	255°58′
λ	20 54
t_{loc}^{Υ}	276°52′ ≈ 277°

$-$ RCB	330°
180°	180
CB	150°
ΔK	+ 3
TB	153° = 27°SE

Set globe in latitude; to do this, raise P_N above point N of the horizon 57°.3 (32°.7 as reckoned from the meridian). To set by time, turn globe to 277° on the meridian ring. Then place crosspiece of vertical circles and turn in azimuth 27°SE, and the index to $h \approx 38°$. We find the star α Aquilae. In the star list of the MAE we have α Aquilae, Altair, No. 146.

Example 3. On 27.09.62 at $\varphi_c = 33°58'$N; $\lambda_c = 148°30'$E at $T_{sh} = 18$h 10m (ZD = 10) observed body: $sr = 28°35'$, $CB = 327°$; $\Delta K = -2°$. Find name of celestial body.

$-\ T_{sh}$	18h 10m
ZD	10
T_{gr}	8h 10m 27.09

t_T^{Υ}	125°38′
Δt	2 30
$+\ t_{gr}^{\Upsilon}$	128 8
λ	148 30
t_{loc}^{Υ}	276°38′ = 276°.5 } φ_c = 34N }

$\pm\ RCB$	327°
180°	180
CB	147°
ΔK	−2
TB	145° = 35°SE } $h = 28°.5$ }

Star not found under index; from table of "Planet Visibility" we see what planet could be in the constellation near the index. But since the index stopped between the two constellations Capricornus and Sagittarius, we have to apply a different procedure. Take the coordinates of the point: $\alpha \approx 308°$, $\delta \approx 20°$S. From the MAE on 27.09 we find that these coordinates belong to Saturn.

II. OBTAINING ALTITUDE AND AZIMUTH OF A BODY FOR A GIVEN TIME

(1) Compute T_{sh} and T_{gr} for the instant of proposed observations, take φ_c and λ_c from map for this time. Stars are mostly observed in twilight, so compute T_{sh} at twilight.

(2) Compute $t^{\Upsilon}_{loc} = t^{\Upsilon}_{gr} \pm \lambda_c$.

(3) Set globe for φ and t^{Υ}_{loc}.

(4) Set crosspiece so that numbered vertical circle is at star; direct index to position of star, then note and record readings of h and A of star.

(5) If it is required to obtain h and A of a planet, first mark its position on the globe by α and δ, as indicated in the preceding section.

Example 4. On 10.10.62 at T_{sh}=6h 05m in φ_c=62°5′N; λ_c=11°57′W. Determine h and A_{*} α Orionis (Betelgeuse).

T_{sh}	6h 05m
ZD	1
T_{gr}	7h 05m 10.10

t^{Υ}_{T}	123°24′
Δt	1 15
t^{Υ}_{gr}	124°39′
− λ	11 57
t^{Υ}_{loc}	112°42′ = 112°.7

Setting globe by φ and t^{Υ}_{loc}, we have $\begin{cases} h_{*} = 32° \\ A_{*} = 30°\text{SW} \end{cases}$

Determining the compass correction ΔK also reduces to this problem. However, here it is first necessary to find the CB of the body and note T_{sh}, φ_c, λ_c, which are used to obtain the TB of this body from the globe.

Example 5. On 12.09.62 in φ_c=52°24′N; λ_c=156°41′E observed $*$ Capella (α Aurigae); RCB=203°.5; T_{sh}=20h 15m (ZD=11E); altitude of star does not exceed 15°. Find ΔK.

Solution.

(1)

T_{sh}	20h 15m
− ZD	11
T_{gr}	9h 15m 12.09

t^{Υ}_{T}	125°53′
Δt	3 46
t^{Υ}_{gr}	129°39′
+ λ	156 41
t^{Υ}_{loc}	286°20′ ≈ 286°.5
φ_c	52°.5

(2) From globe for $*$ α Aurigae we have A=19°.5NE

(3)

TB	19°.5
CB	23 .5
ΔK	−4°.0

When determining position by two stars, the difference of their azimuths must be as close as possible to 90°; when using three stars, the difference in each pair should be close to 120°, and for four stars,

close to 180° in each pair, and close to 90° between pairs. Besides that, take into account the brightness of the horizon, its visibility in these parts, and so forth. The altitudes of the stars should not exceed 60° to 70°. These are the conditions that should be borne in mind when choosing stars. Otherwise, the problem consists in obtaining and recording h and A of stars, which means that it reduces to the preceding problem. The positions of the planets are marked beforehand. It is particularly important to predetermine h and A of the stars (planets) for observations immediately after sunset, when the stars are not visible to the naked eye and have to be found in the sextant telescope.

Example 6. On 10 May, 1962, in $\varphi_c = 9°38'$S; $\lambda_c = 98°15'$E choose three stars for observations in morning twilight. $T_{twi} = 5$h 45m (ZD=7); $\Delta K = -5°$.

T_{twi}	5h 45m	10.05	t_T	197°14′
−ZD	7		Δt	11 17
T_{gr}	22h 45m	9.05	t_{gr}^{Υ}	208°31′
			$+\lambda$	98 15
			t_{loc}^{Υ}	306°46′ ≈ 306°.8; $\varphi \approx 9°.5$S.

Over S, place the *south celestial pole* (P_S) at an altitude of 9°.5 and set the globe for t_{loc}^{Υ}. Using the vertical circles, choose the stars and tabulate the data found.

No.	Constellation and star	h	A	CB
1	α Lyrae (Vega)	35°	28°NW	337°
2	α Piscis Aust. (Fomalhaut)	50	54 SE	131
3	α Scorpii (Antares)	30	64 SW	249

III. DETERMINING TIME OF ARRIVAL OF A CELESTIAL BODY AT A GIVEN POSITION (AT RISING, PRIME VERTICAL, TRANSIT, ETC.)

(1) Take φ_c and λ_c from map for proposed T_{sh} of phenomenon.
(2) Set globe to latitude.
(3) Turn sphere and bring indicated star or planet to required position (on horizon, on prime vertical, etc.).

(4) Take reading of $t_{loc}^{\Upsilon} = S_{loc}$ on upper branch of observer's meridian ring.

(5) Compute $t_{gr}^{\Upsilon} = t_{loc}^{\Upsilon} \mp \lambda_{W}^{E}$ and with aid of MAE obtain T_{gr} and T_{sh} of the phenomenon (see Sec. 47).

Example 7. On 27.09.62, in the evening in $\varphi_C \approx 44°N$, $\lambda_C \approx 137°20'E$, find the time (ZD = 10E) when ✱ Sirius rises.

Solution. Set globe for $\varphi_C = 44°N$ and reduce ✱α Canis Majoris (Sirius) to eastern part of horizon, then take at the meridian ring $S_{loc} = t_{loc}^{\Upsilon} = 27°.5$.

	$-\ t_{loc}^{\Upsilon}$	27°.5
	λ_E	137 .3
	t_{gr}^{Υ}	250°.2
MAE	t_{T}^{Υ}	245 .9 ...
Table 1	Δt^{Υ}	4°.3 ...

T'_{gr}	16h
ΔT	17m
T_{gr}	16h 17m
$+$ ZD	10
T_{sh}	2h 17m 28.09

By computation we find that the star will rise next at 2h 17m on 28.09; the previous time was at 2h 21m on 27.09

Somewhat simpler is the approximate solution of this ploblem using the time of transit of Aries (T_{tr}^{Υ}). In this case, the sidereal time obtained from the globe, t_{loc}^{Υ}, is added to T_{tr}^{Υ} taken from the MAE and we obtain the approximate (to within $\pm$ 6m) local time (T_{loc}^{Υ}) of the phenomenon. For instance, in Example 6 we will have

T_{tr}^{Υ}	23h 35m
t_{loc}^{Υ}	1h 50m (27°.5)
T_{loc}	1h 25m 28 .09
ZD — λ	51
T_{sh}	2h 16m 28 .09

This solution is most advantageously used when the correction (ZD — λ) is constant, that is, when standing for a long time.

SEC. 88. AIDS THAT REPLACE THE CELESTIAL GLOBE

The celestial globe, despite the small size of the sphere, is still somewhat unwieldy, and it is inconvenient to work with in confined spaces on shipboard or in aircraft. *Flat images* of the celestial sphere

are then used in the form of special charts and grids or, finally, special tables for star identification. The solutions obtained are not so accurate or pictorial, but there are other advantages, such as compactness and the possibility of comparison with the sky (in some types). Such aids are called star charts and star finders.

I. THE SOVIET STAR CHART FOR AIR NAVIGATION (БКН)

In this chart (Fig. 140), part of the sphere, which is visible in the given latitude, is depicted in a polar equidistant projection, with the brightest stars, the outlines of constellations and the principal circles of the sphere indicated. The equator and meridian of Aries ♈ are graduated in 10° intervals, so that planetary positions can also be indicated approximately. Around the circumference of the chart is a date scale computed from values of right ascension of the mean sun ($\alpha_{\oplus}$). БКН charts are designed for medium latitude of a definite zone. Three (sometimes more) zones are taken for the territory

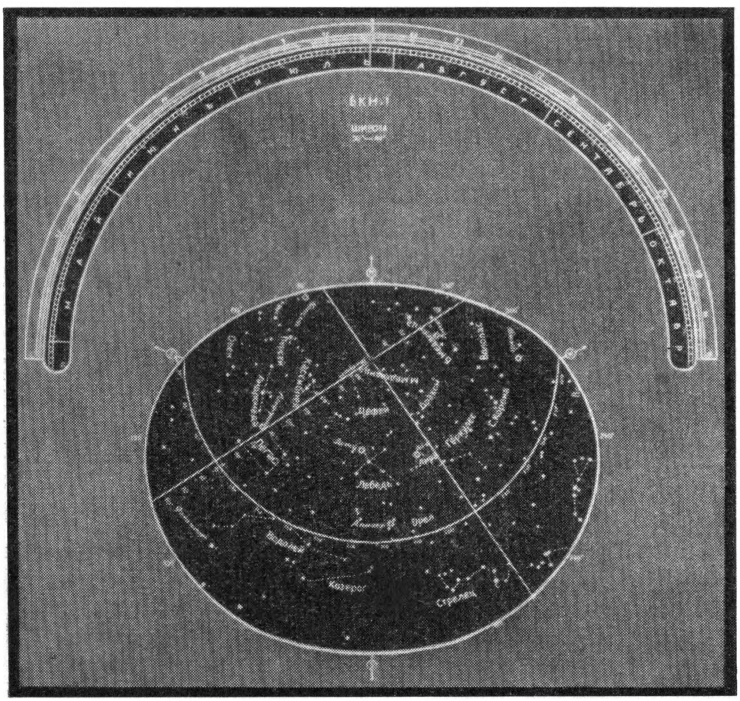

Fig. 140

of the U.S.S.R.: I for φ from 30° to 44°N, II for 46° to 60°N, III for 62° to 76°N. The appropriate chart is chosen closest to the given φ_c.

The chart comes in a cardboard case, in which it can be rotated about the centre (P_N). In the case is an oval slot, the edge of which depicts the horizon for mid-latitude of the zone. It contains the points N, E (O^{st}), S, W and scale of azimuths every 30°. The horizon slot is based on the declinations of points of the horizon ($h = 0$) which are computed from the formula $\tan \delta = -\cot \varphi \cdot \cos t$, where t represents the hour angles of points of the horizon and δ their declinations.

At the top of the case, round the periphery is a slot for the scale of dates; near the slot is a scale of local civil time, T_{loc}.

To set the chart to the time,

(a) compute T_{loc} from formula (8.28);

(b) rotate chart to bring the date (on the date scale) to T_{loc} of the time scale.

The chart sky will then approximate the actual sky. To identify constellations, put the chart over your head and, aligning it N—S by compass, look at the chart and the sky.

The БКН chart yields only rough azimuths and altitudes of celestial bodies, particularly for bodies at small altitude. This is due to a lack of correspondence of latitudes and to distortions, and makes identification of individual stars difficult. Using the БКН chart, one can make a rapid and easy determination of time of transit of a star and the approximate time that it rises and sets.

II. STAR IDENTIFIER*

The star identifier shown in Fig. 141 consists of two charts of stars on two sides of a plastic sheet: one side contains the northern hemisphere including a belt to $\delta = 60°$S, the other, the southern hemisphere to 60°N. On the outer circumferences of the charts are scales of sidereal time. The charts are compiled in polar equidistant projection and contain the brightest navigational stars and also a grid of parallels with 10° intervals. In addition, the separate transparent plastic templates contain seven grids of vertical circles and parallels of altitude in the same projection at 5° intervals of h and A for medium latitudes 0°, 10°, 20°, 30°, 40°, 50°, 60°, that is, for 10° latitude zones. The outer curve of these grids represents the observer's horizon.

When using the star finder, choose the grid for the latitude closest to the computed one. The 180°-360° line of the grid is set in the

* By way of illustration we describe the "Star Identifier" widely used in the U.S.A. and elsewhere. It is similar to the "Rude Star Finder" (H. O. No. 2102-D).

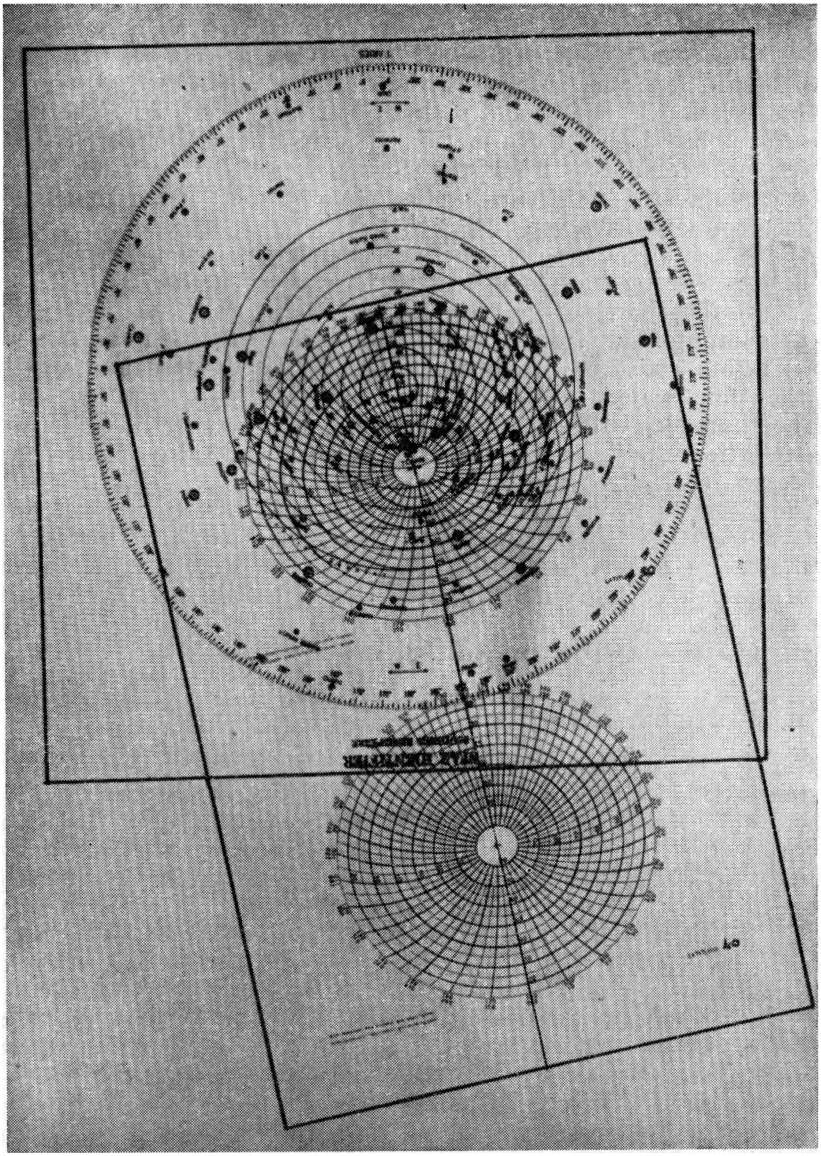

Fig. 141

direction: pole—computed sidereal time S_{loc}. The centre of the grid, indicated by a cross ($\times$) and depicting the zenith, is placed by eye on a parallel of the chart equal to φ. The altitudes are taken from the grid of parallels of altitude, while the azimuths are taken from the grid of vertical circles. Accuracy obtained by a star identifier is of the order of $\pm$ 3°-5°. Its advantage lies in handiness

Example 8. On 10 July, 1962, at $\varphi = 49°S$; $\lambda = 61°W$; $T_{sh} = 17h\ 34m$ (ZD = 4W). Determine h and A of $\ast$ Spica.

T_{sh}	17h 34m	t_T^{Υ}	243°18′
ZD	4	Δt^{Υ}	8 31
T_{gr}	21h 34m 10.07	t_{gr}^{Υ}	251°49′
		$-\lambda$	61
		$t_{loc}^{\Upsilon} \approx$	190°8

Choose a grid for $\varphi = 50°$. Set the 180°-360° line of the grid on the reading $t_{loc}^{\Upsilon} = 190°.8$ on the chart, the cross + (zenith) between the parallels 40° and 50° of the chart (on 49°). Through the grid, find Spica on the chart and record $h = 50°$, $A = 14°$ from the grid (see Fig. 141).

The foregoing are only a few of many aids in the identification of constellations and stars. The basic principles are much the same in all of them.

III. STAR IDENTIFICATION BY MEANS OF TABLES

An observed star may also be identified by means of special or conventional tables and lists of stellar coordinates. To do this, measure the altitude and azimuth of the star, note the watch time, and take φ_c, λ_c. From known h, A, φ_c it is possible to solve the astronomical triangle for $\delta_\ast$ and $t_\ast$ using formulas (2.5) and (2.6) or the tables TBA-57; enter these tables with h instead of δ, and azimuth in semicircular reckoning in place of t. Having computed t_{loc}^{Υ} for the noted instant T_{gr}, we readily get: $\alpha_\ast = t_{loc}^{\Upsilon} - t_\ast$; then with $\delta_\ast$ and $\alpha_\ast$ we choose the name of the observed star from a list of stars (in the MAE, for instance) or from any star chart. Solution via formulas (2.5) and (2.6) or TBA-57 is rather involved. Instead, one can use special rude "tables of star identification" that appear in a number of publications, for example, H.O. No. 214, which contains the values of δ and t on the basis of A and h at 4° intervals for each degree of latitude. For the same problem, one can use a coordinate grid, or simply any hand drawing of the sphere to scale.

PART THREE

METHODS OF NAUTICAL ASTRONOMY

CHAPTER 17

ASTRONOMICAL DETERMINATION OF THE COMPASS CORRECTION

SEC. 89. THE FUNDAMENTALS OF ASTRONOMICAL DETERMINATION OF THE COMPASS CORRECTION

Compass readings, due to a number of internal and external causes, are subjected to systematic and random errors. Ordinarily, systematic errors greatly exceed random errors in magnitude, and to eliminate them from compass readings it is necessary to introduce a so-called total compass correction ΔK.

When sailing near land, the total correction is found by leading beacons or from navigational determinations. When sailing in the *open sea*, the total compass correction can only be found by taking bearings of celestial bodies, in other words, by *methods of nautical astronomy*. The magnitude and sign of ΔK are defined by the formula

$$\Delta K = TB_{body} - CB_{body} \tag{17.1}$$

where TB_{body} is the true bearing of the celestial body equal to its azimuth in *circular* reckoning (0° to 360°)

CB_{body} is the bearing of the body obtained by compass.

Unlike navigation, where TB of an object (range) is obtained from a map or is indicated in pilot books at sea, the true bearing of a celestial body is obtained by computing from formulas or with special tables and grids.

Thus, the problem of determining the compass correction is in principle solved very simply astronomically: take the compass bearing of the body and compute its azimuth for that instant; the difference will be the compass correction.

Formulas for computing azimuth (true bearing) of a body are derived from the astronomical triangle ZP_NC (Fig. 142) on the basis of general formulas of spherical trigonometry. Two methods may be used here: the time azimuth method and the altitude azimuth method.

(1) The time azimuth method. If when taking the bearing of a body we note the instant and pick off a chart of the coordinates φ and λ, then the triangle is solved by the resultant φ, δ, t_{loc} by means

of the cotangent formula

$$\cot A = \tan \delta \cdot \cos \varphi \cdot \operatorname{cosec} t - \sin \varphi \cdot \cot t_{loc} \tag{17.2}$$

This is the formula that is usually used to compute A when determining the compass correction.

(2) The altitude azimuth method. If when taking the bearing of a body we determine the altitude h, then with φ, δ, h we obtain from the cosine formula of a side (after transformations)

$$\cos A = \sin \delta \cdot \sec \varphi \cdot \sec h - \tan \varphi \cdot \tan h \tag{17.3}$$

This formula is applied in special cases for determining ΔK when the altitude of the celestial body is known, for example, at instant

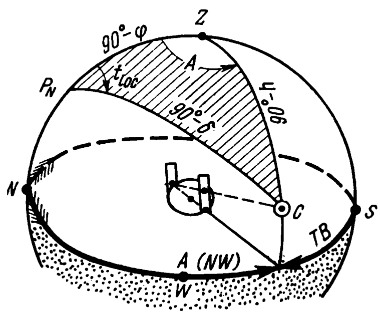

Fig. 142

of rising or setting. But in the general case, the time azimuth method is much more convenient and is the principal method used to determine the compass correction.

SEC. 90. THE EFFECT OF ERRORS IN D.R. LATITUDE AND LONGITUDE ON THE AZIMUTH BEING COMPUTED OF A CELESTIAL BODY. MOST FAVOURABLE CONDITIONS OF OBSERVATION

The coordinates φ and λ, which as a rule are computed, enter into formula (17.2). The azimuth of a body obtained from these coordinates will obviously also be D.R. coordinates, that is, it will contain errors of sailing. Let us consider the magnitudes of these errors, their effect on the result, and let us find the most favourable conditions for observations.

(1) The Effect of Errors in the D. R. Latitude on the Azimuth Being Computed

We determine the increment in azimuth as a function of the increment in latitude, for which purpose we differentiate (17.2) with respect to A and φ

$$-\frac{dA}{\sin^2 A} = -\sin\varphi \cdot \tan\delta \cdot \operatorname{cosec} t d\varphi - \cos\varphi \cdot \cot t d\varphi$$

Passing to finite increments and simplifying the formula, to do which we first apply formula $\sin h$ and then: $-\sin A : \sin t = \cos\delta : \cos h$, we get

$$\Delta A = \sin^2 A\left(\frac{\sin\varphi \cdot \sin\delta}{\sin t \cdot \cos\delta} + \frac{\cos\varphi \cdot \cos t}{\sin t}\right)\Delta\varphi$$

$$= \sin^2 A\left(\frac{\sin\varphi \cdot \sin\delta + \cos\varphi \cdot \cos\delta \cdot \cos t}{\sin t \cdot \cos\delta}\right)\Delta\varphi$$

$$= \frac{\sin^2 A}{\sin t} \cdot \frac{\sin h}{\cos\delta}\Delta\varphi = \frac{\sin A \cdot \sin h}{\cos h}\Delta\varphi = \tan h \cdot \sin A \cdot \Delta\varphi$$

that is,

$$\Delta A = \tan h \cdot \sin A \cdot \Delta\varphi \tag{17.4}$$

From (17.4) it is seen that for $h = 0$ and $A = 0$ the error in azimuth is equal to zero; otherwise the error ΔA is not zero.

Let us say that in the general case the error $\Delta\varphi$ in the computed latitude does not exceed 20 miles, i.e., $\Delta\varphi \approx 0^\circ.3$.

If we take $\sin A_{max} = 1$, the error in azimuth will depend solely on the altitude of the body. For altitudes of bodies (whose bearings are being taken) less than 18°, we have an error $\Delta A \leqslant 0^\circ.1$; for altitudes up to 35° we get $\Delta A \leqslant 0^\circ.2$. An ordinary azimuth circle (bearing finder) permits observing bodies without a mirror only up to 15°, other azimuth circles, up to 35°. Thus, for ordinary sailing conditions the error in D.R. (dead reckoning) latitude has hardly any effect on the azimuth.

(2) The Effect of Errors in Longitude on Azimuth Being Computed

When analyzing variations of horizontal coordinates due to the diurnal motion of the sphere (Ch. 3, Sec. 11), we obtained formulas (3.18) and (3.19)

$$\Delta A = -\cos\delta \cdot \cos q \cdot \sec h \cdot \Delta t \tag{*}$$

$$\Delta A = -(\sin\varphi - \cos\varphi \cdot \cos A \cdot \tan h)\,\Delta t \tag{**}$$

Regarding Δt as an error in the local hour angle $t_{loc} = t_{gr} \pm \lambda_W^E$, we come to the conclusion that this error can occur either due to an error in the chronometer correction and in t_{gr}, or due to errors

in the D.R. longitude. In the former case, the error will not exceed $\pm 2s$ and it may be neglected. We will therefore consider the error in the hour angle as due solely to errors in the D.R. longitude, that is,

$$\Delta t + \pm \Delta\lambda_c \tag{17.5}$$

and formulas (*) and (**) will take the form

$$\Delta A = \mp \cos\delta \cdot \cos q \cdot \sec h \cdot \Delta\lambda_c \tag{17.6}$$

or

$$\Delta A = \mp (\sin\varphi - \cos\varphi \cdot \cos A \cdot \tan h)\, \Delta\lambda_c \tag{17.7}$$

From these formulas it is seen that ΔA is zero only when $\delta = 90°$ (for Polaris, $\delta \approx 89°$) or $q = 90°$, that is, for bodies in elongation; otherwise, ΔA is not equal to zero. The error ΔA has smallest values for positions of the body after rising to the prime vertical (or after the prime vertical prior to setting), as was shown in Chapter 3, Sec. 11. In addition, ΔA has a relative minimum in special positions of the body: on the prime vertical for $A = 90°$ or $270°$ and at rising when $h = 0°$.

Let us consider the possible magnitude of the error ΔA. For ordinary sailing conditions, the dead reckoning error in departure does not exceed 20 miles, that is for medium latitudes $\Delta\lambda_c \leqslant 0°.3$-$0°.5$. If in formula (17.6) we take the mean values for q and δ and put $h \leqslant 20°$, then ΔA will be less than $0°.2$. Consequently, the error in D.R. longitude does not affect the azimuth for small altitudes of the body. In high latitudes ($\varphi > 75°$) for the same conditions, the error ΔA may attain $\pm 1°.5$. For this reason, the error in departure here should not exceed $3'$ to $5'$.

From the foregoing analysis we can draw some general conclusions:

(a) The errors in D.R. latitude and longitude are least if the celestial body is located *at low altitude.*

(b) For ordinary conditions of sailing, assuming that the errors in D.R. φ and λ do not exceed $0°.3$ and for the bearings of bodies taken at altitudes up to 35°, we may regard the *computed azimuth* as practically equal to the *true azimuth of the body.*

(c) In low northern latitudes (up to 35°) one of the best objects for sighting to determine ΔK is the Pole Star.

SEC. 91. SYSTEMATIC AND RANDOM ERRORS IN THE COMPASS BEARING OF A CELESTIAL BODY. CONDITIONS FOR TAKING BEARINGS

Taking the bearings of terrestrial objects and celestial bodies involves random and systematic errors, like any other instrumental measurements.

Let us examine the causes of these errors and their effects on the result.

I. SYSTEMATIC ERRORS IN AZIMUTH CIRCLE OF BODIES

A. Instrument Errors of Azimuth Circle

When taking the bearings of terrestrial objects and particularly of celestial bodies, one should bear in mind the errors of the azimuth circle itself. These errors are systematic in character, are found both in bearing finders for magnetic compasses and in bearing finders of gyrocompasses.

Reading errors in old-type azimuth circles are due mainly to incorrect positions of elements: the prism, sighting vane, azimuth vane, and also the eccentricity and free play of the circle. The total error may exceed 2°-3°. These errors are reduced by adjusting the azimuth circle (this is discussed in detail in courses of compass deviation and nautical instruments). When it is impossible to make an adjustment or when a partial adjustment is made, find the total error from range observations or by other methods and then introduce it as a correction. If the error of the azimuth circle has not been eliminated or determined, it will completely enter the compass correction ΔK being determined, and will cause a similar error in the true course when correcting the compass course. It is therefore advisable, when determining the compass correction (if reliability of the azimuth circle is not assured) to determine ΔK twice on a given compass: with the main azimuth circle and with a reserve circle (or from another repeater). If there is an appreciable discrepancy between the corrections, the ΔK obtained is unreliable.

B. The Effect of Errors Due to Position of Mirror

When taking the bearings of celestial bodies, mainly the sun, at altitudes greater than 15°, all azimuth circles, with few exceptions, are equipped with mirrors. In such cases, the bearing is taken of the reflection of the body in the mirror. It is rather difficult to set the mirror perfectly, and the position can change, all of which results in a systematic error in the bearing.

As studied by Prof. Kavraisky, this error δA is expressed by the formula

$$\delta A = \left(k + l \tan \frac{h}{2} + p \sec \frac{h}{2}\right) \tan h \qquad (17.8)$$

where k is the angle of tilt of the axis of rotation of the mirror to the plane of the azimuth circle

l is the angle between the axis of rotation of the mirror and a perpendicular to the sighting plane

p is the angle between the axis of rotation of the mirror and its reflecting plane.

Taking $k = l = p = 0°.5$, we get for $h = 15°$ an error $\delta A = 0°.3$, for $h = 45°$, $\delta A = 1°.4$, for $h = 60°$, $\delta A = 2°.4$. Thus, even slight errors in the position of the mirror bring about perceptible errors in the compass bearing of the body for small altitudes and prohibitive errors for large altitudes. Therefore, use the mirror only in exceptional cases, when there is no other way out; the correction obtained should be regarded with great caution.

When using optical azimuth circles whose field of view is limited, choose celestial bodies the bearings of which may be taken without a mirror or, if such are lacking, then bodies at the lowest possible altitude for bearing-taking with a mirror. Do not think that the mirror of such an azimuth circle has to be put to use and is free of errors.

C. Error in Compass Bearing of Bodies Due to Tilt of Sighting Plane of Azimuth Circle

An error will appear in the bearing reading if, when taking a bearing we tilt the azimuth circle sideways and thus take the sighting plane out of the vertical circle of the body.

On an auxiliary sphere (Fig. 143), let HH' be the plane of the azimuth circle of a magnetic compass, ZM the sighting plane of the

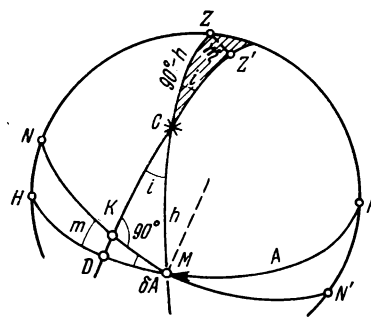

Fig. 143

azimuth circle that coincides with the vertical circle of the celestial body for a correct position of the azimuth circle. Now if we tilt the plane of the azimuth circle (bowl) an angle m and again point the sighting plane to the body C, we will have the reading $A + \delta A$ instead of A. Let us find δA. When tilted, the vertical axis of the azimuth circle is shifted from Z to Z' by an amount of arc equal to the angle of inclination m. From the elementary spherical triang-

In $DC'M$, using the formula of cotangents, we have

$$\cot i \cdot \sin 90^\circ = \cot \delta A \cdot \sin h - 0$$

or

$$\tan \delta A = \tan i \cdot \sin h$$

replacing the tangents of small angles by the angles themselves, we get

$$\delta A = i \cdot \sin h \tag{17.9}$$

From the elementary triangle CZZ' we similarly have

$$m = i \sin (90^\circ - h) = i \cdot \cos h$$

Dividing (17.9) by the latter formula, we finally get

$$\delta A = m \cdot \tan h \tag{17.10}$$

It will be seen that the error $\delta A = 0$ if the body is on the horizon.

When the azimuth circle of a gyrocompass repeater is tilted, the plane of its compass card is likewise tilted and occupies a position NN' (Fig. 143). The azimuth reading will also change, but somewhat less: by MK.

Indeed, from the small triangle MKD we have

$$\delta A_{gyr} = MK = \delta A \cdot \cos m = m \cdot \tan h \cdot \cos m \tag{17.11}$$

For small angles of inclination, $\cos m \approx 1$, and we get expression (17.10).

From an analysis of formulas (17.10) and (17.11), it is seen that reading errors of bearings of compass and gyrocompass depend on the altitude of the body. If under calm conditions with a level, the angle of inclination m of the azimuth circle does not exceed 0°.25*, and without a level there are possible inclinations from 1° to 3°, we get the following errors in bearing (Table 8).

Table 8

h \ m	0°.25	1°	3°	h \ m	0°.25	1°	3°
5°	0°.02	0°.09	0°.26	35°	0°.18	0°.70	2°.1
10	0 .04	0 .18	0 .50	45	0 .25	1 .0	3 .0
15	0 .07	0 .27	0 .80	55	0 .36	1 .4	4 .3
25	0 .12	0 .47	1 .4	70	0 .69	2 .7	8 .2

From the table it is seen that even with a level that ensures an accuracy of 0°.25, the bearing errors for altitudes greater than 35°

* Sensitivity of the level is 15′.

exceed the accuracy of the reading (0°.2). If the ship is rolling (or pitching), the level accuracy falls to 0°.5-1°.0 and bearing errors for altitudes greater than 15° exceed 0°.3.

From the analysis we conclude that:

(1) when taking the bearings of bodies, one should hold the sighting plane of the azimuth circle strictly on the vertical circle of the celestial body; to do this, always use a level; when there is no level, do not tilt the bowl of the compass with your hands;

(2) choose bodies with altitude no more than 15° for taking bearings. This is particularly important when the ship is rolling (or pitching).

II. RANDOM ERRORS IN TAKING THE BEARINGS OF CELESTIAL BODIES

Random errors are inherent in every operation of this nature:

(a) errors in aiming the hair of the sighting vane towards the celestial body or its centre;

(b) errors in bearing reading on compass card;

(c) errors due to aiming hair and reading compass card at different instants. These errors are appreciable for inexperienced observers.

The magnitudes of total random errors for a magnetic compass and gyrocompass differ and, what is more, depend on the conditions of observation, mainly the rolling (pitching) of the ship. Experiment has yielded ε_{CB} from 0°.3 for good conditions to $\pm$1°.5 for considerable rolling or pitching.

To reduce random errors, take bearings with great care and try to read the compass card at the *exact time* of aiming, and take three or more bearings with subsequent averaging. This is particularly important in the case of appreciable rolling (pitching). It is also advisable to take several bearings in order to avoid blunders in readings.

The following are conclusions drawn from a consideration of the principal errors made in taking bearings:

(1) When working with a compass the navigator should be confident that the azimuth circle has been adjusted and its correction is zero or known.

(2) When taking bearings, try to avoid using the mirror.

(3) To reduce errors due to tilt of bowl, use an azimuth-circle level; if none is available, try not to tilt the bowl with your hands.

(4) The effect of the foregoing errors will be less if the bearings are taken of a body at low (up to 15°) altitude.

(5) When taking bearings of the sun or moon, see that the hair of the azimuth circle cuts the disc in half. Always read the bearing at the instant the vane hair is aimed.

(6) To reduce random errors take three bearings of a body, under poor conditions five bearings.

SEC. 92. DETERMINING COMPASS CORRECTION IN THE GENERAL CASE

The compass correction is determined, as far as possible, during every watch for a constant course and, in addition, for each change of course. When using the sun, it is best to find the compass correction in the morning and evening for small altitudes; in the daytime (high altitude of sun) the correction is determined with less accuracy and it should be regarded with caution; in this case do not determine the correction when the ship is rolling (pitching) heavily. At night, the correction is determined from any star or planet whose altitude does not exceed 15°; altitudes less than 5° are likewise inconvenient due to poor visibility of the body. It is not difficult to choose such a body visually or with the help of a globe or other aid.

Compass corrections are usually determined in the following sequence of operations.

I. Prior to Observations

Choose a celestial body at low altitude. For an ordinary azimuth circle this altitude should not exceed 15°, for the Kavraisky and other azimuth circles it can, if necessary, be increased to 35°. For the sun, find the time when its altitude is less than 15°.

II. Observations

1. Take a round of three bearings of the body, taking the readings to within $\pm 0^{\circ}.2$ and note the chronometer or watch time to within 10 seconds.
2. Note and record T_{sh}, lr and course of the ship.
3. Take D.R. of ship (φ_c, λ_c) from the map for the time of observation with an accuracy of $1'$.

III. Working Sights

1. Compute the arithmetic means of bearings and instants. If RCB is found, convert it to $CB_{av} = RCB \pm 180°$.
2. From the mean instant compute T_{gr} and with the help of the MAE or other almanacs (for instance "Constant Ephemerides" of sun and stars*) obtain t_{loc} and δ of the celestial body.
3. Using arguments φ_c, δ and t_{loc} with the aid of (17.2), special tables, nomograms or instruments, obtain the azimuth A of the

* Given in the tables "ТИПС-56" and elsewhere (see Appendix VII, for example).

body and convert it to circular reckoning (0° to 360°), that is, to the true bearing of the body. Some aids give the true bearing directly in circular reckoning.

4. Compute $\Delta K = TB - CB$.

5. Check variability of total correction as compared with earlier accepted correction. For a magnetic compass, obtain deviation for the given course equal to $\Delta K - d$, where d is the magnetic declination taken from a map, and compare it with the tabulated value.

6. If there is a considerable discrepancy between the ΔK correction and the accepted value, check computations for blunders. If there are no blunders, check azimuth circle and compass, and then determine ΔK a second time. Take the new value of ΔK only if the second determination yields a close result.

It is best to check blunders in computing A with different tables or graphs.

Example 1. On 13.09.68 at $T_{sh}=17\text{h}\ 20\text{m}$; $lr=72.5$; $\varphi_c=49°12'\text{N}$; $\lambda_c=151°37'\text{E}$; $u_{ch}=+5\text{m}\ 50\text{s}$; course $=44°$; $d=-7°.5$ (W); the observed bearings of sun:

RCB	T_{sh}
94°.8	7h 14m 20s
95 .6	7 15 05
95 .4	7 15 40
av. 95°.3	7h 15m 02s

Determine ΔK and the deviation.

Solution. Perform computations with four-place logarithmic tables.

T_{sh}	17h 20m	
$-$ ZD	10	
T_{gr}	7h 20m	13.09

T_{ch}	7h 15m 02s
$+\ u_{ch}$	5 50
T_{gr}	7h 20m 52s

$$\cot A = \overset{+}{\cos}\varphi \cdot \overset{+}{\tan}\delta \cdot \underset{+}{\operatorname{cosec}}\, t - \underset{+}{\sin}\varphi \cdot \underset{+}{\cot}\, t$$
$$(+\mathrm{I} - \mathrm{II})$$

$\delta_T^{\odot}$	3°45'.0 ($\overline{1.0}$)
$\Delta\delta$	−0 .3
$\delta_{\odot} \approx$	3°45'N

$t_T^{\odot}$	286°01'.4 (0.5)
Δt	5°13'.1
$t_{gr}^{\odot}$	291°14'.5
$+\ \lambda$	151 37
$t_{loc}^{\odot} \approx$	82°51'W

$\varphi = 49°12'$	cos	9.8152	sin	9.8791
$\delta = 3\ 45$	tan	8.8165	—	
$t = 82\ 51$	cosec	0.0034	cot	9.0984
	I	8.6351	II	8.9775
	−II	8.9775	β	9.7367
	Arg	0.3424	cot A'	8.7142

$$A' = 180° - A = 87°02' = 87°.0$$

$$A = 93°.0NW = 267°.0'$$

$$CB = 275.3$$

$$\Delta K = -8°.3'$$

$$\text{Deviation} = -8°.3 + 7°.5 = -0°.8.$$

SEC. 93. SPECIAL TABLES FOR COMPUTING AZIMUTH (OR BEARING) OF A CELESTIAL BODY

Instead of computing the true azimuth from formula (17.2), one can make use of special tables that give ready azimuths computed from some formula at specified intervals of φ_c, δ, t. There are many such tables; we shall confine ourselves to the tables of A. P. Yushchenko, A. P. Demin and, in general outline, to Burdwood's Tables.

I. AZIMUTH TABLES OF CELESTIAL BODIES (PROF. YUSHCHENKO)

The most commonly used azimuth tables in the U.S.S.R. are those of Professor A. P. Yushchenko, first published in 1935. These tables are constructed as follows.

To determine A from the astronomical triangle, three elements (φ, δ and t) must be known. For this reason, three arguments are used in compiling the tables. For most tables then available it was necessary to interpolate in all three arguments which is extremely inconvenient. If one of the variables (arguments) is taken to be constant, the interpolation and the construction of the tables are simplified. For this purpose, it is necessary to know which of the arguments changes slowest. Variations of azimuth for increments in the quantities φ and t were obtained in Sec. 90 above

$$\left.\begin{aligned} \Delta A_1 &= \tan h \cdot \sin A \cdot \Delta\varphi \\ \Delta A_2 &= -\cos q \cdot \cos\delta \cdot \sec h \cdot \Delta t = -(\sin\varphi - \tan h \cdot \cos\varphi \cdot \cos a)\,\Delta t \end{aligned}\right\} \quad (17.12)$$

To obtain the variation of azimuth with change in declination of a celestial body, differentiate (17.2) with respect to A and δ. After simplifications, we get

$$\Delta A_3 = -\sin q \cdot \sec h \cdot \Delta\delta \qquad (17.13)$$

An analysis of these increments shows that for average conditions ΔA_1 changes least due to increments in latitude, particularly at small altitudes.

As a consequence, the latitude is taken as the constant or "leading" argument, which means that all azimuths are computed for constant latitude (the mean latitude for a given zone). Thus, only two arguments remain (δ and t) which are used to pick out A. This procedure was proposed by the Russian navigators Daragan and Struisky in two small azimuth tables. To reduce the azimuth to a given latitude, we introduce a small correction ΔA_φ, whose values are obtained from the same tables. This principle is now the accepted one in all Soviet azimuth tables.

In Yushchenko's tables the latitude zones are equal to 10° and the azimuths are computed for the mean latitudes of each zone $\varphi_m = 5°, 15°, \ldots 85°$.

The declination interval is 30′, hour angle interval 20′ (1 minute in the first editions). The declinations of celestial bodies are from 0° to 30°N and S so that the tables are suitable for all bodies of the solar system and about one half of the navigational stars. The values of the hour angles are taken within the limits of the diurnal above-horizon path of the body.

Yushchenko's tables consist of 9 volumes at 10°-interval latitude zones, which is to say they include all latitudes from 0° to 90°. Sometimes several volumes are bound together into one book. Each volume is divided into two parts: the first containing azimuths for declinations of same name as latitude, the second, those of contrary name. This makes it possible to use the tables for both north and south latitudes.

Each part is in turn divided into sections of 10° of the hour angle (at 1h intervals in earlier editions). Each section is subdivided inside: into columns for declination (from 0° to 30° at 30′ intervals) and into rows for hour angles (every 20′); a complete section occupies six pages. This system of division greatly simplifies finding the required value of A; the latitude gives the volume including φ_c; the name of declination gives the part of the tables; tens of degrees of hour angle give the section; the value of declination gives the column of the section, and the hour angle gives the row.

Interpolation for minutes of declination and hour angle is performed mentally by proportions between four values of A; it may be neglected in certain cases.

At the bottom of each page are tables of the values of azimuth corrections (ΔA_1) for changing latitude by $\pm 1°$. The corrections are given in the same columns of δ, but every 2° of the hour angle; the upper correction in each column refers to a latitude greater than the mean latitude φ_m of the zone, the lower correction refers to a latitude less than φ_m.

If the computed latitude φ_c differs from the mean latitude, then multiply the chosen correction $\Delta A_1°$ with its sign into the absolute value of the difference $\varphi_c - \varphi_m = \Delta\varphi$ taken with an accuracy of 0°.1. As a result, we get the azimuth correction ΔA_φ from the latitude, that is,

$$\Delta A_\varphi = \pm \Delta A_{1°} |\Delta\varphi| \tag{17.14}$$

This correction with its sign is then applied to the azimuth chosen from the tables.

In certain cases (indicated by an asterisk at the bottom of the page), to obtain A with an accuracy of 0°.1-0°.2, it is necessary to introduce into the chosen azimuth an additional correction for the second differences; this correction may be computed from the formula

$$\Delta_{II} = -\Delta\varphi (\Delta A_+ + \Delta A_-)(0.5 - 0.1\Delta\varphi) \tag{17.15}$$

where ΔA_+ and ΔA_- are corrections for latitude greater and less than mean.

The azimuth obtained will be in *semicircular reckoning*, so the first letter of its designation is the same as the latitude of the place, the second is that of the hour angle of the celestial body. The azimuth thus obtained is converted to circular reckoning, that is, to true bearing.

The Yushchenko tables are the best azimuth tables because they yield high accuracy of azimuth (of the order of $\pm$0°.1-0°.2*), are simple to handle and require little time to obtain azimuth (about three or four minutes with selection of hour angle included). These tables have the following disadvantages: size—the whole edition includes 9 volumes (about 1,000 to 1,200 pages), it cannot be applied to stars with $\delta > 30°$, and the inconvenience of semicircular reckoning of azimuth, namely: the necessity of naming A and converting to circular reckoning.

Example 2. On 12.09.68 at about $T_{sh} = 16$h 45m (ZD=3E) three bearings of sun taken by gyrocompass: $CB_{av} = 252°.3$; av. $T_{ch} = 1$h 35m 20s; $u_{ch} = +$ 1m 23s; $\varphi_c = 67°16'$N; $\lambda_c = 40°31'$E; course=30°. Determine ΔC.

---

* With interpolation.

(1)

$-$ T_{sh}	16h 45m
ZD	3
T_{gr}	13h 45m 12.09
δ_T	4°02′.2 ($\overline{1.0}$)
$\Delta\delta$	— 0 .7
$\delta_{\odot} \approx$	4°01′.5N

T_{ch}	+ 1h 35m 20s
u_{ch}	4 23
T_{gr}	13h 39m 43s
t_T	15°57′.5(0.5)
Δt	9 55 .9
t_{gr}	25°53′.4
λ	40 31
$t^{\odot}_{loc} \approx$	66°24′

(2) $\varphi_m = 65°$, $\Delta\varphi = \varphi_c - \varphi_m = 2°.3$.

(3) From $\delta_{\odot}$ and $t_{\odot}$ ...

A_T	110°.0 ($\Delta A_1 = +0°.20$)
$\Delta\varphi \times \Delta A_1 = \Delta A_{\varphi}$	+ 0 .5
A	N 110°.5W
TB	249 .5
CB	252 .3
ΔC	—2°.8

II. A. P. DEMIN'S "TABLES OF TRUE BEARINGS OF CELESTIAL BODIES" (ТИПС-51, ТИПС-56)

The constructive principle of these tables is absolutely analogous to that of Yushchenko's tables, the only difference being that the intervals of latitudes have been increased to 15° (with exception of the 0°-10° zone), and the intervals of declinations and hour angles increased to 1°. Also, the values of declination are limited to ±24°, the values of hour angle, to values of altitude of the body from 0° to 30°. Due to these changes, the size of the tables has been reduced to 100 pages, but the accuracy of azimuths is inferior and interpolation is complicated.

Demin's tables come in a single volume divided into two parts: the first for φ and δ of the same name; the second, for contrary names. Each part has six categories as to latitude zone: the first at 10°, the second at 12°.5, and the remaining 15° each with mean latitudes 5°, 15°, 30°, 45°, 60°, and 75°.

In each section there are two tables: for the hour angles of the body at setting ($t_W < 180°$) and at rising ($t_W > 180°$). For declination and hour angle, these tables give, at 1° intervals, the values of *true bearings* of a body, that is, the azimuths measured from N in circular reckoning.

Since the intervals of declination are double those of the Yushchenko tables, and the hour angles are three* times as great, it is necessary when picking out the bearing to interpolate both with respect to hour angle and declination, otherwise the errors will become prohibitive.

The bearing correction (ΔB) due to change of latitude by 1° is located (via the same arguments δ and t) at the bottom of each page. The correction indicates only one sign, so that when obtaining the correction ΔB_φ due to a discrepancy between the computed and tabulated latitudes, one has also to take into account the *sign of the difference*: $\Delta\varphi = \varphi_c - \varphi_{tab}$, i.e.,

$$\Delta B_\varphi = (\pm \Delta B) \cdot (\varphi_c - \varphi_{tab}) \tag{17.16}$$

where the sign of ΔB_φ is obtained with allowance made for the signs of the factors, after which the correction is applied to the bearing.

Appended to the ТИПС-56 tables are Tables No. 1-5 of the "Constant Ephemerides" of the sun, the quantities $R = \alpha_\oplus \pm 180°$, t and δ for stars with $\delta < 24°$.

From these tables we can get the hour angles and declination of the sun, the sidereal time, t and δ of stars without the MAE with an accuracy sufficient for determining ΔK for 30 years. The accuracy of the true bearing obtained from the ТИПС-56 tables will be of the order of $\pm 0°.3$ for the sun and $\pm 0°.4$ for stars.

The ТИПС tables may be used in south latitude as well, but the TB_S is converted to circular reckoning by the formula

$$TB = 540° - TB_S \tag{17.17}$$

in which 360° is dropped if $TB_S < 180°$.

Among the advantages of the ТИПС tables are their small size and the circular reckoning of azimuth used, which (in northern latitudes) obviates determining the name of azimuth and conversion to circular reckoning. An added convenience are the "Constant Ephemerides" that take the place of an almanac; they could, incidentally, be appended to any tables or manuals for determining azimuth. The disadvantages of the ТИПС tables are: a more complicated interpolation, lower accuracy, of the order of $\pm 0°.4$ in true bearing, and considerable limitations in celestial bodies ($\delta < 24°$) and their positions ($h < 30°$).

Example 4. On 26.03.68 at about $T_{sh} = 19$h 30m observed ✳ Arcturus. $RCB = 263°4$; av. $T_{ch} = 8$h 4m 25s; $u_{ch} = +25$m 18s; $\varphi_c = 48°25'$N; $\lambda_c = 15°03'.0$W; course $= 78°$ (RCB); $d = 14°.5$ (W). Find ΔK and deviation.

---

* Six times in ТИПС-51.

(1)

T_{sh}	19h 30m
$+$ ZD	$+1$
T_{gr}	20h 30m 26.03
T_{ch}	8h 4m 25s
u_{ch}	25 18
T_{gr}	20h 29m 43s

(2) By MAE

t_T^{Υ}	124°19′
Δt	7 27
t_{gr}^{Υ}	131°46′
$-$ λ	15 03
t_{loc}^{Υ}	116°43′
τ_{*}	146 27
t_{loc}^{*}	263°10′

$\delta_{*} = 19°21'$N

By ephemerides tables from ТИПС

Table 1	T_{gr}	307°.4
Table 3	R_0	182 .9
	ΔR_0	$+1$ .4
	t_{gr}^{Υ}	491°.7 = 131°.2
	$-$ λ	15 .0
	t_{loc}^{Υ}	116°.7
$\delta_{*} = 19°.4$N	τ_{*}	146 .7
	t_{loc}^{*}	263°.4

(3) From known $\varphi_m = 45°$N; $\delta = 19°4$N; $t = 263°.2$; we get

B_T	71°.7
ΔB_δ	-0 .3
ΔB_t	$+0$ .1
B	71°.5
ΔB_φ	$+0$ .5
TB	72°.0
CB	83 .4
ΔK	$-11°.4$

$\Delta\varphi = \varphi_c - \varphi_m = +3°.4$

$\Delta B_1 = +0$ .14

$\Delta B_\varphi = +0°.5$

ΔK	$-11°.4$
$-$ d_W	$+14$ .5
Dev.	$+3°.1$

III. BURDWOOD'S "SUN'S TRUE BEARINGS OR AZIMUTH TABLES"

These tables are compiled for latitudes 30°-64°, declinations from 0 to 24° and hour angles within the limits of the visible part of the diurnal circle of the body. Since the azimuth in the tables is computed for latitudes, declinations and hour angles* at 1° intervals, interpolation is required with respect to three arguments, and for latitudes, with the next section of latitudes, which is very inconvenient.

The tables consist of two parts: the first for declinations of the same name as latitude; the second, for declinations of contrary name to latitude.

Between the two parts are the constant ephemerides of the sun ($\delta_{\odot}$ and η) for each day of four consecutive years, so that one can dispense with an almanac when using the Burdwood tables.

For entering the tables, the west hour angles less than 180° (12′) are indicated in the extreme right-hand column headed p.m. (post meridiem). But if $t_W > 180°$ (12h), then one has to compute t^E $360° - t_W$ and with this argument enter the same right-hand column without using the left-hand a.m. (ante meridiem) column. In this case, the azimuth will be 0° to 180° (semicircular) and its name will be determined by the latitude (first letter) and hour angle (second letter). When extracting material from the table, it is most convenient to interpolate azimuth from the nearest integral value of φ, δ and t, adding corrections with respect to each argument separately, as usual.

The Burdwood tables are among the oldest azimuth tables (the first edition was published in 1852). These tables are considerably less convenient and azimuth accuracy is lower than in the Yushchenko tables.

IV. FUNDAMENTALS OF CALCULATING CHARTS AND GRAPHS FOR OBTAINING TRUE AZIMUTHS (BEARINGS) OF CELESTIAL BODIES

The azimuth of a body may be computed on the basis of φ_c, δ, t not only from formulas or tables, but also from calculating charts or graphs. As a rule, graphic solutions are much more simple than tabular solutions, and for appropriate scales, graphs ensure accuracy of azimuth to within $\pm 0°.2$-$0°.3$, which is sufficient for navigation.

At one time, various types of calculating charts and graphs were widely used to determine azimuth and also altitude and azimuth in the British and German merchant marines, for instance Weier's "Azimuth Diagram", Schütte's "Azimuth graphs" and so on.

* Hour angles are taken at 4m = 1° intervals, in some places at 2m or 8m intervals.

At the present time diagrams are published in Great Britain (Admiralty Chart No. 500), in the U.S.S.R. (Calculating Chart No. 290).

Let us examine one such calculating chart for determining azimuth. The Admiralty Chart No. 500 is a grid chart with binary field, called the Weier Diagram (1890) after its compiler. This chart embraces the entire circle of 360° with respect to A and t, so that no conversion to circular reckoning (0° to 360°) is needed. Around

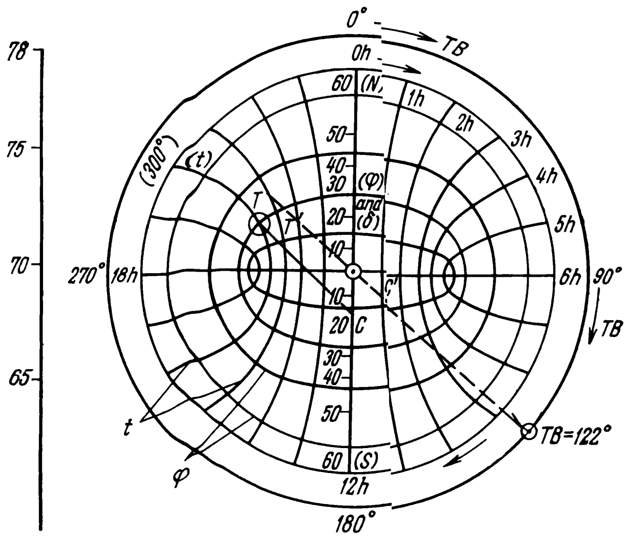

Fig. 144

the outer circumference of the chart, which is in the form of a compass card (Fig. 144), are the true bearings at 1° intervals numbered every 5°. The hour angles are indicated on the inner circumference: in black figures for northern latitudes and in green for southern latitudes (from south point, reading anticlockwise).

This numbering refers to the hyperbolic system constructed every 4m (1°) of hour angle. The vertical diameter 0°-180° of the chart, which is the general scale of latitudes and declinations, is divided into 1° intervals and numbered every 5°, the numbers going upwards from the centre and referring to north φ and δ, the numbers going downwards and referring to south φ and δ.

This numbering refers both to the system of elliptic curves of latitude constructed at 1° intervals and to the rectilinear vertical scale of declinations. The binary field is formed by systems of hyper

bolas (t) and ellipses (φ). The chart is designed for use in latitudes from 0° to 65°N and S and for declinations within these same limits. The scale of declinations may be extended beyond the frame of the calculating chart (by means of a measurer and a scale situated in the margins to the side of the chart) and the limits of declinations may be increased to 78°N and S.

Working with a calculating chart requires a parallel ruler or two large set squares.

The rules for using a calculating chart are as follows:

(1) for a given declination, say $\delta = 20°S$ (see Fig. 144), plot point C on the vertical scale;

(2) from the latitude find the elliptic curve (say $\varphi = 30°N$);

(3) from the hour angle find the hyperbola (say $t_W = 300°$);

(4) at the intersection of these curves (φ and t) plot the point T on the binary field;

(5) placing the parallel ruler on points T and C, transfer the TC direction to the centre O of the chart ($T'C'$);

(6) extending $T'C'$ (in the direction from T' to C') to its intersection with the outer circumference (scale TB), we get the TB reading= 122° on this scale in circular reckoning (0°-360°).

The whole procedure of obtaining true bearing takes about 2 minutes and does not require any arithmetic, while the error does not exceed $\pm 0°.5$. One of the disadvantages in using this chart is the necessity of the parallel ruler, which is not always convenient in close quarters. Another thing is that in certain parts of the chart, due to the density of the grid, it becomes difficult to plot T. However, with a little practice this can be overcome and the chart becomes a handy tool for computing true bearing. The calculating chart is especially good to use in checking solutions from tables.

In some fleets, "Azimuth graphs" compiled by K. Schütte in 1942 have been in use. These graphs are constructed by latitude zones at 1° intervals from $\varphi = 0°$ to $\varphi = 88°$ (for $\varphi > 80°$ every 2°) for any declinations and hour angles, with the exception of altitudes greater than 80°. The graphs come in one book of 360 pages.

A point is plotted on the sheet of the graph corresponding to φ_c with respect to t, from the vertical frame and δ from the appropriate curve. Using this point we take the azimuth in semicircular reckoning (0°-180°) from the horizontal frame. It is extremely easy to handle graphs and the accuracy of azimuth is nearly the same as given by Yushchenko's tables.

The true azimuths of celestial bodies can also be computed instrumentally. Very simple and accurate azimuths are obtained by the German ARG-3 instrument; somewhat more involved are computations performed with a cylindrical rule. These instruments are considered in Sec. 108.

In addition to the tables already considered, there are numerical tables of altitude and azimuth used to compute azimuth, for example BAC-58 (see Sec. 107), H.O.-214, and others.

SEC. 94. SPECIAL CASES IN DETERMINING THE COMPASS CORRECTION FROM THE SUN AND POLARIS

Obtaining the true azimuth is greatly simplified in the case of certain special positions of celestial bodies on the sphere.

For example, if we observe the sun at the instant of true or apparent sunrise, we can use the "altitude method" which does not require selecting the hour angle of the body, and all computations are simplified.

I. DETERMINING THE COMPASS CORRECTION AT THE INSTANT OF TRUE SUNRISE

At the instant of true rising of celestial bodies, including the sun, the altitude of the body's centre $h = 0$ (Fig. 145) and formula (17.3)

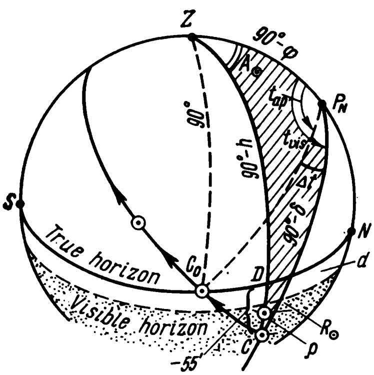

Fig. 145

obtained from the triangle ZP_NC_0 takes the form

$$\cos A = \sec \varphi \cdot \sin \delta \tag{17.18}$$

To apply this formula to the sun, it is necessary to find the instant that its centre arrives at the celestial horizon. Since at sea we observe apparent directions, for an observer with height of eye 6.1 metres, this will occur when the lower limb of the sun (C_0) is at a distance of about $0.7 \approx 3/4$ of its apparent vertical diameter above the visible horizon. Here, the azimuth of the sun may be computed from

formula (17.18) or, still better, it may be taken from a table compiled from it; for example, such tables are given at the end of each volume of the azimuth tables of Yushchenko and others.

This method was once rather common, but now it is hardly ever used because it is more convenient to take bearings of the sun at a more definite instant of apparent sunrise or sunset of the upper limb.

II. DETERMINING THE COMPASS CORRECTION FROM THE SUN'S AZIMUTH AT THE TIME OF APPARENT SUNRISE OR SUNSET OF THE UPPER LIMB

When the upper limb of the sun becomes tangent to the (visible) horizon, the centre of the sun (C) will be located somewhat lower than the horizon (Fig. 145) due to the following factors:

(a) dip of the visible horizon relative to the celestial horizon; taking an average height of eye $e = 6.1$ metres (20 feet), we get $d = -4'.4$;

(b) astronomical refraction on the visible horizon; this raises the sun's image an average of $\rho = -35'.5$;

(c) the semidiameter of the sun, equal to an average of $16'.0$, and a parallax of about $+0'.15 \approx 0'.1$.

Consequently, the true geocentric altitude of the sun at the instant when its upper limb arrives at the visible horizon will be

$$h_{\odot} = -4'.4 - 35'.5 - 16'.0 + 0'.1 = 55'.6$$

If we take $e = 0$, then

$$h_{\odot} = -34'.4 - 16'.0 + 0'.1 = -50'.3$$

In the MT-53, $h_{\odot} = -55'$, in MT-63, $h_{\odot} = -50'.3$ (from sea level).

When compiling azimuth tables for these cases it is more convenient to transform (17.3) by replacing $\cos A = 1 - 2\sin^2\frac{A}{2}$, and the secants and tangents by sines and cosines, thus

$$1 - 2\sin^2\frac{A}{2} = \frac{\sin\delta - \sin\varphi\cdot\sin h}{\cos\varphi\cdot\cos h}$$

or

$$2\sin^2\frac{A}{2} = 1 - \frac{-\sin\varphi\cdot\sin h + \sin\delta}{\cos\varphi\cdot\cos h}$$

Reducing the right side to a common denominator and simplifying, we finally get

$$\sin^2\frac{A_{\odot}}{2} = \frac{\cos(\varphi - h) \mp \sin\delta}{2\cos\varphi\cdot\cos h} \tag{17.19}$$

For φ and δ of the same name, the numerator of this formula will be subtractive, for contrary names, additive.

In the MT-53 tables, formula (17.19), in which $h = -55'$, is used to compute Tables 20a and 20б for latitudes from 0° to 75° and declinations of the sun from 0° to 24°. Table 20a is used for φ and δ of the same name, Table 20б for φ and δ of contrary names. Formula (17.19) or Table 20 yields azimuth in semicircular reckoning, that is, the first letter of the designation is determined from the latitude, the second, depending on whether the sun is rising (E) or setting (W); a simplified rule for the designation of A is given at the bottom of Table 20.

In the MT-63 tables, similar tables 20a and 20б are computed with $h_{\odot} = -50'.3$. There are additional tables 20в and 20г for computing corrections to the azimuth for the magnitude Δh. The altitude correction is computed from the formula

$$\Delta h = -d - \rho_0 \pm h_t \pm \Delta h_U$$

From the spherical triangle ZP_NC (see Fig. 145) we write the formula

$$\sin\delta = \sin\varphi\cdot\sin h + \cos\varphi\cdot\cos h\cdot\cos A$$

Differentiating it with respect to h and A and passing to increments, we obtain

$$0 = (\sin\varphi\cdot\cos h - \cos\varphi\cdot\sin h\cdot\cos A)\,\Delta h - \cos\varphi\cdot\cos h\cdot\sin A\cdot\Delta A$$

Since h is small, we can take $\cos h = 1$, $\sin h = 0$. Then we have

$$\Delta A' = \tan\varphi\cdot\operatorname{cosec} A\cdot\Delta h' \qquad (17.20)$$

Table 20в is computed from $K = \dfrac{\tan\varphi\cdot\operatorname{cosec} A}{60}$. The correction $\Delta A^\circ = K\cdot\Delta h'$ is taken from Table 20г; the sign of the correction ΔA° is the same as that of $\Delta h'$.

The accuracy of the compass correction obtained by the altitude-azimuth method is lower than in the general time-azimuth case. This is because it is difficult to catch the exact instant of apparent rising or setting of the upper limb; in addition, it is impossible to reduce the effect of random errors or detect a blunder in the sole bearing. In very high latitudes, for large values of δ, that is, on the very edge of Table 20, linear interpolation cannot be applied and the azimuth is inaccurately determined. Therefore, do not use Table 20 in such conditions.

For this reason, a correction obtained from the azimuth of apparent sunrise or sunset (particularly when the ship is rolling or pitching) must be regarded as approximate and must be refined at the first opportunity by more reliable observations by the "time azimuth" method.

An advantage of the altitude azimuth method is the extreme simplicity and the possibility of getting along without the MAE confining oneself to picking $\delta_{\odot}$ out of the tables of constant ephemerides.

The sequence of operations in determining ΔK is as follows:

(1) Observe sun bearing at the instant its upper limb appears (or disappears) on the horizon, and convert RCB to CB.

(2) Note the time T_{sh} to within 5 to 10 m; compute T_{gr} and take from the MAE or tables of constant ephemerides $\delta_{\odot}$ with an accuracy of 0°.1. For this same instant, take φ_c with the same accuracy. In high latitudes and for considerable $\delta_{\odot}$, it is best to obtain $\delta_{\odot}$ and φ_c to within 3′ (0°.05).

(3) Enter Table 20a with φ and δ of same name, or 20б with contrary names and pick out the value of A_T nearest to φ and δ. Interpolate separately with respect to δ and φ and apply the correction to A_T. Prefix designation to azimuth obtained and convert it to circular reckoning.

(4) Compute $\Delta K = TB - CB$ and compare with earlier value.

Example 5. On 7.06.68 at $T_{sh} \approx$ 21h 45m (ZD=2E); $\varphi_c \approx 55°.6$N; $\lambda_c \approx 18°$E; took bearings of sun at setting of upper limb $RCB_{\odot} = 137°5$; course=87°. Find ΔK.

(1)

$-\ T_{sh}$	21h 45m	7.06
ZD+1	2	
T_{gr}	19h 45m	7.06

(2) from MAE $\delta_{\odot}$=
=22°49′.2N ≈ 22°.8N

(3) $\left.\begin{array}{l}\varphi_T = 56° \\ \delta_T = 23°\end{array}\right\}$

A_T	43°.7	... Table 20a
ΔA_{φ}	+0 .6	
ΔA_{δ}	+0 .5	
$A_{\odot}$	N 44 .8W	
$TB_{\odot}$	315 .2	
$CB_{\odot}$	317 .5	
ΔK	−2°.3	

III. DETERMINING COMPASS CORRECTION FROM OBSERVATIONS OF POLARIS

Polaris (α Ursae Minoris) describes in its diurnal motion (Fig. 146) a parallel of extremely small spherical radius, $\Delta \approx 53'$.

For this reason, in latitudes up to 35°N the azimuth of Polaris changes only from 0° to 1°.1NE and NW, and in latitudes up to 70°N, from 0° to 2°.6; the formula for computing A may be simplified.

From the astronomical triangle ZP_NC we have

$$\frac{\sin A}{\sin \Delta_*} = \frac{\sin t_{loc}}{\cos h}$$

or, taking into account that $t_{loc} = S_{loc} - \alpha$, we have

$$\sin A = \sin \Delta_* \cdot \sec h \cdot \sin (S_{loc} - \alpha_*)$$

Due to smallness of the polar distance Δ_* and the azimuth A, we replace the sines by the first terms of the series, and h is taken equal to φ, since their difference does not exceed 53′. Introducing simplifications, we obtain

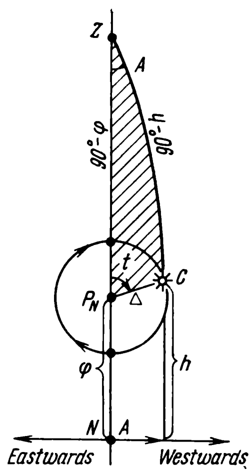

Fig. 146

$$A^\circ = \Delta^\circ_* \cdot \sec \varphi \cdot \sin (S_{loc} - \alpha_*) \quad (17.21)$$

This formula is used in Almanacs, taking for the given year the mean values of Δ_* and α_* of Polaris (in 1968, $\Delta_* = 53'$ and $\alpha_* = 30°$), to compute a table of the "Azimuths of Polaris" for latitudes from 5° to 70°N (in MAE of the old type, up to $\varphi = 35°$N). The arguments for entering the table are the local sidereal time $S_{loc} = t^{\Upsilon}_{loc}$ given every 5°, and the latitude of the place given every 5° too. The name of the azimuth is given at the bottom of the tables, but it can easily be established from Fig. 146. Indeed, it is seen that from the instant of upper transit of Polaris to lower transit, the azimuth will be NW. Here, on the basis of formula $S_{loc} = t_* + \alpha_*$ with t_* equal to 0° and 180°, we have $S_{loc} = \alpha_* = 30°$ and $S_{loc} = \alpha_* + 180° = 210°$. Consequently, for t^{Υ}_{loc} from 30° to 210° the azimuth of Polaris will be NW, and for t^{Υ}_{loc} from 210° to 30°, it will be NE.

To obtain the compass correction from Polaris, take 2 or 3 bearings, note T_{sh} to within 5 to 10 minutes and take the coordinates of the ship φ_c, λ_c to within 1°. Taking t^{Υ}_{gr} from the MAE and obtaining t^{Υ}_{loc}, enter MAE, choose A and convert it to circular reckoning.

This method is applicable to latitudes from 0° to 15°-17°N when taking bearings without a mirror and up to $\varphi = 35°$N with a mirror or with other azimuth circles.

However, even in larger latitudes one should never neglect approximate orientation by Polaris; for example, when on course N or S, Polaris should be located in the centre-line plane of the ship, that is, in a line of masts.

Example 6. On 13.09.68 at $T_{sh}=21\text{h}\,30\text{m}$ (ZD$=1$W) three bearings taken of Polaris av. $RCB=195°.5$; $\varphi_c=33°$N; $\lambda_c=21°5'$W; course $=230°$. Determine ΔK.

(1)

T_{sh}	21h 30m	13.09
+ ZD_W	1	
T_{gr}	22h 30m	13.09
t_T	322°57′	
Δt	7 31	
t^{Υ}_{gr}	330°28′	
− λ_W	21° 5′	
$t^{\Upsilon}_{loc} \approx$	309°.4	

(2) $\left.\begin{array}{l} S_{loc}=309° \\ \varphi_c\ =\ 30° \end{array}\right\}\ldots$

A_T	1°.1NE
TB	1 .1
− CB	15 .5
ΔK	−14°.4

SEC. 95. COMPILING DEVIATION TABLES FROM THE BEARINGS OF CELESTIAL BODIES

To obtain a new table of deviation from the observations of celestial bodies (the sun is the best), choose a region where a circulation can be executed and a time during which the altitude of the sun does not exceed 15°, that is, ordinarily morning or evening. Other bodies may be observed under the same conditions, but their declinations should be less than 30°, since Yushchenko's azimuth tables have $\delta \leqslant 30°$.

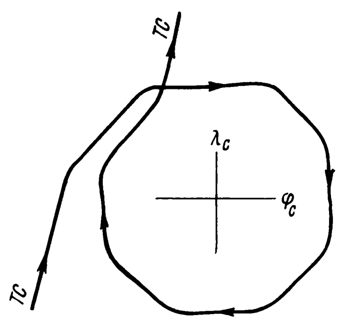

Fig. 147

The work consists in taking a sequence of bearings of the body on the eight principal and four quadrantal courses. To do this, at the time of observation T_{sh}, take the closest course (Fig. 147), take two or three bearings of the body, and note the chronometer time (to within 5s). When passing to another course, it is best to overlap by 10° to 15° and then return to the given course and stay on for 3 to 4 minutes so that the magnetic state of the ship becomes steady.

After observing bearings on all courses, it is advisable to repeat the observation on the first course, and then head for destination. The whole job takes about 50 minutes.

To work the sights, take the coordinates of the centre of circulation, compute the mean RCB and the instants for each course; then, for the instant corresponding to the mean (fifth) course, we

obtain δ and t of the body from the MAE. For the other instants we consider that the change in hour angle is equal to the change in time and the declination remains unchanged. When obtaining azimuth from the Yushchenko tables, φ and δ remain unchanged, therefore the azimuth and the correction ΔA_{φ} will only change due to change in the hour angle. All computations are tabulated in two ways, as shown below.

To eliminate deviation by means of magnetic bearings of the sun, these same schemes may be used, but take the instants of the prospective time of work and convert precomputed true bearings to magnetic bearings.

Example 7. On 9.09.68 at about $T_{sh} = 5$h 55m (ZD $=$ 2W) started determining the residual deviation from course 135°. The coordinates of the centre of swing: $\varphi = 42°40'$N, $\lambda = 27°25'$W. Observations tabulated in Scheme 1. Worked by Yushchenko tables with instant T_5 heading 315° and given in Scheme 2; $u_{ch} = +0$m46s, magnetic declination (from map) $d = -8°.7$(W).

(1) *Scheme No. 1*

Course	*RCB*	T_{ch}	Av. *RCB*	Av. T_{ch}	ΔT min, sec	$\Delta T°$
135°			276°.6	7-55-08	21m 5s	—5°16
180			279 .2	8-00-23	15 50	—3 57
225			277 .2	8-05-38	10 35	—2 40
270			276 .5	8-10-43	5 30	—1 20
315°			278°.6	8-16-13	0m 00s	0°0′
360° (0)			279°.6	8-22-48	6m 35s	+1°40′

45 . . . .

(2)

$+\,T_{sh}$	5h 55m 9.09	T_{ch}	8h 16m 13s	
ZD	2	u_{ch}	+ 0 46	$\varphi_c - \varphi_m = 2°.3$
T_{gr}	7h 55m 9.09	T_{ch}	8h 16m 59s	
δ_T	N 5°15′.3 (0.9)	$t_T^{\odot}$	300°40′.6 (0.5)	
$\Delta\delta$	— 0 .3	Δt	4 14 .8	
$\delta_{\odot} \approx$	5°15′N	$-\,t_{gr}^{\odot}$	304°55′.4	
		λ_W	27°25′	
		$t_{av} = t_{loc}^{\odot}$	277°30′W=82°30′E	

$d = -8°.7$

(b) *Scheme No. 2*

Course	$t_{loc} = t_{av} + \Delta T°$	A_{tab}	ΔA_{φ}	$A_{\odot}$	$TB_{\odot}$	$CB_{\odot}$	ΔK	δ	δ from table
135°	87°40′	87°.8NE	0°.2	87°.6NE	87°.6	96°.6	−9.0	−0.3	−1°.3
180	86 30	88 .8	−0 .3	88 .5	88 .5	99 .2	−10.7	−2.0	−3 .1
225	85 10	89 .7	−0 .3	89 .4	89 .4	97 .2	−7.8	+0.9	0 .0
270	83 50 ↑	90 .6	−0 .3	90 .3	90 .3	96 .5	−6.2	+2.5	+1 .2
315	$\boxed{82°30'E}$	91 .5	−0 .4	91 .1	91 .1	98 .6	−7.5	+1.2	+0 .3
360 (0)	80 50 ↓	92 .8	−0 .4	92 .4	92 .4	99 .6	−7.2	+1.5	+2 .4
45	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

CHAPTER 18

ESSENTIALS OF ASTRONOMICAL DETERMINATION OF POSITION AT SEA

SEC. 96. THE RELATIONSHIP BETWEEN THE POSITION OF A SHIP AND THE POSITION OF ITS ZENITH ON THE SPHERE

The most important task of navigation is determining the true position of a ship on the earth's surface. Since maps reproduce the surface area of the earth with sufficient accuracy, this problem reduces to *determining the actual position of a ship on a map*. In navigation, the problem of determining position is solved by observing various terrestrial objects. But if such objects are lacking, determination of position both at sea and in coastal areas is done by **astronomical methods,** that is, by observing celestial bodies. Navigation maps are constructed in the system of geographic coordinates φ and λ; for this reason any determination of position reduces either to a determination of the coordinates φ and λ with subsequent plotting of the point on the map, or to obtaining the position on the map directly.

Nearly all navigational methods of determination of position serve to illustrate the direct way: by bearings, angles, distances. Here the coordinates φ and λ are used only for indicating places of terrestrial objects, while the position itself is obtained relative to these objects by the lines of position method.

Up to the middle of last century, nautical astronomy only used methods for separate determination of the coordinates φ and λ. After the discovery, in 1837-1843, of the method of altitude lines of position, simpler procedures were used for determining positions on maps from the lines of position. Such procedures are in principle analogous to navigational methods. At the present time these methods are dominant in nautical astronomy.

In astronomy, determining the position of a ship or its coordinates φ and λ reduces to finding the ship's zenith (or its coordinates) on the celestial sphere with subsequent conversion to geographic coordinates or directly to the position on a map.

Let us establish a relationship between the position of an observer on the earth's surface and that of his zenith on the celestial sphere.

As already mentioned at the beginning of the course (Secs. 3 and 4), the celestial coordinates α, δ are related to the geographic coordinates φ, λ via an analogy of these coordinate systems, the planes of the principal circles of which (equator and meridian) are common.

Referring to Fig. 148, which depicts the celestial sphere with centre at the centre of the earth, let M or ("loc") be a ship on the

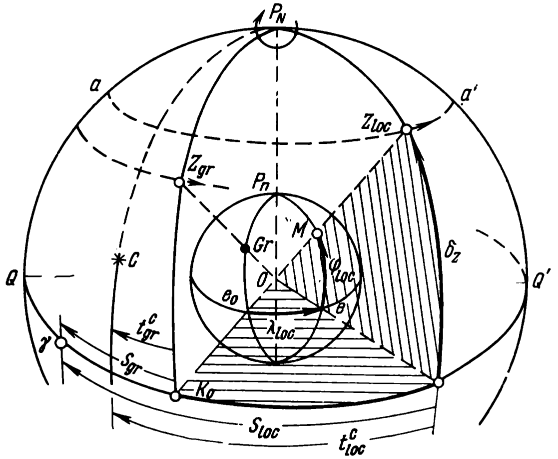

Fig. 148

earth, with coordinates φ_{loc} and λ_{loc}, MO being the plumb line of the observer.

Extending the plumb line to its intersection with the celestial sphere, we get the zenith of the ship Z_{loc}; similarly for Greenwich we have Z_{gr}. Also indicate the meridian of the celestial body C and the point of Aries (γ). From the figure it will be seen that the position of the ship is the projection of its zenith on the earth's surface and its coordinates are connected by the relations below, which follow from the angular equality of the arcs $KZ_{loc} = eM$ and $K_0K = e_0e$

$$\left.\begin{aligned} \varphi_{loc} &= \delta_Z; \quad S_{loc} = \alpha_Z \\ \lambda_{loc} &= (\alpha_Z - \alpha_{Zgr}) = S_{loc} - S_{gr} \end{aligned}\right\} \tag{18.1}$$

If the hour angles of the body C are used in place of the hour angles of Aries, we get the formula in a more general form

$$\lambda_{loc} = t^C_{loc} - t^C_{gr} \tag{18.2}$$

where λ is reckoned to the east with a plus (+) sign and to the wes with a minus (—) sign.

Due to the diurnal rotation of the celestial sphere, the zenith is constantly moving and in a 24-hour period describes the parallel aa'; we must therefore define the "instantaneous position of zenith", which is the zenith at the given instant. Hence, when determining a position, always note the time by chronometer or watch; this is then converted to S_{gr} or t_{gr}^{C}.

If the instantaneous zenith has been determined, then from formulas (18.1) we get the coordinates φ and λ for the same instant or, if we project on the earth's surface the constructions which served to determine Z, then graphically we get the position on the earth.

If in a particular case the celestial body C transits exactly in the zenith Z_{loc}, we obviously have

$$\varphi = \delta_c; \ \lambda = \alpha_c - S_{gr} = -t_{gr}^{C} \tag{18.3}$$

From the foregoing it will be seen that if in place of the coordinates of the zenith δ_Z and t_{loc} we introduce the geographic coordinates of the place φ_{loc} and $t_{loc} = t_{gr} + \lambda_{loc}$ into the astronomical triangle of the body, then we can determine the position of the ship or its coordinates directly instead of determining the position of the zenith.

SEC. 97. GENERAL PRINCIPLES FOR DETERMINING THE ZENITH ON THE CELESTIAL SPHERE OR THE OBSERVER'S POSITION ON THE EARTH

Fundamentally, it is possible to determine the coordinates of the zenith or its position on the sphere by an analytic solution of the astronomical triangle, graphically, or, finally, by means of instruments. Both for solving the triangle and in other cases, we must know (in addition to the coordinates of the body taken from the MAE) two more elements of the triangle, for example, the altitude and azimuth of the body, two altitudes, and so forth. If we get two elements from observations, we determine both coordinates of the zenith $\delta_Z = \varphi$ and t_{loc}; if one is obtained (h, for instance), then only one coordinate, φ or λ (t_{loc}), need be found.

Thus, to determine the coordinates of the zenith it is necessary to make observations of the coordinates of the body, and to obtain t_{gr}^{C} and δ_C from the MAE, note the chronometer time.

For these purposes, one can observe:

(a) the altitudes of one or two bodies;

(b) the difference of altitudes or equal altitudes;

(c) the azimuths of one or two bodies;

(d) the difference of azimuths or the equal azimuths;

(e) the altitude and azimuth of body;

(f) the parallactic angles q.

With modern instruments, only the altitudes of bodies or their differences may be obtained with sufficient accuracy from observations at sea. Now the azimuths of bodies, and all the more so the angles q cannot be obtained from observations with the same accuracy. For that reason, we shall consider only those solutions which involve measured altitudes.

(1) Analytic determination of the coordinates of the zenith position of a ship. The problem of finding the coordinates of the zenith

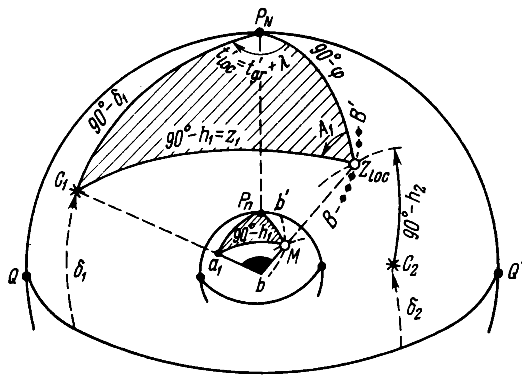

Fig. 149

(or, what is the same thing, the coordinates of the place φ and λ) reduces, from a purely analytical point of view, to solving the following set of equations

$$u_1 = f_1(\varphi, \lambda); \quad u_2 = f_2(\varphi, \lambda) \tag{18.4}$$

where the functions u_1 and u_2 are obtained from observations.

To determine both coordinates φ and λ, it is necessary to form two equations (18.4), which means having at least two observations. Only one equation is sufficient to find a single coordinate, φ or λ; in this equation, one of the coordinates must be taken as known. If the number of observations and, hence, the number of equations (18.4) is greater than two, you can solve the problem by the least-squares method and refine the determination of φ and λ. For an analytic expression of the equations (18.4), let us take the solution of the astronomical triangle $Z_{loc}P_NC_1$ (Fig. 149) from altitude observations of the bodies.

Suppose we observed the altitudes h_1 and h_2 of two bodies C_1 and C_2, whose coordinates δ_1, t_{gr1} and δ_2, t_{gr2} are taken from the

MAE for the noted instants T_1 and T_2. Then from the astronomical triangles for the bodies C_1 and C_2 we get

$$\left.\begin{aligned}\sin h_1 = \sin\varphi\cdot\sin\delta_1 + \cos\varphi\cdot\cos\delta_1\cdot\cos(t_{gr1}\pm\lambda)\\ \sin h_2 = \sin\varphi\cdot\sin\delta_2 + \cos\varphi\cdot\cos\delta_2\cdot\cos(t_{gr2}\pm\lambda)\end{aligned}\right\} \quad (18.5)$$

Determining the coordinates φ and λ from observations of the altitudes of two bodies is called the "two-altitude problem".

From the set of equations (18.5), it is possible in principle to determine φ and λ, that is, to solve the two-altitude problem.

A direct analytic solution of the two-altitude problem is considerably more involved and has been given by Gauss as a solution of three spherical triangles. It is not used in practical astronomy because of its complexity, but it may be applied in a computerized solution of the problem.

In nautical astronomy, only special cases of the analytic solution of one of the equations (18.5), $\sin h$, are used for finding φ or λ. As indicated above, when determining φ we specify λ, that is, we take $\lambda = \lambda_c$, and when determining λ, we specify $\varphi = \varphi_c$.

(a) *The principle of determining latitude.* If a body is located on the meridian, then $h = H$; $t_{loc} = 0$ and equation (18.5) takes the form

$$\sin H = \cos(90^\circ - H) = \cos Z = \cos(\varphi - \delta)$$

whence

$$\varphi = Z + \delta \quad (18.6)$$

where δ is obtained from the noted instant T_{ch}.

If the body is located near the meridian, its altitudes are ordinarily "reduced" to the altitude H, and then the solution is by formula (18.6).

(b) *The principle of determining longitude.* From equation $\sin h$ we find

$$\cos t_{loc} = \frac{\sin h - \sin\varphi\cdot\sin\delta}{\cos\varphi\cdot\cos\delta}$$

and

$$\lambda = t_{loc} - t_{gr}$$

where t_{gr} and δ are obtained for the noted time T_{ch} and h is measured near the prime vertical.

On Determining a Position by Observing the Altitude and Azimuth of a Celestial Body

Suppose that both the altitude of a celestial body and the azimuth have been measured with the same accuracy by some instrument. Then the coordinates λ and φ are determined by observation of this

one body by solving the astronomical triangle, which is divided into two by a perpendicular dropped from the position of the body onto the meridian. The solution may be performed by the formulas

$$\left.\begin{array}{l}\sin t_{loc} = \cos h \cdot \sec \delta \cdot \sin A \\ \lambda = t_{loc} - t_{gr} \\ \tan k = \cot h \cdot \cos A \\ \sin (\varphi + k) = \sin \delta \cdot \operatorname{cosec} h \cdot \cos k\end{array}\right\} \quad (18.7)$$

where the auxiliar quantity k is the zenith distance of the foot of the perpendicular.

Due to inaccuracies in determining azimuth at sea, this and other "azimuthal" methods are not used at the present time.

(2) Determining the position of zenith (position of ship) graphically from lines of position. If on the surface of a sphere (Fig. 149) of radius $C_1Z_{loc} = 90° - h_1$ we draw from the position of celestial body C_1 an arc BB' of a small circle, we get the locus of zeniths of observers for which $90° - h_1$ (or h_1) remains constant. This arc may be regarded as the *line of position of observers' zeniths* for which h_1 const and φ and λ (t_{loc}) are different.

Project the entire drawing on the surface of the earth; C_1 will be projected onto point a_1, the zenith onto the position of the observer M, the side C_1Z_{loc} of the triangle onto arc a_1M, which in degrees is equal to $90° - h_1$. Since $1'$ of arc of a great circle on the earth is equal to 1 nautical mile, arc $(90° - h_1)$ (expressed in minutes) equals arc a_1M (expressed in nautical miles). In place of the line of position of the zenith BB', we will have an arc bb' representing the line of position of the ship. To construct this arc we need to know only the position of a_1 on the earth's surface (which is the centre of the circle bb') and its radius equal to arc $a_1M = (90° - h_1)$.

This whole construction will hold only for the given instant; the next instant there will be a change in the altitude of the body and the position of point a_1.

To determine the position of the zenith or the place on the earth, one more observation of the altitude of another body C_2 is required for the same instant. The second line of position of the zenith is constructed in similar fashion from the place of the body C_2 with radius $(90° - h_2)$; the position of the zenith on the sphere or the place on the earth will be obtained at the point of intersection of these two lines of position (see Fig. 149). This method is thus fully analogous to the determination of position in navigation by distances, only in place of beacons we observe celestial bodies, and the distances are the quantities $(90° - h)$ in miles.

(3) Determining the place of zenith by instruments. In addition to the above-described analytic and graphical methods of determi-

ning the zenith, there are methods which may be termed "instrumental".

In these methods, observations and their reduction are combined by appropriate positioning of special instruments.

(a) *The principle of the "instrumental" determination of the zenith position.* Suppose that on a transparent sheet we have a precise map (a) of circumzenith regions of the stellar sky in a coordinate system φ and S_{loc} to a certain scale (Fig. 150). The direction of the plumb line is determined, let us say, by the bubble (b) of a level, a gyrolevel, or in other ways. If we set the map and sheet horizontally and bring to coincidence the directions to 2 or 3 stars with

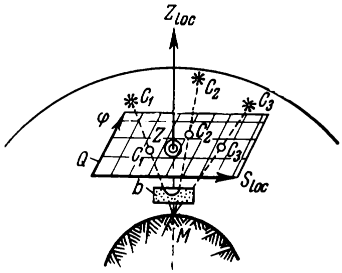

Fig. 150

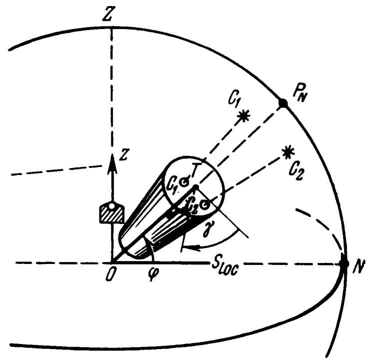

Fig. 151

their positions on the map, the instantaneous position of the zenith on the map will be at point z', the intersection of the map with the plumb line.

The zenith may be noted on the map by the bubble of the level or in other ways, and its coordinates φ and S_{loc} are taken from the lateral and lower frames of the map. The longitude of the place is determined from formula (18.1) at T_{gr} noted at the instant of observations.

This is the underlying principle of a number of instruments, of which the best known are the "zenithometer" of A. Mikhailov proposed in 1944 and the "instrument for night determination of position" of the Jugoslav astronomer L. Randich proposed in 1955-1956. Due to the difficulties of bringing into coincidence celestial bodies and the bubble index, and also due to considerable errors inherent in bubble levels and gyrolevels when the ship is rolling or pitching, these instruments are not suited to nautical astronomy.

A fundamental drawback of this method is the restricted choice of bodies which for a given latitude and a given time pass near the

zenith. These are frequently faint stars that require a strong zenith telescope. Another thing is that it is very difficult to take the observations.

(b) *The principle of an instrumental determination of directions to the pole and the zenith at one and the same time.*

Suppose we have a map of the sky similar to that considered above but constructed for the circumpolar region and placed in the field of view of a telescope OT (Fig. 151). We bring to coincidence the stellar images C_1, C_2, etc. on the map with the directions towards them, OC_1 and OC_2, in space; this is done by tilting the telescope and turning the map about the OT axis; in this way we get the direction of the celestial axis. If by means of a bubble or gyro-level we determine the direction of the plumb line Oz (or the plane of the horizon), we can determine the position of the meridian and obtain φ and S_{loc} and the direction N or S.

Instruments constructed on this principle have been proposed a number of times. In 1948, V. Kavraisky proposed a "pole-finder" to determine A from the stars α and β Ursae Minoris (Polaris and Kochab). In 1957 engineer Dubovetsky offered a similar instrument for determining φ and S_{loc} (λ). The principle of a mechanical determination of the direction of the celestial axis is also utilized in projects and models of "Gyrolatitude", in the "Spherant" instrument (1929, U.S.A.), and others.

All these instruments are absolutely useless at sea without a perfectly stabilized platform. What is more, they have one common defect: extremely complicated observations.

Of all the foregoing methods for determining the position of a ship at sea or its coordinates, use is made almost exclusively of the lines of position method and also occasionally of analytical procedures for determining latitude.

SEC. 98. MOST FAVOURABLE CONDITIONS IN ARRANGEMENT OF CELESTIAL BODIES FOR DETERMINING POSITION AND FOR SEPARATE DETERMINATION OF ITS COORDINATES φ AND λ

From navigation we know that the positions of beacons relative to a ship are very important for an accurate determination of position. Similarly in nautical astronomy the positions of celestial bodies on the celestial sphere relative to one another and the observer's meridian affect the accuracy of any determination. Let us find the conditions for which errors of observation (the total systematic and random errors) have the least effect on determining the position of the ship.

I. DERIVATION OF THE BASIC EQUATION OF ERRORS

When solving the astronomical triangle by formulas (18.5), we are given h, δ and t_{gr}, and we seek the observed coordinates φ and λ. Here, the quantities δ and t_{gr} taken out of MAE will be practically exact; so the errors that arise in φ_0 and λ_0 depend exclusively on the errors in the measured altitude. Taking error Δh as an increment, we find the increments $\Delta\varphi$ and $\Delta\lambda$ of the observed coordinates.

The analytical relationship between these coordinates is established by the familiar equation

$$\sin h = \sin\varphi\cdot\sin\delta + \cos\varphi\cdot\cos\delta\cdot\cos t_{loc} \qquad (*)$$

The total differential of a composite function of the form $u = f(\varphi, \lambda)$ will be

$$du = \frac{\partial u}{\partial\varphi}d\varphi + \frac{\partial u}{\partial\lambda}d\lambda$$

or, for our case,

$$dh = \frac{\partial h}{\partial\varphi}d\varphi + \frac{\partial h}{\partial t}dt_{loc} \qquad (18.8)$$

The partial derivatives $\frac{\partial h}{\partial\varphi}$ and $\frac{\partial h}{\partial t}$ are found by differentiating formula (*) with respect to the variables φ and t_{loc}.

$$\cos h\,\partial h = (\cos\varphi\cdot\sin\delta - \sin\varphi\cdot\cos\delta\cdot\cos t_{loc})\,\partial\varphi$$

or

$$\frac{\partial h}{\partial\varphi} = \frac{\cos\varphi\cdot\sin\delta - \sin\varphi\cdot\cos\delta\cdot\cos t_{loc}}{\cos h} \qquad (18.9)$$

From the astronomical triangle (see Fig. 149) and the formula of five elements (see Appendix III, 2), we write $\sin(90° - h)\cdot\cos A = \sin(90° - \varphi)\cos(90° - \delta) - \cos(90° - \varphi)\sin(90° - \delta)\cos t_{loc}$

or

$$\cos h\cdot\cos A = \cos\varphi\cdot\sin\delta - \sin\varphi\cdot\cos\delta\cdot\cos t_{loc} \qquad (18.10)$$

Substituting this expression in (18.9), we get

$$\frac{\partial h}{\partial\varphi} = \frac{\cos h\cdot\cos A}{\cos h}\cos A \qquad (18.11)$$

The partial derivative $\frac{\partial h}{\partial t}$ was obtained in Sec. 11 in the form

$$\frac{\partial h}{\partial t} = -\cos\varphi\cdot\sin A \qquad (18.12)$$

Since $t_{loc} = t_{gr} \pm \lambda_W^E$ and t_{gr} is practically exact*,

$$dt_{loc} = \pm\, d\lambda_W^E \text{ and } \frac{\partial h}{\partial t} = \frac{\partial h}{\partial \lambda}$$

Putting into (18.8) the values of the partial derivatives, passing to finite increments, and taking $\Delta t_{loc} = \pm\, \Delta\lambda_W^E$, we obtain

$$\Delta h = \cos A\, \Delta\varphi \mp \cos\varphi \cdot \sin A\, \Delta\lambda_W^E \tag{18.13}$$

If for the sake of simplicity we take the error $\Delta\lambda$ westwards, we finally get

$$\Delta h = \cos A\, \Delta\varphi + \cos\varphi \sin A\, \Delta\lambda \tag{18.14}$$

or, replacing the error in the longitude $\Delta\lambda$ by the error in departure $\Delta\sigma = \Delta\lambda \cos\varphi$, we get the relationship in a different form

$$\Delta h = \cos A\, \Delta\varphi + \sin A\, \Delta\sigma \tag{18.15}$$

We can call formula (18.14) or (18.15), which connects the error of measurements of altitude Δh with errors in the coordinates of position, $\Delta\varphi$ and $\Delta\lambda$ or $\Delta\sigma$, the *basic error formula* in determining position.

II. MOST FAVOURABLE ARRANGEMENT OF BODIES FOR DETERMINING A POSITION FROM TWO LINES OF POSITION

The lines of position of the zenith (and place), as shown above, are determined by measured altitudes h; for this reason, an error in altitude will cause a displacement of the line and errors in the coordinates will be expressed by the equation (18.14).

To find the position of a ship (obtain a fix) we need two lines of position that correspond to two measured altitudes of bodies; and so to find the errors $\Delta\varphi$ and $\Delta\lambda$ or $\Delta\sigma$, set up an error equation for each of the two lines

$$\left.\begin{aligned} \Delta h_1 &= \cos A_1\, \Delta\varphi + \sin A_1\, \Delta\sigma \\ \Delta h_2 &= \cos A_2\, \Delta\varphi + \sin A_2\, \Delta\sigma \end{aligned}\right\}$$

Solving this set of equations by means of determinants, we get

$$\Delta\varphi = \frac{\begin{vmatrix} \Delta h_1 & \sin A_1 \\ \Delta h_2 & \sin A_2 \end{vmatrix}}{\begin{vmatrix} \cos A_1 & \sin A_1 \\ \cos A_2 & \sin A_2 \end{vmatrix}}; \qquad \Delta\sigma = \frac{\begin{vmatrix} \cos A_1 & \Delta h_1 \\ \cos A_2 & \Delta h_2 \end{vmatrix}}{\begin{vmatrix} \cos A_1 & \sin A_1 \\ \cos A_2 & \sin A_2 \end{vmatrix}}$$

or

$$\Delta\varphi = \frac{\Delta h_1 \sin A_2 - \Delta h_2 \sin A_1}{\cos A_1 \sin A_2 - \sin A_1 \cdot \cos A_2} = \frac{\Delta h_1 \sin A_2 - \Delta h_2 \sin A_1}{\sin (A_2 - A_1)} \tag{18.16}$$

* Errors in the instant T_{gr} are considered errors of altitude (see Sec. 79).

and

$$\Delta\sigma = \Delta\lambda \cos\varphi = \frac{\Delta h_2 \cos A_1 - \Delta h_1 \cos A_2}{\sin(A_2 - A_1)} \tag{18.17}$$

We get the displacement ΔM in the ship's position on the map due to the errors $\Delta\varphi$ and $\Delta\sigma$ from the triangle M_1M_2D (Fig. 152)

$$\Delta M^2 = \Delta\varphi^2 + \Delta\sigma^2 \tag{18.18}$$

Substituting here the values $\Delta\varphi$ and $\Delta\sigma$ and performing simple manipulations, we get

$$\Delta M = \frac{\sqrt{\Delta h_1^2 + \Delta h_2^2 - 2\Delta h_1 \Delta h_2 \cdot \cos(A_2 - A_1)}}{\sin(A_2 - A_1)} \tag{18.19}$$

Here, the quantities Δh_1 and Δh_2 represent the total errors in altitude (both systematic and random), which are not the same either in magnitude or sign.

From an analysis of (18.19) it will be seen that displacement of the position of the ship depends, aside from the errors themselves,

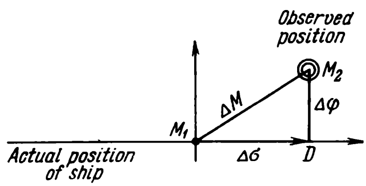

Fig. 152

also on the *difference of azimuths of the celestial bodies*, that is, on the angle of intersection of the lines of position. The least error in position will be for an azimuth difference of $A_2 - A_1 = 90°$, irrespective of the azimuths themselves, that is, regardless of the positions of the celestial bodies relative to the meridian.

III. MOST FAVOURABLE POSITION OF STARS FOR DETERMINING THE LATITUDE OF A POSITION

From the error equation (18.14) we determine the error in latitude

$$\Delta\varphi = \frac{\Delta h}{\cos A} - \cos\varphi \cdot \tan A \, \Delta\lambda \tag{18.20}$$

The error in latitude depends on the errors of measured altitudes and on the error $\Delta\lambda$ in the observer's longitude taken for these calculations.

Let us analyze this formula part by part:

(a) Assuming $\Delta\lambda = 0$, we get

$$\Delta\varphi = \frac{\Delta h}{\cos A} \tag{18.21}$$

From (18.21) it will be seen that for $A \approx 90^\circ$ (270°) the errors $\Delta\varphi$ increase without bound, and for $A = 0^\circ$ (180°), $\Delta\varphi = \pm\Delta h$. Thus, errors in altitude have the least effect on the desired latitude if the body is located near the observer's meridian.

(b) Assuming $\Delta h = 0$, we have

$$\Delta\varphi = -\cos\varphi \cdot \tan A \, \Delta\lambda \tag{18.22}$$

From this formula it is seen that for $A = 90^\circ$ (270°), the errors increase without bound, while for $A = 0^\circ$ (180°) they will be zero. It is also seen that the latitude is more accurately determined in high latitudes.

Consequently, it is best to determine *latitude* by bodies located close to the observer's meridian; and the latitude cannot be found by bodies with $A = 90^\circ$ (270°).

IV. MOST FAVOURABLE POSITIONS OF CELESTIAL BODIES FOR DETERMINING LONGITUDE

From the error equation (18.14) we determine the error in departure or longitude

$$\Delta\lambda \cos\varphi = \frac{\Delta h}{\sin A} - \cot A \, \Delta\varphi \tag{18.23}$$

The error $\Delta\lambda$ depends, as we see, on the errors Δh in the measured altitude and on the errors $\Delta\varphi$ in the latitude taken for the calculations.

(a) Assuming $\Delta\varphi = 0$, we get

$$\Delta\lambda \cos\varphi = \frac{\Delta h}{\cos A} \tag{18.24}$$

From this formula it is seen that for $A = 0^\circ$ (180°) the error $\Delta\lambda$ increases without bound, and for $A = 90^\circ$ (270°), $\Delta\lambda = \pm\Delta h \cdot \sec\varphi$, that is, it will be least.

(b) Assuming $\Delta h = 0$, we get

$$\Delta\lambda \cos\varphi = -\cot A \, \Delta\varphi \tag{18.25}$$

whence it is seen that for $A = 0^\circ$ (180°) the error $\Delta\lambda$ also increases without bound, and for $A = 90^\circ$ (270°) it will be zero. It is also obvious that the longitude is best determined near the equator and cannot be found at all near the poles.

CONCLUSIONS

1. A ship's position on a map is best determined from two astronomical lines of position for an azimuth difference of the bodies about 90°, regardless of the azimuths themselves.

2. The latitude of a place is best determined by stars located near the observer's meridian ($A = 0°$ or $180°$).

3. Longitude is best determined by stars located near the prime vertical ($A = 90°$ or $270°$).

From the foregoing analysis it is evident that the method of lines of position is independent of the position of the celestial body, whereas ways of determining φ and λ are drastically limited.

SEC. 99. CIRCLE OF EQUAL ALTITUDES

1. On lines of position. The concept of a line of position lies at the heart of graphic solutions of the problem of determining a ship's position. Let us see what this term means. To every astronomical and navigational observation on the earth's surface there corresponds a definite geometric place. These geometric places are distinguished by the fact that they correspond to an equal distribution of numerical values of a certain function $u = f(\varphi, \lambda)$ of the variables φ and λ and graphically—on a globe or map—represent lines of equal value or the *isolines* of this function. If we observe some physical quantity on the earth and use the observations to compute the value of the function $u = a$, we can then construct on a map or globe a section of this isoline $a = f(\varphi, \lambda)$ closest to the computed (dead-reckoning) position, which will be the *line of position of the ship*. Thus, the *line of position is a segment of an isoline of the function of some physical quantity, which segment is constructed near the dead-reckoning (D.R.) position and corresponds to the value of this quantity measured by the observer at the given instant.* In the general case, a line of position is a curve, but sometimes its segment near the D.R. position may be replaced by a straight line. Since only the altitudes of celestial bodies can be measured at sea with sufficient accuracy, nautical astronomy makes use only of lines of position obtained from the altitudes of celestial bodies. Let us see what an isoline of altitude is.

2. Circle of equal altitudes. Suppose a body C (Fig. 153) is seen on a given part of the earth and is located at such a distance that the light rays coming from it may be considered parallel.

Let observer M on the earth, whose plumb line is ZM and horizon H-H', obtain for celestial body C an altitude h_1 or zenith distance z_1. Since the light rays are parallel, the angle ZOa at the centre O of the earth will also be equal to z_1.

If we construct a number of angles z_1 near the line Oa, we will get a small circle MM_1.

Observers M_1M and all other observers located on this circle will have the same zenith distance z_1 or altitude h_1 of C at a given instant; this circle MM_1 is called a circle of equal altitudes. In other words, *a circle of equal altitudes is a locus of points on the earth's surface from*

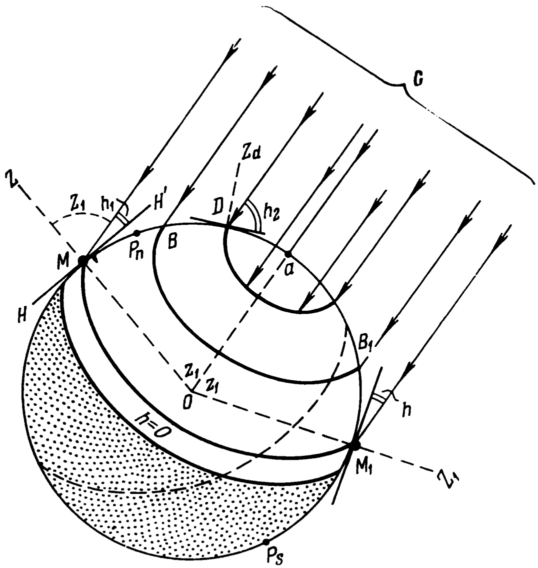

Fig. 153

which a given celestial body is seen at the same altitude at one and the same instant. This latter definition is exact, since it also takes into account the irregularity of the earth's shape and peculiarities in the position of the plumb line on the earth.

Thus, the isolines of altitude on the globe are circles of equal altitudes, and the line of position of a ship is a segment of that circle which corresponds to the altitude measured from the ship.

3. **Geographic position (GP).** From Fig. 153 it is seen that the greater the altitude h of a body, the smaller the circle of equal altitude; at point a the altitude $h = 90°$ or $z = 0°$, which means that an observer located at this point sees C in the zenith. This point is the *projection of the body* on the earth's surface at a given instant and is called the *geographic position of the body* or *the substellar* (subsolar, sublunar) *point.* If we reduce the altitude h, then for $h = 0$ the circle MM_1 turns into a great circle that divides the earth's surface into

two parts, in one of which the body C is visible, and in the other it is not.

Projecting a circle of equal altitudes on the celestial sphere (Fig. 154), we also get a circle of equal altitudes $Z_{loc}Z_i$, which is sometimes called a circle of equal zenith distances. From Fig. 154

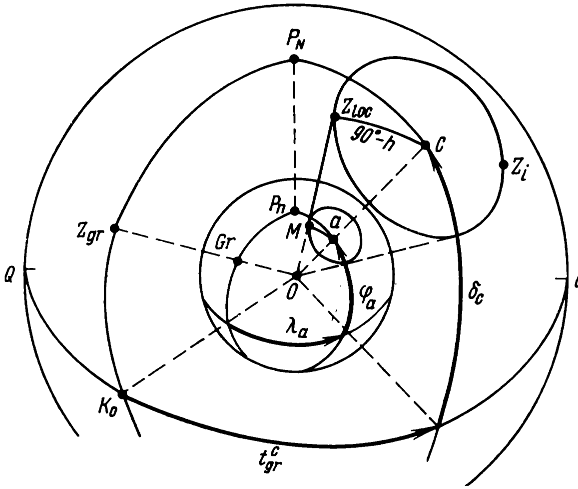

Fig. 154

it is seen that this circle may be drawn on the celestial sphere from the place of the body with a spherical radius $z = 90° - h$, and on the earth with the arc a_{loc}, equal to $90° - h$ in miles, as shown in Sec. 97.

The geographic position of a body on the earth is defined by the geographic coordinates φ_a and λ_a, while the position of the celestial body C on the celestial sphere is given by its equatorial coordinates δ^C and t^C_{gr} if the hour angles of the body are reckoned from Greenwich.

From Fig. 154 and also on the basis of formulas (18.3), Sec. 96, it is seen that

$$\left.\begin{aligned} \varphi_a &= \delta^C \\ \lambda_a &= t^C_{gr} \end{aligned}\right\} \qquad (18.26)$$

which states that the *latitude of the geographic position is equal to the declination of the body, and the longitude is equal to the Greenwich hour angle of this body at the given instant.* Since t^C_{gr} increases rapidly (in proportion to time), the geographic positions of all celestial bodies are in rapid westward motion over the earth with slight change in declination.

From the relationships obtained it may be said that the parameters which define the position of a circle of equal altitudes on the celestial sphere and on the globe are the coordinates of the body (the geographic position) and the spherical radius of the circle $z = 90° - h$.

4. Equation of a circle of equal altitudes on the celestial sphere. Suppose Z_{loc} is one of the points of a circle of equal altitudes (see Fig. 149), the equatorial coordinates of which are δ_Z (φ) and t_{loc} (λ). Constructing for this point the astronomical triangle $Z_{loc}P_NC_1$ by the formula of the cosine of a side, we get the familiar formula

$$\sin h_1 = \sin \varphi_{loc} \cdot \sin \delta_1 + \cos \varphi_{loc} \cdot \cos \delta_1 \cdot \cos (t_{gr1} + \lambda_{loc}) \quad (18.27)$$

If this same expression is written for other points of the circle HH', we will see that δ and t_{gr} remain unchanged (the body remained the same), the altitude h_1 likewise remains unaltered and only φ and

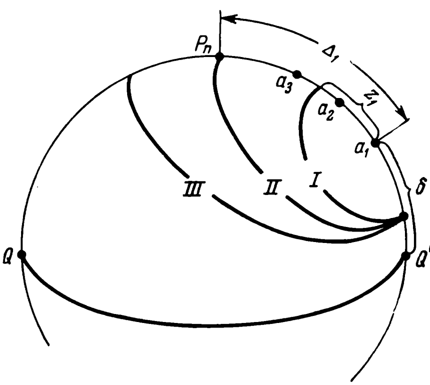

Fig. 155

λ of the points change. On this basis, formula (18.27) may be called the *equation of a circle of equal altitudes on the celestial sphere.* If we are given the value of one of the coordinates, say φ, then from equation (18.27) we can compute a series of values of t_{loc} and determine $\lambda = t_{loc} - t_{gr}$, which will be the longitudes of the corresponding points of the circle of equal altitudes. Similarly, we can find the latitudes from given λ_i of the corresponding points of the circle of equal altitudes.

5. Equation of a circle of equal altitudes on a Mercator map. On a Mercator map the circles of equal altitudes are depicted in the form of far more complicated curves called *cyclic curves.* The form of a cyclic curve (from the properties of a Mercator projection) depends on the position of the circle of equal altitudes relative to the earth's pole P_N.

Fig. 155 shows three circles of equal altitudes with geographic positions at points a_1, a_2 and a_3, spherical radii $z_i = 90° - h_i$ and polar distances $\Delta_i = 90° - \delta_i$. The first circle of equal altitudes is peculiar in that it does not include the pole P_N, that is, for it $z_1 < \Delta_1$, the second one passes through the pole, or $z_2 = \Delta_2$, and the third includes the pole, $z_3 > \Delta_3$.

On a Mercator map, the first circle is shown as a closed oval curve of the first type I′ (Fig. 156), the second circle is an open curve of

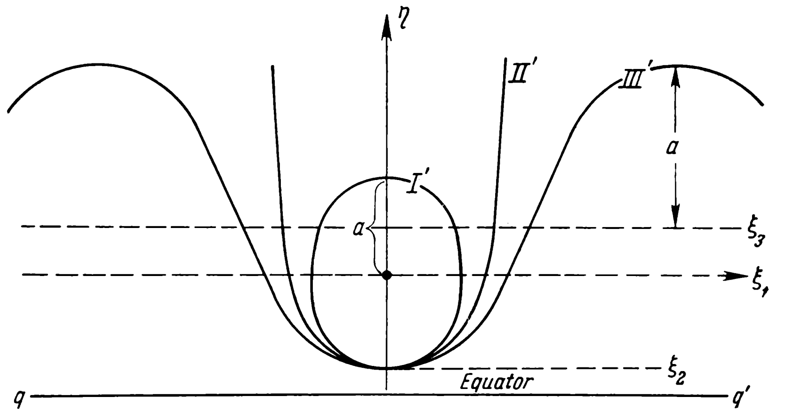

Fig. 156

the second type II′ in the form of a catenary, the third circle is an open curve of the third type III′ like a sine curve.

In the system of rectangular coordinates η and ξ, which coincide with the axes of symmetry of the curves, the equations of these curves will have the form

$$\left.\begin{aligned} \text{For curve I}'\text{:}\quad \cosh\eta &= \cos\xi\cdot\cosh a; \\ -\sinh\eta &= \cos A\cdot\sinh a \\ \text{For curve II}'\text{:}\quad e^{-\eta} &= \cos\xi = \cos A \\ \text{For curve III}'\text{:}\quad \sinh\eta &= -\cos\xi\cdot\sinh a; \\ \cosh\eta &= \cos A\cdot\cosh a \end{aligned}\right\} \qquad (18.28)$$

Here, sinh x and cosh x are the hyperbolic sine and cosine of the angle x

η and ξ are the ordinates and obscissas of points of the curve

a is the semiaxis of the curve

A is the azimuth of the celestial body.

From Fig. 156 and an analysis it follows that the curvature will be considerable only for curves of the first type in cases of very large h; in most cases, however, the curvature of cyclic curves is small and within 30 to 50 miles the curve hardly differs at all from a straight line.

SEC. 100. FUNDAMENTALS OF VARIOUS METHODS OF DETERMINING THE POSITION OF A SHIP FROM CIRCLES OF EQUAL ALTITUDES

The general concept of determining position by circles of equal altitudes (considered in Sec. 97) is specifically embodied in a variety of ways of plotting arcs of circles. The principles of these methods are given below.

I. USE OF CIRCLES OF EQUAL ALTITUDES ON THE CELESTIAL SPHERE

In this case, a circle of equal altitudes is the line of position of the observer's zenith on the sphere, and the position is obtained in

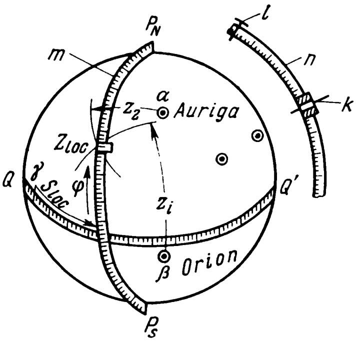

Fig. 157

equatorial coordinates. Then the relationships (18.1) yield the coordinates φ and λ of the position. This principle is utilized in the spherograph and in star altitude curves.

A *spherograph* is an instrument designed to give an approximate determination of the coordinates of a place. It represents the celestial sphere in the form of a star globe (Fig. 157) with depressions at star positions, an equator (with a scale of local sidereal time) and a movable meridian (m) with a scale of declinations or latitudes. Arcs of circles of equal altitudes are drawn from stellar positions by a special arc of the vertical circle (n) with a scale $z = 90° - h$,

a needle (l) on one end and a pencil (k) on a carriage that moves along the arc n. To determine position, measure the altitudes of two stars and note the instants: after the usual altitude correction, set readings $90° - h_1$ and $90° - h_2$ successively on the arc of the vertical circle n, and on the surface of the spherograph draw two arcs of circles of equal altitudes from the positions of the first and second stars. Of the two points of intersection, choose the one closer to the computed

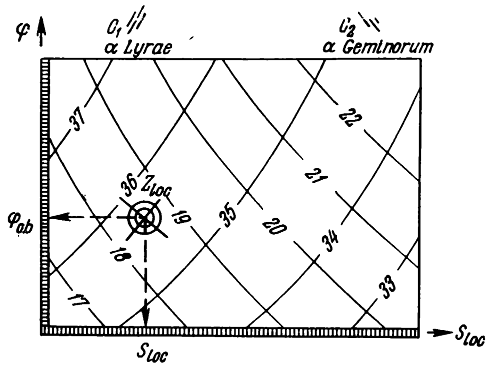

Fig. 158

point as the zenith. Using the movable meridian, take from the arc its $\delta_z = \varphi$, and from the equator $t^{\Upsilon}_{loc} = S_{loc}$. After computing t^{Υ}_{gr} from the noted T_2, we have $\lambda = t^{\Upsilon}_{loc} - t^{\Upsilon}_{gr}$.

Coordinates are determined quickly and easily, but accuracy is very poor. Indeed, some spherographs of foreign make have a diameter of 35 cm; thus, 1′ of arc (or 1 nautical mile) is equal to 0.05 mm. This means that 1 mm represents 20 miles and the overall accuracy is of the order of 0°.3 to 0°.5. To obtain an accuracy of the order of 1′, a mile must be no less than 1 mm; $\pi D = 360° \cdot 60' \cdot 1$ mm/′ and the diameter of the sphere $D = 6.9$ metres. This would be impractical.

Position approximations of this type are obtainable on an ordinary star globe with the aid of dividers.

Star altitude curves. Let us suppose that systems of circles of equal altitudes for various h have been plotted on a celestial sphere for stars C_1 and C_2 in a way similar to that done for the spherograph. The surface of the sphere together with the circles of equal altitudes and a grid of meridians and parallels is depicted on a plane in Mercator projection (Fig. 158). The resultant *chart of this section of the sky with the grid of isolines of altitude is called a star-altitude-curve chart* and serves for determining the coordinates of a place. If the

altitudes of two stars C_1 and C_2 are measured on a ship, working at night amounts to finding the isolines on the chart that correspond to those altitudes. The zenith will be at the point of intersection of these lines; its coordinates are taken from the lateral (φ) and lower (S_{loc}) frames of the chart; we then get $\lambda = S_{loc} - S_{gr}$. The accuracy depends mainly on the scale of the chart. This aid is in rather common use (see Sec. 136).

161. USE OF CIRCLES OF EQUAL ALTITUDES ON AN EARTH GLOBE

Unlike methods of obtaining position on the celestial sphere, here the ship's position is found directly on a globe (or map), but it is required to find and plot the geographic positions of the observed celestial bodies.

To determine the place of the ship on the globe, measure the altitudes of any two celestial bodies and note the instants. From T_{gr1}

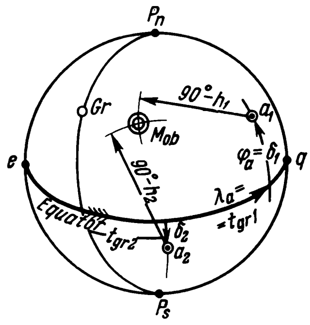

Fig. 159

and T_{gr2} thus obtained, take out of the MAE δ_1, t_{gr1} and δ_2, t_{gr2}, which, on the basis of the equalities (18.26), are the coordinates of the geographic positions of these bodies. Using these coordinates, plot the geographic positions a_1 and a_2 (Fig. 159); using the radii $(90° - h_1)'$ and $(90° - h_2)'$, draw the arcs of circles of equal altitudes to global scale; their intersection will give the position of the ship on the globe. However, to obtain a position with an accuracy of about $1'$, the diameter of the globe, as was shown for the spherograph, must be about 7 metres.

A similar construction may be applied to a chart; indeed, if one takes the quantities $(90° - h)'$ less than about 2°, the cyclic curves will be close to circles and they may be constructed on the chart just as they are on the globe. This procedure for constructing circles is used in low latitudes for altitudes of the sun greater than 88°; it cannot be used in ordinary conditions.

III. USING CIRCLES OF EQUAL ALTITUDES PLOTTED ON A CHART FOR DETERMINING POSITION

Arcs of a circle of equal altitudes may be constructed on a chart with high accuracy on the basis of several points; these curves will be the lines of position of the ship, but their construction is exceedingly involved. However, a small portion of a cyclic curve may be considered a straight-line segment. This considerably simplifies construction. This *straight line, which replaces a portion of the circle of equal altitudes, is called an altitude line of position* (or simply, *position line*). One method of plotting it was proposed by Captain Thomas H. Sumner, an American shipmaster, who in 1837-1843 discovered the principle of altitude lines of position, for which reason the altitude line of position is sometimes called a Sumner line.

Subsequently, several other ways of plotting altitude lines were suggested. We shall consider some of them here.

(1) Sumner's Method (the Longitude Method) of Plotting an Altitude Line of Position

This method may be called *longitudinal* because we compute the longitudes of points of a circle of equal altitudes.

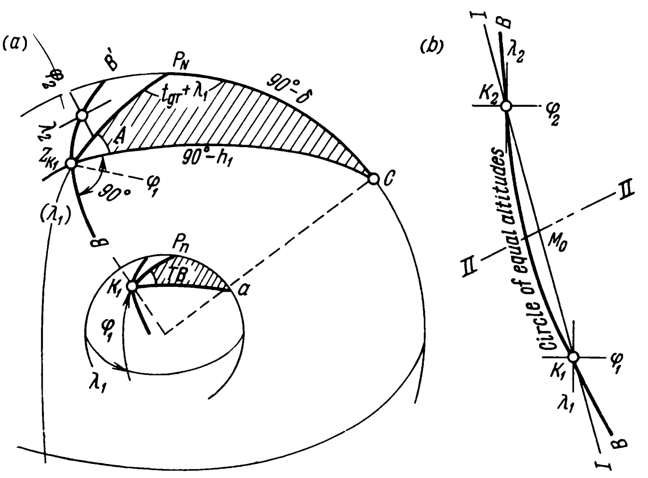

Fig. 160

Suppose that a measurement is made of the altitude h_1 of a body and its coordinates δ and t_{gr} have been chosen; then from equation (18.27) of the circle of equal altitudes, given the latitude φ_1, it is

possible to compute the longitude λ_1. This will be the longitude of the point K_1 (Fig. 160*a*) lying at the intersection of the circle of equal altitudes BB' and the parallel φ_1.

If we change the latitude and with the value of φ_2 from the same equation compute a new value of longitude λ_2, we will get a second point of the circle of equal altitudes. Joining these points on the chart with a straight line K_1K_2, we get an altitude position line *I-I* for the altitude h_1 (Fig. 160*b*). Thus, in Sumner's method, the position line is the chord of a small arc of a circle of equal altitudes.

To obtain the position M_0, measure a second altitude and plot a second line of position *II-II* in the same way.

Sumner suggested changing the latitude by 10′ to 20′ and computing t_{loc} from formula (2.8) $\sin^2 \frac{t_{loc}}{2}$, and then obtaining $\lambda = t_{loc} - t_{gr}$. The longitude method, in accord with the most favourable conditions for determining λ, is only applicable within the limits $\pm 45^\circ$ about the prime vertical; for $A < 45^\circ$ and $A > 135^\circ$ (in semicircular reckoning), the results become inaccurate.

(2) The Latitude Method

For the construction of position lines at azimuths where the Sumner method cannot be applied, that is, near the meridian ($45^\circ > A > 135^\circ$), the Danish navigator Paludan proposed (in 1852) a *method of latitudes*. The latitude of a point on a circle was obtained from the measured altitude h, t_{gr} and δ of the celestial body using the formulas

$$t_{loc}^{C} = t_{gr} + \lambda_1; \ \tan x_1 = \cot\delta\cdot\cos t_1 \ \text{and} \ \sin(\varphi_1 + x_1) = \\ = \sin h_1 \operatorname{cosec}\delta\cdot\cos x$$

The two points (φ_1, λ_1 and φ_2, λ_2) thus obtained of the circle were joined by a straight line.

(3) Akimov's Method (Azimuth Method)

An entirely different and more refined and simple principle for plotting altitude position lines was proposed by the Russian sea officer M. Akimov in 1849. In place of computing two points on a circle of equal altitudes, Akimov suggested computing *one point* of the circle (longitude method) and the direction towards the geographic position, that is, the *azimuth* of the body.

From Fig. 160*a* it is seen that the arc of the vertical circle $Z_{h_1}C' = 90^\circ - h_1$, which we shall call the azimuth line makes an angle of 90° with the circle of equal altitudes BB' (that is, it is the

radius of the circle). If at point K_1 we draw a tangent to the circle of equal altitudes, this will be the line of position that always passes at an angle of 90° to the azimuth line. Therefore, construction of an altitude position line on a chart (Fig. 161) reduces to plotting the point K_1 from φ_c, λ_1; to drawing the azimuth line at an angle A to the meridian, and to drawing through the point K_1 the position line *I-I* perpendicular to the azimuth line.

For computing A and t_{loc}, Akimov proposed using the semiperimeter formulas* in the form

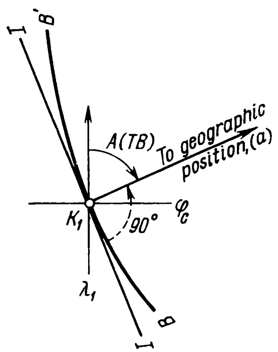

Fig. 161

$$\tan \frac{t_{loc}}{2} = \sqrt{u \cdot y}; \quad \tan \frac{A}{2} \sqrt{\frac{u}{y}}$$

where

$$u = \frac{\sin (S - \delta)}{\cos (S - z)}; \quad y = \frac{\sin (S - \varphi_c)}{\cos S}$$

and

$$S = \frac{\delta + z + \varphi_c}{2}; \quad z = 90° - h;$$

and then

$$\lambda_1 = t_{loc} - t_{gr}$$

The second line of position is plotted in the same way from the altitude of the second body.

In England, Akimov's method is known as Johnson's method. In 1862 Johnson published tables for computing t and A from Akimov's formulas.

Akimov's method cannot be used near the observer's meridian; for such cases a different method was proposed.

(4) Method of Latitude and Azimuth

For celestial bodies located near the meridian $135° < A < 45°$ the following formulas may be used with the computed longitude λ_c, δ, t_{gr} and h_{ob} of a body to compute the latitude φ_1 of a point of the circle and the azimuth to the geographic position:

$$t_{loc} = t_{gr} \pm \lambda_W^E; \quad \tan y = \cot \delta \cdot \cos t_{loc}; \quad \cos v = \sin h_{ob} \cdot \cos y \cdot \operatorname{cosec} \delta;$$

$$\varphi_1 = 90° - (y + v) \text{ and } \sin A = \cos \delta \cdot \sec h \cdot \sin t_{loc}$$

(5) Method of Marcq Saint-Hilaire

A fundamentally new method of plotting an altitude line—*from dead-reckoning position*—was offered in 1875 by the French seaman Marcq Saint-Hilaire.

* See Appendix III, 4.

Let us suppose that the altitude h_c is computed for the D.R. position M_c (φ_c, λ_c) with coordinates δ and t_{gr} of the observed body. This is the altitude an observer would have measured if he were in the D.R. position M_c. However, we have actually measured the altitude of the body h. The altitude difference $h - h_c$ will represent the distance in miles between the computed and actual circles of equal altitudes (Fig. 162). To construct a line of position from

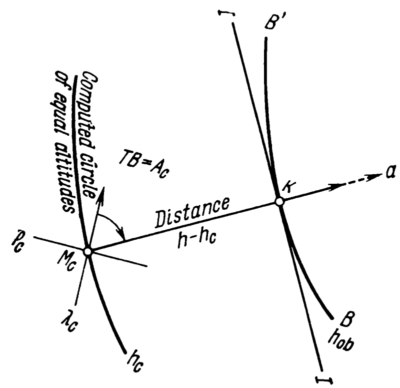

Fig. 162

the D.R. position M_c, compute the azimuth A_c of the body, which is the direction from the D.R. position to the geographic position of the body or the direction of the vertical circle of the body. Drawing an azimuth line from the D.R. position M_c and laying off on it a distance $(h - h_c)'$, we find the point K of the circle of equal altitudes. The altitude line of position I-I will pass through K perpendicular to the azimuth line as a tangent to the circle of equal altitudes. In contrast to other methods, this one is suitable for any azimuths; it is a universal method. What is more, the lines are laid down from the D.R. position, which is more convenient for navigators. This method gradually became predominant in all fleets and at present is the accepted method for plotting altitude lines on charts. A detailed analysis of this method is given in the next chapter.

From the foregoing consideration of methods of plotting position lines it is evident that in the first two methods this line is the chord of an arc of a cyclic curve, while in the latter three it is a tangent to this curve.

SEC. 101. THE ESSENCE OF THE GENERALIZED METHOD OF LINES OF POSITION

On the earth's surface, to each value of an observed quantity (altitude, say) there corresponds a definite isoline (see Sec. 100) of some function $u = f(\varphi, \lambda)$. If the isoline is obtained for a value

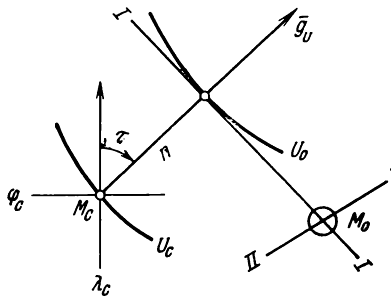

Fig. 163

of this quantity observed from a ship, the segment near the D.R. position is the line of position of the ship at that instant, that is

$$u_0 = f(\varphi_0, \lambda_0) \qquad (*)$$

The value of the function u_c for the D. R. position M_c located close to this line (Fig. 163) will be

$$u_c = f(\varphi_c, \lambda_c) \qquad (**)$$

which differs but slightly from u_0, that is,

$$u_0 = u_c + \Delta u$$

Since the arcs of isolines are small, they may be replaced by straight-line segments tangent to the isolines, which are the approximate *lines of position*. This kind of line of position may be plotted on a chart or paper by means of the *gradient of the function u* in exactly the same way for any observed quantity and any function. The gradient of the function $u = f(\varphi, \lambda)$ at a given point M_c is a vector $\bar{g}_u$ normal to the isoline u_c and directed towards an increase in the function, and is numerically equal to $\lim\limits_{n \to 0} \frac{\Delta u}{n}$, where $\Delta u = u_0 - u_c$ is the increment in the function u for a movement of the isoline through a distance n. The expression for g_u is derived beforehand for each method of determination and should be familiar to the observer.

Construction of a line of position reduces to the following operations:

(1) measure the value of u_0;
(2) compute the value of u_c for a known point M_c (φ_c, λ_c);
(3) form the difference $u_0 - u_c$;
(4) obtain the direction τ of the gradient and compute g_u;
(5) compute $n = \frac{u_0 - u_c}{g_u}$, where the expression for g_u is known;
(6) on the map, construct the gradient g_u for the D.R. position; find the determining point K and plot the line of position I-I perpendicular to the gradient.

To determine the position, plot two lines of position; the point of their intersection gives the observed position M_0 on the map.

When observing altitudes, the gradient is unity; indeed, a change in altitude of 1′ causes the isoline of altitudes to move 1 mile, that is, 1′ and $g_u = \frac{1'}{1'} = 1$. Its direction is A_c, and $n = h - h_c$, where h is measured and h_c is obtained by computation. From the foregoing it is evident that the Saint-Hilaire method is a special and elementary case of the general method of lines of position.

Analytically, the generalized method of lines of position reduces to finding the corrections $\Delta\varphi$ and $\Delta\lambda$ to the computed coordinates. If in equation (*) written for two lines we substitute the values $\varphi_0 = \varphi_c + \Delta\varphi$ and $\lambda_0 = \lambda_c + \Delta\lambda$ and confine ourselves to the first terms of the Taylor's series, we have

$$f_1(\varphi_0, \lambda_0) = f_1(\varphi_c, \lambda_c) + \frac{\partial f_1}{\partial \varphi}\Delta\varphi + \frac{\partial f_1}{\partial \lambda}\Delta\lambda + \ldots$$

$$f_2(\varphi_0, \lambda_0) = f_2(\varphi_c, \lambda_c) + \frac{\partial f_2}{\partial \varphi}\Delta\varphi + \frac{\partial f_2}{\partial \lambda} + \Delta\lambda \ldots$$

whence

$$\left.\begin{aligned} \frac{\partial f_1}{\partial \varphi}\Delta\varphi + \frac{\partial f_1}{\partial \lambda}\Delta\lambda &= f_1(\varphi_0, \lambda_0) - f_1(\varphi_c, \lambda_c) = \Delta u_1 \\ \frac{\partial f_2}{\partial \varphi}\Delta\varphi + \frac{\partial f_2}{\partial \lambda}\Delta\lambda &= f_2(\varphi_0, \lambda_0) - f_2(\varphi_c, \lambda_c) = \Delta u_2 \end{aligned}\right\} \quad (18.29)$$

where the values of the partial derivatives are obtained for the assumed position. From these equations it is easy to determine the differential corrections $\Delta\varphi$ and $\Delta\lambda$, and then we also get the observed coordinates $\varphi_0 = \varphi_c + \Delta\varphi$; $\lambda_0 = \lambda_c + \Delta\lambda$.

It is thus evident that the equations (18.29) are linear and represent equations of straight lines which we take to be the lines of position.

SEC. 102 THE EQUATION OF AN ALTITUDE LINE OF POSITION. OBTAINING CORRECTIONS $\Delta\varphi$ AND $\Delta\lambda$ TO THE COORDINATES φ_c, λ_c ANALYTICALLY

For the origin of the coordinate system we take the dead-reckoning position M_c (see Fig. 162) and the directions of axes along a meridian and parallel. To obtain the equation of the altitude line of position, apply one of the linear equations (18.29) in the form

$$\frac{\partial h}{\partial \varphi}\Delta\varphi+\frac{\partial h}{\partial \lambda}\Delta\lambda=h-h_c \tag{18.30}$$

The values of the partial derivatives were obtained when deriving the error equation in Sec. 98:

$$\frac{\partial h}{\partial \varphi}=\cos A; \quad \frac{\partial h}{\partial \lambda}=-\frac{\partial h}{\partial t}=\cos\varphi\cdot\sin A$$

Substituting them in (18.30) gives

$$\cos A\cdot\Delta\varphi+\sin A\cdot\cos\varphi\cdot\Delta\lambda=h-h_c$$

or, denoting $\Delta\sigma=\Delta\lambda\cdot\cos\varphi$ and $h-h_c=\Delta h$, we have

$$\cos A\cdot\Delta\varphi+\sin A\cdot\Delta\sigma=\Delta h \tag{18.31}$$

Expression (18.31) is the equation of a *straight line* (the altitude line of position) in the standard form. Indeed, analytic geometry yields the standard equation of a straight line (Fig. 164):

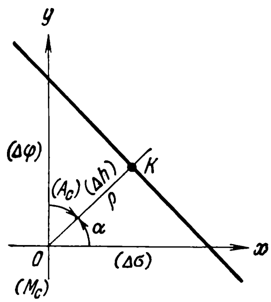

Fig. 164

$$x\cdot\cos\alpha+y\cdot\cos\alpha-p=0$$

Assuming $y=\Delta\varphi$, $x=\Delta\sigma$, $p=\Delta h$, and $\alpha=90°-A$, we have equation (18.31).

Thus, to a first approximation, a small segment of a curve of equal altitudes may be regarded as a straight line tangent to the curve at the point K.

The equation of a position line (18.31) differs from the error equation (18.15) in one way only: the coordinate origin in the former case is situated in the computed point and Δh is equal to $h-h_c$, while in the latter case, it is in the actual position of the ship and Δh is equal to the sum of altitude errors. Thus, the error equation is also an equation of a straight line (the altitude position line) but shifted due to errors. The equation of the position line including errors is shown below in Sec. 110.

For large segments of the curve in formulas (18.29), that is, in series expansions, one has to take into account terms of second and higher powers, which is a practical inconvenience. Errors due to substitution of the curve of equal altitudes by a straight line—the position line—may be obtained by computing the magnitude of suppressed terms of the series (18.29).

If we form equations (18.31) for two position lines:

$$\Delta h_1 = \Delta\varphi \cdot \cos A_1 + \Delta\sigma \cdot \sin A_1$$
$$\Delta h_2 = \Delta\varphi \cdot \cos A_2 + \Delta\sigma \cdot \sin A_2$$

and solve them as shown above in Sec. 98, we get the coordinate corrections $\Delta\varphi$ and $\Delta\lambda$:

$$\left.\begin{aligned} \Delta\varphi &= \frac{\Delta h_1 \cdot \sin A_2 - \Delta h_2 \cdot \sin A_1}{\sin (A_2 - A_1)} \\ \Delta\sigma = \Delta\lambda \cdot \cos\varphi &= \frac{\Delta h_2 \cdot \cos A_1 - \Delta h_1 \cdot \cos A_2}{\sin (A_2 - A_1)} \end{aligned}\right\} \qquad (18.32)$$

Here, A_1 and A_2 are expressed in circular reckoning (0° to 360°) and $A_2 > A_1$; $\Delta h_1 = (h - h_c)_1$; $\Delta h_2 = (h - h_c)_2$; $\Delta\varphi$ and $\Delta\lambda$ are considered plus (+) to the N and E, respectively.

Observed coordinates are obtained by introduction of the corrections $\Delta\varphi$ and $\Delta\lambda$:

$$\varphi_0 = \varphi_c + \Delta\varphi; \qquad \lambda_0 = \lambda_c + \Delta\lambda$$

Determining φ_0, λ_0 by computing the corrections $\Delta\varphi$ and $\Delta\lambda$ to the D.R. coordinates is called the method of differential corrections; it has not yet been used in practice but is suitable for use by computers.

CHAPTER 19

THE METHOD OF ALTITUDE LINES OF POSITION (METHOD OF MARCQ SAINT-HILAIRE)

SEC. 103. ELEMENTS OF AN ALTITUDE LINE OF POSITION

Of all the methods of plotting position lines on charts, the method of Marcq Saint-Hilaire (or intercept method) in its modern form is the only one in use. We shall now examine it in more detail.

Suppose (Fig. 165) that at the instant of observation the D.R. position of a ship on the earth is located at the point M_c; obviously, its zenith will be at Z_c on the celestial sphere. The position Z_c will be located on a certain circle of equal altitudes $h_c h_c'$ which may be drawn from the position of the celestial body B with a radius $z_c = 90° - h_0$, that is, on the isoline corresponding to M_c $(\varphi_c \lambda_c)$. Let the altitude h_0 be measured on the ship; then using the radius $90° - h_0$ we can draw the arc $h_0 h_0'$ of a circle of equal altitudes, which arc represents the actual line of position of the observer's zenith. If we construct an arc of the vertical circle of the body $Z_c B$, which arc projected on a map will be called the *azimuth line*, then it will intersect the circle of equal altitudes $h_0 h_0'$ at the point K, nearest to Z_c, called the *determining point*. Drawing, at this point, a tangent to the arc $h_0 h_0'$ perpendicular to the arc of the vertical circle (azimuth line), we get the *altitude line of position I-I*. Therefore, in this method, the altitude line of position, or simply the *position line, is a tangent to the circle of equal altitudes at the point K closest to the D.R. position.*

The place of the position line relative to Z_c is determined, as seen from Fig. 165, by the direction of the vertical circle KB and the arc $Z_c K$, that is, by the distance Δh to the determining point K. The direction of the vertical circle (the "azimuth line") is found from the magnitude of A_c, while the distance Δh, as seen from the figure, will be equal to the difference of the radii of the circles $h_c h_c'$ and $h_0 h_0'$, that is,

$$\Delta h = (90° - h_c) - (90° - h_0) = h_0 - h_c$$

On the earth's surface, the place of the position line *I-I* relative to the D.R. position M_c (Fig. 165) will be determined, respectively, by the bearing (true bearing) *TB* to the geographic position *a*, reckoned in the same way as A_c, and the distance M_cK, which, by virtue of the equal angles of the arcs Z_cK and M_cK will be $\Delta h = h_0 - h_c$ in minutes of arc, that is, nautical miles. The position line will have the very same position on a chart (Fig. 162).

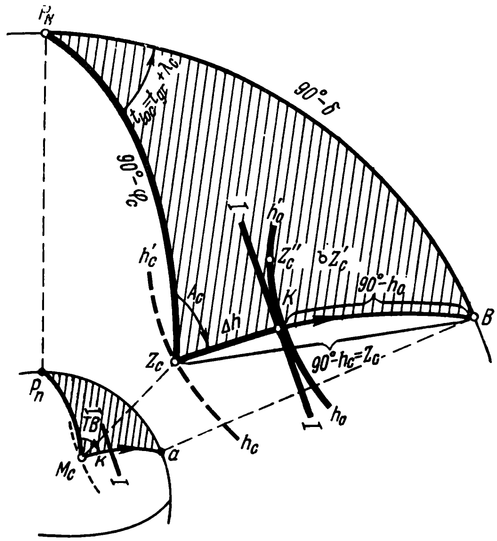

Fig. 165

The quantity $\Delta h = h_0 - h_c$ is sometimes called the intercept: it shows how much the computed line $h_c h_c'$ must be shifted to become an actual line of position of the ship.

Thus, to plot a position line on a chart we must have $A_c = TB$ and the intercept $\Delta h = h_0 - h_c$. These quantities are called *elements of the position line.*

The elements of a position line include: observed altitude h_0 (or h); altitude h_c, which corresponds to the dead-reckoning (computed) position M_c (or Z_c) and called the *computed altitude*, and, finally, A_c, similarly called the *computed azimuth.*

Methods for measuring and correcting observed altitude have already been studied; let us now examine certain methods for computing the quantities h_c and A_c.

SEC. 104. THE ASTRONOMICAL TRIANGLE AND ITS SOLUTION BY FORMULAS OF SPHERICAL TRIGONOMETRY

By constructing the vertical circle of a body Z_cB and the meridian of D.R. zenith P_NZ_c, we get (Fig. 165) on the sphere the astronomical triangle P_NZ_cB, which, in contrast to the ordinary astronomical triangle constructed for the actual place of zenith Z, is called the *computed astronomical triangle* of the body. Two vertices of this triangle—the pole P_N and the place of the body—coincide with the vertices of the ordinary astronomical triangle, while the third vertex—the "computed" zenith Z_c—corresponds to the computed or D.R. (assumed) position.

The elements of the computed astronomical triangle are:

δ and t_{gr} = the actual values of the equatorial coordinates of the celestial body B taken from the MAE for T_{gr};

φ_c and λ_c = the D.R. coordinates of the position taken from a chart at the instant of observation;

h_c and A_c = the "computed" altitude and azimuth that correspond to the D.R. position M_c (Z_c).

Here, the first four quantities that correspond to two sides and an angle of the computed triangle are known, the last two (the side $90° - h_c$ and the angle A_c) are to be sought.

The quantities h_c and A_c may be found by solving the triangle by the basic formulas of spherical trigonometry, by transformed formulas, or by special tables made for this purpose. Finally, special instruments and even computers are used to transform the coordinates. More complicated methods of determining h_c and A_c will be considered at the end of this chapter. For the present, let us confine ourselves to solution of the triangle by the basic formulas.

I. FIRST SYSTEM OF FORMULAS

Applying to the triangle Z_cP_NB formulas of the cosine of a side and of sines, similar to the formulas (2.1) and (2.3), we have

$$\sin h_c = \sin \varphi_c \cdot \sin \delta + \cos \varphi_c \cdot \cos \delta \cdot \cos (t_{gr} \pm \lambda_c) \qquad (19.1)$$

and

$$\sin A_c = \sin t_{loc} \cdot \cos \delta \cdot \sec h_c \qquad (19.2)$$

By computing the azimuth from the simpler formula $\sin A_c$ in place of the exact formula $\cot A_c$, we depart from the principle of computing known quantities independently of one another. But in the given case, this departure is possible because A_c (which is needed for plotting the azimuth line on the map) need be known

only to 0°.1, whereas the quantity h_c introduced into formula (19.2) is obtained with an accuracy of the order of 0′.1-1′.0.

The computations that these formulas involve may be performed with the help of tables of logarithms, tables of natural values of trigonometric functions, or, finally, by a combination method: products by a table of logarithms, sums by tables of natural values. This method, which does not require tables of logarithms of sums

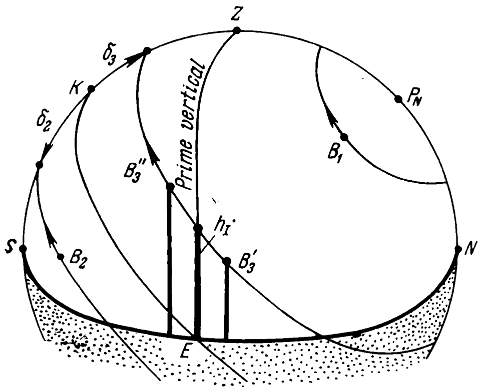

Fig. 166

and differences (Gauss tables), is in common use in many countries. In the Soviet Union, only the logarithmic method of computation is used.

Investigating the sign of formula sin h is explained in Sec. 7 and in the accompanying example.

Determining the name of A_c. In formula (19.2) the factors are always positive; therefore, when investigating the sign, the quadrant in which A_c is located is not determined. Thus, after computing A_c from (19.2) and tables, we always get a value less than 90°, which is azimuth in quadrantal reckoning.

In quadrantal reckoning, the first letter of the designation indicates the portion of the meridian (N or S) from which reckoning begins, the second, the hemisphere (E or W) in which the celestial body is located. For this reason, the name of the quadrant of the horizon may be determined from the peculiarities of diurnal motion of bodies in different latitudes.

The *second letter* in the designation of quadrantal azimuth *will always be the same as that of the hour angle* t_{loc}. To determine the *first letter* of the name of quadrantal azimuth, let us consider the above-horizon part of the diurnal circle of three groups of bodies

(Fig. 166) that differ as to the relationship between the δ of the body and the φ of the position.

(a) Celestial bodies (B_1) whose declinations are of the same name as the latitude and greater than it ($\delta > \varphi$ and same name) have the first letter of the azimuth designation always the *same* as that of latitude.

(b) Celestial bodies (B_2) whose declination is of name contrary to latitude (δ of contrary name to φ) have the first letter of azimuth designation always *contrary* to latitude.

(c) Celestial bodies (B_3') whose δ is of same name as φ and $\delta < \varphi$, will cross above-horizon portion of prime vertical and may be located in the northern or southern parts of the horizon; for this reason, the first letter depends on whether they have crossed the prime vertical or not. The distinguishing feature is the altitude of the body on the prime vertical h_{I} (or the hour angle t_{I}).

If the body (B_3') has $h_c < h_{\mathrm{I}}$, then the first letter of the azimuth designation is of the same name as φ; but if the body (B_3'') has $h_c > h_{\mathrm{I}}$, then the first letter of A is contrary to φ.

The altitude of the body on the prime vertical is determined from the formula $\sin h_{\mathrm{I}} = \operatorname{cosec} \varphi_c \cdot \sin \delta$ and is given in Table 21, MT-63. The foregoing discussion about determining the name of A_c is given in Table 9 and also in the explanations to the tables of MT-63.

Table 9

Name δ	Magnitude of δ	Magnitude of h_c	1st letter, A_c	2nd letter, A_c
Contrary to latitude	No value		Contrary to latitude	Always same name as (practical) hour angle
Same as latitude	$\delta < \varphi$	$h_c > h_{\mathrm{I}}$	Ditto	
Same as latitude	$\delta < \varphi$	$h_c < h_{\mathrm{I}}$	Same name as latitude	
Ditto	$\delta > \varphi$	No value	Ditto	

On board ship, the name of the azimuth may be determined by compass in all cases, except when the celestial body is very close to the prime vertical. To avoid mistakes in this case, it is best to use this rule.

The sequence of computations for h_c and A_c is similar to that given in Sec. 7 and is shown in Example 1.

Example 1. Determine h_c and A_c, using $\varphi_c = 60°2'.5\text{N}$; $\delta = 28°44'.9\text{N}$; t 108°44.'2E.

$$\sin h_c = \overset{+}{\sin}\, \varphi_c \cdot \overset{+}{\sin}\, \delta + \overset{+}{\cos}\, \varphi_c \cdot \overset{+}{\cos}\, \delta \cdot \overset{-}{\cos}\, t_{loc} \qquad \text{(I-II)}$$

$$\sin A_c = \cos \delta \cdot \sin t_{loc} \cdot \sec h_c$$

φ_c 60°2'.5	sin	9.93772	cos	9.69842	sec h		0.01721
δ 2844.9	sin	9.68211	cos	9.94287	cos		9.94287
t 10844.2			cos	9.50680	sin		9.97635
(71°15'.8)	I	9.61983	II	9.14809	sin A_c		9.93643
	β	9.82119	*Arg*	0.47174	A_c		59°45'
					A_c	$\approx$	59°.8NE
	sin h_c	9.44102					
	h_c	16°1.5					

Establish the name of azimuth. In this example, $\delta < \varphi$ and of the same name, that is, the body crosses the prime vertical. From Table 216, using φ and δ, we take $h_{\text{I}} \approx 33° > h_c$, which means the body has not passed the prime vertical and the first letter of the name is the same as φ_N. The second letter with respect to t_{loc} is E.

Computation of h_c from formula $\sin h_c$ has the advantage that it is done by the general formula and ordinary tables of logarithms; all investigations and extractions are performed in accord with general mathematical rules, thus allowing for a certain amount of checking (see Sec. 7). But this formula also has certain drawbacks: low accuracy in computation of altitudes exceeding 30° with four-place tables; when working with logarithms, the need for tables of sums and differences, which in the case of four decimals are small (6 pages) and simple, but with five decimals are unwieldy (19 pages) and inconvenient.

II. SECOND SYSTEM OF FORMULAS

The basic formula $\sin h_c$ may be reduced (see Sec. 7, Ch. 1) to the function $\sin^2 \frac{z^2}{2}$, which is more exact than $\sin h$ for large angles. Applied to the computed astronomical triangle, this formula has the form

$$\sin^2 \frac{z_c}{2} = \sin^2 \frac{\varphi_c \approx \delta}{2} + \cos \varphi_c \cos \delta \cdot \sin^2 \frac{t_{loc}}{2} \qquad (19.3)$$

In this case, the computed azimuth is obtained from (19.2) or, after replacing h_c by z_c, by the formula

$$\sin A_c = \sin t \cdot \cos \delta \cdot \operatorname{cosec} z_c \qquad (19.4)$$

In formula (19.3), for φ and δ of same name, the smaller is subtracted from the larger quantity; for contrary names, they are combined. There are two ways of computing z_c from (19.3): (a) mixed solution by tables of logarithms and tables of natural values of functions, (b) solution solely by tables of logarithms.

Solution by the first procedure is common in many countries and does not require tables of sums, but it yields lower accuracy and is less convenient than with logarithms.

In the logarithm solution used in the Soviet fleet, formula (19.3) has an advantage over (19.1) in that it only requires tables for sums. Indeed, both terms of this formula will be positive for all values of φ, δ, t, and no investigation of sign in (19.3) is required. Tables of $\log \sin^2 \frac{\alpha}{2}$ are given in MT-63 (Tables 5a and 5б) for the argument α from 0° to 180°; note that when entering these tables, α should not be halved. The natural values of $\sin^2 \frac{\alpha}{2}$ are not given in MT-53 (in MT-43, Table 8), so the mixed procedure cannot be applied with MT-53.

Formula (19.3) yields more precise results than (19.1) for h greater than 30°. This should be kept in mind when solving problems with the help of four-place tables of logarithms.

Thus, computations by this formula have certain advantages: no investigation of sign is needed, no Gauss tables of differences either; precision for large angles is higher than for small angles, but five-place tables suffice for all angles. And the amount of arithmetic involved in computations with (19.3) and (19.1) is about the same.

Example 2. Find h_c and A_c using formulas (19.3) and (19.4) and knowing $\varphi = 21°54'.5$S; $\delta = 5°19'.8$N; $t_{loc} = 36°51'.4$E.

$$\sin^2 \frac{z_c}{2} = \sin^2 \frac{\varphi + \delta}{2} + \cos\varphi \cdot \cos\delta \cdot \sin^2 \frac{t}{2},$$

$$\sin A_c = \sin t \cdot \cos\delta \cdot \operatorname{cosec} z_c$$

$t = 36°51'.4$E			$\sin^2$	8.99970	sin	9.77802
$\varphi_c = 21°54'.5$S			cos	9.96744	—	9.99812
$\delta = 5°19'.8$N			cos	9.99812	cos	
					cosec Z	0.14892
$\varphi_c + \delta = 27°14'.3$	$\sin^2$	8.74386	II	8.96526		
	Arg	0.22140	α	0.20429	sin A	9.92506
			$\sin^2 \frac{z_c}{2}$	9.16955	A_c	57°3NE
			z_c	45°12'.7	h_c	44°47'.3

III. THIRD SYSTEM OF FORMULAS: "TANGENT FORMULAS"

Like any spherical triangle, the computed astronomical triangle of a body is solvable not only by the two formulas (19.1) and (19.3), but also by formulas reduced to logarithmic form. These formulas are obtained from the basic ones by introducing auxiliary quantities, or if we examine the problem geometrically, by dividing the astronomical triangle into two right-angled triangles. Of the several methods of dividing a triangle, we shall consider only one: by means of a spherical perpendicular arc $CD = p$ dropped from the place of the body C onto the meridian of the observer (Fig. 167). Solving triangles P_NDC and ZDC in succession, we find the desired h_c and A_c. Denote by x the arc KD from the equator to D (the foot of the perpendicular), then the arc $ZD = (90° - x) - (90° - \varphi) = \varphi - x$.

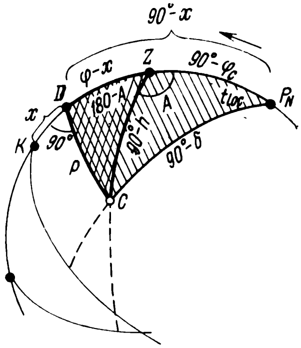

Fig. 167

From the spherical triangle P_NDC in which the angle D is equal to 90° we get, using the cotangent formula (or Napier's mnemonic rules),

$$\underbrace{\cot 90° \cdot \sin t_{loc}}_{0} = \cot (90° - \delta) \cdot \sin (90° - x) - \cos (90° - x) \cos t_{loc}$$

whence the formula for computing the auxiliary quantity x will be

$$\tan x = \tan \delta \cdot \sec t_{loc} \tag{19.5}$$

An investigation of (19.5) and Fig. 167 shows that the name of x is always the same as that of declination, while the value of x depends on t: for $t < 90°$, $x < 90°$, for $t > 90°$, $x > 90°$.

Applying the cotangent formula to the angle t_{loc} (clockwise), from the same triangle P_NDC we get p:

$$\cot t \cdot \sin 90° = \cot p \cdot \sin (90° - x) - \underbrace{\cos 90° \cdot \cos (90° - x)}_{0},$$

whence

$$\tan p = \cos x \cdot \tan t \tag{19.6}$$

Having determined x and p, we pass to the solution of the triangle ZCD, from which we can find A_c and h_c. Applying the cotangent formula to the angle $180° - A_c$, we have

$$\cot (180° - A_c) \cdot \sin 90° = \cot p \cdot \sin (\varphi - x) - \underbrace{\cos 90° \cdot \cos (\varphi - x)}_{0},$$

whence

$$-\tan A_c = \tan p \operatorname{cosec}(\varphi - x)$$

or, after substituting the values of tan p from (19.6), we get

$$\tan A_c = -\tan t \cdot \cos x \cdot \operatorname{cosec}(\varphi - x)$$

In order to get rid of the minus sign and reduce to the function sec α, substitute $\operatorname{cosec}(\varphi - x) = -\sec[90° + (\varphi - x)]$ and $\cos x = \frac{1}{\sec x}$; then we finally get

$$\tan A_c = \frac{\tan t \cdot \sec[90° + (\varphi - x)]}{\sec x} \tag{19.7}$$

From the same triangle ZCD we have h_c (by way of A_c)

$$\underbrace{\cot 90° \cdot \sin(180° - A_c)}_{0} = \cot(90° - h_c) \cdot \sin(\varphi - x) -$$

$$-\cos(\varphi - x)\cos(180° - A_c)$$

whence

$$\tan h_c = -\cot(\varphi - x) \cdot \cos A$$

or, substituting

$$\tan[90° + (\varphi - x)] \text{ and } \cos A_c = \frac{1}{\sec A_c} \text{ for} - \cot(\varphi - x)$$

we get

$$\tan h_c = \frac{\tan[90° + (\varphi - x)]}{\sec A_c} \tag{19.8}$$

In the formulas obtained we have $\varphi - x$; similarly, we can obtain $x - \varphi$ and $\varphi + x$ depending on the relationship of the quantities and the names of φ and x; that is, depending on the portion of the meridian where point D is located. Therefore, in the general form we may write $x \sim \varphi$, where the sign $\sim$ means that for φ and x of the same name, subtract the smaller quantity from the larger one, and combine for contrary names of φ and x.

Formulas (19.5), (19.7), (19.8) are called "formulas of tangents" and were previously obtained in somewhat different form by K. Gauss.

These formulas have the following advantages: they are all logarithmic, which means they do not require tables of sums and differences; the desired quantities are computed by the most favourable tangent function, which makes them equally suitable for small and large altitudes. In practice, h_c and A_c are not directly computed

by MT-63 tables and from these formulas, but they are used in solutions based on special tables of altitudes and azimuths, such as the Tables TBA-52 and TBA-57 of Yushchenko and in other cases.

SEC. 105. INSTANCES OF D.R. POSITION RELATIVE TO CIRCLE OF EQUAL ALTITUDES. LAYING DOWN THE POSITION LINE

The computed (D.R) position, or D.R. zenith, may happen to be outside the circle of equal altitudes, inside it or right on the circle.

If the D.R. zenith is located outside the circle of equal altitudes (see Z_c in Fig. 165), the determining point K is located relative

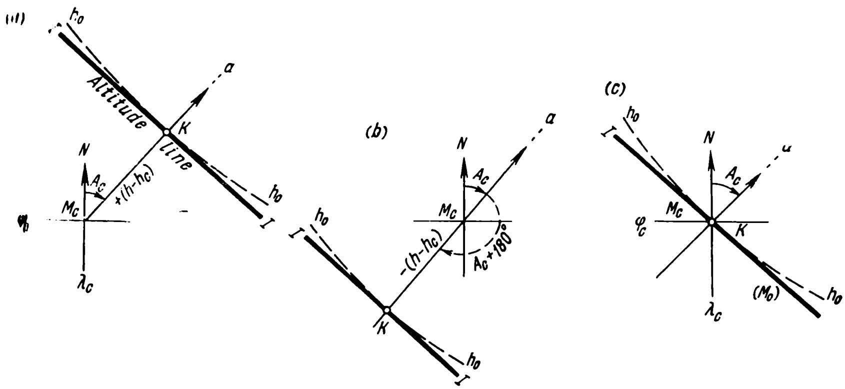

Fig. 168

to Z_c in the direction of the celestial body B. In this case $(90° - h_c) > (90° - h_0)$ or arc $z_c >$ arc z_0, that is, $h_0 > h_c$ or the difference $h_0 - h_c > 0$. Consequently, if the altitude difference $h_0 - h_c$ has the sign "+", the magnitude is laid off from M_c (Fig. 168*a*) along the azimuth line in the direction of the geographic position or the celestial body; this direction is noted by an arrow.

But if the D.R. zenith lies inside the circle of equal altitudes (see Z_c' in Fig. 165), the determining point K will be in a direction opposite the bearing towards the geographic position, that is in the direction $A + 180°$. In this case, arc $z_c <$ arc z_0, whence $h_0 < h_c$ and the difference $h_0 - h_c < 0$. Hence, if the altitude difference $h_0 - h_c$ has the minus sign, its magnitude is laid off from M_c along the azimuth line extended in the direction of $A_c + 180°$, or away from the celestial body (Fig. 168*b*). Finally, if the D.R. position lies right on the circle of equal altitudes (see Z_c'' in Fig. 165), then $z_c = z_0$ or $h_0 = h_c$ and $h_0 - h_c = 0$. In this case

the position line will pass through the D.R. position perpendicular to the line of azimuth (Fig. 168*c*). This case does not yet mean that the D.R. and observed positions have coincided, since the second line may also cross the first at some distance from M_c.

From the foregoing we get rules for plotting position lines on a chart.

(1) The computed altitude h_c is always subtracted from the observed altitude h_0 (denoted by h); in other words, we find the difference $h - h_c = \Delta h$ (intercept) with its sign.

(2) The azimuth line is drawn from the D.R. position in the direction of the celestial body if the sign of the difference Δh is positive, in the reverse direction if negative. It is advisable to arrow the direction towards the body.

(3) The quantity $h - h_c$ is laid off "towards the celestial body" when the sign is "+" and "away from the body" when the sign is "—". We get a determining point K lying on the line of position. For $h - h_c = 0$, the point K coincides with D.R. position M_c.

(4) The position line is drawn through K perpendicular to the azimuth line.

The pencil work associated with plotting an altitude line on a chart or on paper is, by analogy with navigation, called *laying down* (plotting) position lines.

I. PLOTTING ALTITUDE LINES OF POSITION ON A NAVIGATION CHART

Let us suppose that two celestial bodies have been observed and the elements of two position lines obtained. In accord with the

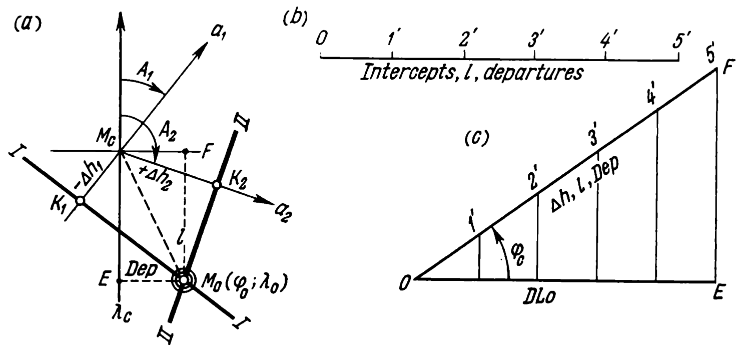

Fig. 169

rules, draw azimuth lines from the D.R. position M_c; to do this, lay off computed azimuths from the meridian of M_c in quadrantal reckoning by means of a protractor to within 0°.2 (Fig. 169*a*). The

values of the intercepts (Δh) in minutes (′) and fractions of minutes are taken by a chart divider from the lateral (vertical) frame of the chart. Having obtained the determining points K_1 and K_2 on the azimuth lines, draw altitude lines through them perpendicular to the azimuth lines and label them *I* and *II*. The observed position M_0 of the ship on the chart is obtained at the intersection of these two lines. Its coordinates (φ_0 and λ_0) are logged from the chart in the usual way.

II. PLOTTING ALTITUDE LINES OF POSITION ON PAPER

When dealing with small-scale charts or in training observations, plotting position lines may be done:

(a) on another chart with a larger scale in the same latitude and in arbitrary longitude or on position-line charts (plotting sheets) with subsequent determination of $\Delta\varphi$, $\Delta\lambda$ and transfer to the course chart;

(b) on unruled or plotting paper. Let us consider this latter procedure in more detail.

Plotting on paper differs from plotting on a chart in that the result is not the desired position of the ship on a chart but only the position relative to the D.R. position; in other words, what is actually found are the corrections $\Delta\varphi$ and $\Delta\lambda$ to the D.R. coordinates, from which we then get the position or its observed coordinates φ_0, λ_0. Therefore, this procedure is fundamentally less refined than plotting on a chart, but it is a rather common practice.

Before starting the construction, choose a *scale* for plotting. In accord with the rules for work with a Mercator chart, two scales are needed: the distance scale in Mercator miles, and the longitude scale in equatorial minutes (less by sec φ times).

First method. For 1′ of distance (lateral frame of map) we take an arbitrary segment of, say, 1 cm. From this scale (Fig. 169*b*) we take Δh, l and departures; to convert departures into difference of longitude (*DLo*), use Table 25, MT-53 (35, MT-43).

Second method. Construct a "scale angle" (Fig. 169*c*) at an arbitrary point O of a sheet of paper; the line OF forms an angle φ with the line OE. Lay off arbitrary segments of 1 to 1.5 cm on the inclined line OF. These segments represent minutes of the lateral frame. Projecting these segments on the horizontal line OE, we get equatorial minutes, since $OE = OF \cdot \cos\varphi$. From the inclined scale we take Δh, l and departure; from the horizontal scale, only longitude differences. It is not advisable to use the "scale angle" method in high latitudes.

Third method. For the distance scale (Δh, l) take a twofold to fivefold minute of the lateral frame of the course chart in the region

of the D.R. latitude; for the scale of longitudes, a minute of the lower frame of the chart increased the same number of times.

After choosing the scale, plotting is done as follows.

An arbitrary point M_c of the sheet (Fig. 169*a*) chosen with direction of the construction in mind is taken as the computed (D.R.) position; through it draw the D.R. meridian and parallel. The position lines are constructed from this point in a manner similar to that shown above for a chart, and the values of Δh are taken from the distance scale (the line OF in the second method).

Having obtained the observed position M_0, we measure the segments M_0F up to the parallel of the D.R. position and M_0E to its meridian. In the distance scale, segment M_0F is equal to l, and M_0E is equal to DLo in the longitude scale. We thus obtain the observed coordinates:

$$\left.\begin{aligned} \varphi_0 &= \varphi_c + l \\ \lambda_0 &= \lambda_c + DLo \end{aligned}\right\} \qquad (19.9)$$

Also, we take the direction of *leeway* C, for instance (in Fig. 169) $C = 150° - 2'.0$.

In plotting position lines, it is more convenient to use a special instrument, for example, a transparent protractor.

SEC. 106. PROPERTIES OF AN ALTITUDE LINE OF POSITION

Let us consider the principal features of an altitude line of position plotted by the method of Marcq Saint-Hilaire.

I. THE APPROXIMATE NATURE OF AN ALTITUDE LINE OF POSITION

When plotting a position line on a chart in the form of a straight line in place of a cyclic curve (see Fig. 168), there is obviously an inaccuracy that increases with the distance between the D.R. position and the observed position. As will be seen further on, this substitution is permissible only for a difference in positions of 20′ to 30′, or up to 0°.5. Hence, the *straight line is an approximate line of position.*

II. INDEPENDENCE OF A POSITION LINE OF ACCOUNTED COORDINATES

Fundamentally, the method of Saint-Hilaire permits taking the coordinates of any point on the earth (and not only those of a D.R. position) to compute the elements of the position line. However, due to the approximate nature of the position line, the accepted

point should not be farther than 0°.5 away from a circle of equal altitudes. Within these limits we can take any coordinates φ_c and λ_c for computing h_c and A_c.

The values of h_c and A_c and Δh will of course change with any change in the place of Z_c on the sphere, but the position of the circle of equal altitudes (which corresponds to h_0) *remains unchanged.* Indeed, let M_c', M_c'' and M_c''' be various D.R. positions (Fig. 170) taken for computing the elements of a position line *I-I* for one and

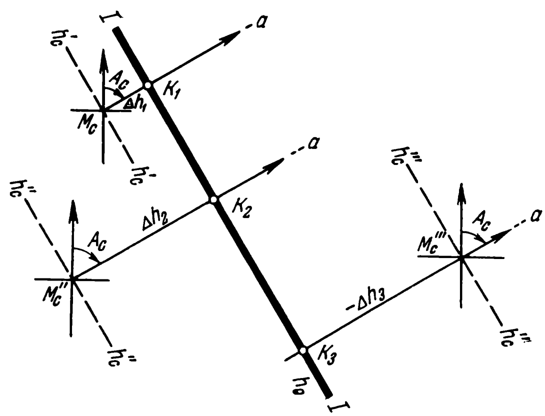

Fig. 170

the same value of h_0, δ and t_{gr} of the celestial body. The computed circles indicated by dashed lines will be different for all these points; therefore, the values $\Delta h = h_0 - h_c^i$ will also be different. In principle, A_c should also be different (for a circle of equal altitudes), but practically speaking these differences are not noticeable within the limits of 0°.5 from M_c.

Having plotted from the points M_c', M_c'' and M_c''', we get one and the same position line *I-I*. This line will include the observed position of the ship at the instant of observations, regardless of where the D.R. position was taken previously. Thus, a position line is quite *independent of the accepted values of D.R. coordinates used for plotting* (to within 0°.5).

This property of altitude lines of position is used in a method of plotting from the so-called chosen or assumed position (C.P. or A.P.) (for integral φ_c and t_{loc}) which is often recommended in the manuals of certain countries to simplify computation of h_c. For this same reason, when computing h_c and A_c, the D.R. coordinates may be rounded off to whole minutes ($'$), particularly when plotting on paper.

III. THE UNIVERSALITY OF THE ALTITUDE LINE OF POSITION

Methods for separate determination of the coordinates φ_0 or λ_0 of ship position are, as we have already mentioned, particular solutions of the equation

$$\sin h_0 = \sin\varphi\cdot\sin\delta + \cos\varphi\cdot\cos\delta\cdot\cos(t_{gr}\pm\lambda) \qquad (*)$$

for a single measured altitude of a celestial body. The longitude is considered known when determining latitude, and conversely.

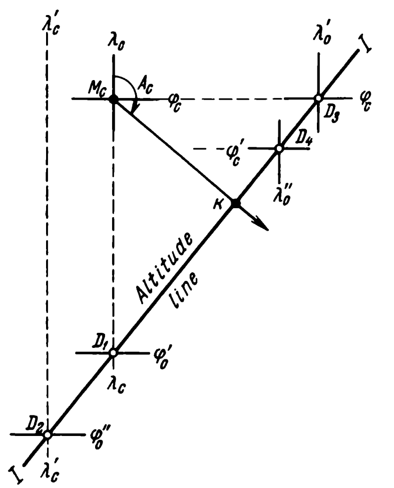

Fig. 171

Since a position line is an approximate but sufficiently accurate graphical expression of equation (*), a single position line is enough to pass graphically to φ_0 or λ_0.

Let the point M_c be a D.R. position (Fig. 171) with coordinates φ_c, λ_c. Plot the position line *I-I* for the values h_0, δ and t of some celestial body. If we use these data and the value of λ_c to compute the observed latitude, its parallel φ_0' will have a common point D_1 with the position line and this point will lie on the meridian of the longitude λ_c which is taken for computing the latitude.

If we change the D.R. longitude and take λ_c', the position of the line I-I will not change, as we know, but φ_0'' computed with the same data will be different, in accord with the point of intersection D_2 of the position line with the meridian λ_c'.

Consequently, the observed latitude of a position corresponds to the point of a position line at which it is intersected by the meridian of the D.R. longitude; thus, φ_0 is a quantity that depends on λ_c.

From the figure it will be seen that the position line coincides with the parallel φ_0 only when $A = 0°$ (180°), that is, when the celestial body is on the observer's meridian. Here, the very same φ_0 will correspond to any accepted longitude, and therefore this position of the celestial body will be the most favourable for determining latitude.

Similarly, if we use the same data, h_0, δ, t and φ_c to compute and plot a position line and then compute the observed longitude, the common point D_3 of the meridian λ_0' and the position line will

lie on the parallel of the accepted latitude φ_c. If this latitude is changed by φ_c', computations yield λ_0'', which corresponds to the point D_4 of the position line *I-I*. The position line will obviously coincide with the meridian λ_0 for $A = 90°$ (270°); these are the most favourable conditions for determining longitude.

It is thus clear that problems in determining the parallel of *latitude* and the meridian of *longitude* of a place are only *special cases* of the general problem of plotting an altitude line of position. The position line is not dependent either on the azimuth of the celestial body or on the coordinates of the position; it is more convenient to use and is more accurate than the meridian λ_0 or the parallel φ_0 and can *replace them in any of its positions.* Thus, an altitude line of position is a truly universal line of position, whereas a parallel or meridian can be taken as lines of position only when $A = 0°$ (180°) and $A = 90°$ (270°).

The foregoing thus demonstrates the *impossibility* of methods for *determining position from a single altitude* worked as a position line and as an observed latitude or longitude. The authors of such proposals believed that the observed position of a ship would be at the points of intersection of the position line and the parallel φ_0 or the position line and the meridian λ_0. These points usually fail to coincide somewhat with the points D_1, D_3 solely because of errors in computation and plotting. Fundamentally, however, all these points belong to a single position line, while the quantities φ and λ, as the moving coordinates of this line, are interdependent. The point D thus obtained yields one coordinate, φ or λ, the second coordinate will be the computed one or will be taken arbitrarily, as the case should be when solving the single equation $u = f(\varphi, \lambda)$.

SEC. 107. SPECIAL TABLES FOR COMPUTING ALTITUDE (h_c) AND AZIMUTH (A_c)

To simplify computations of h_c and A_c, over 50 special types of tables have been proposed since the discovery of the method of Marcq Saint-Hilaire. Mathematically, all these tables are tables of conversion of spherical coordinates from one system to another, and so nearly all of them are applicable for solving other navigational problems, such as computing the hour angle, sailing along the arc of a great circle, computing the elements for plotting radio bearings, etc.

To simplify orientation in this large number of tables, Professor N. Matusevich has suggested dividing them into three classes according to the principle of construction and operation.

1. "Trigonometric" tables, which are small tables of logarithms or the natural values of several trigonometric functions adapted

to the solution of the astronomical triangle in accord with specific systems of formulas.

2. "Artificial" tables, which are tables of artificially converted trigonometric functions for solving triangles by means of artificially transformed formulas.

3. "Numerical" tables that yield numerical values of h and A and corrections to them at specified intervals of φ, δ, t.

We shall now consider the most common tables using this classification.

I. "TRIGONOMETRIC" TABLES

The formulas used in the compilation of this class of tables are obtained by dividing the astronomical triangle into two right triangles or into a number of related triangles. This type includes "Tables for Finding Altitude and Azimuth", by V. Fus, GGU Publishing House, 1901, the tables of Radle-de-Aquino, Dreisonstok (H.O. No. 208), Ageton (H. O. No. 211), A. Yushchenko (TBA-57), and many others.

The principal merits of this type of table are that it is compact and in most cases sufficiently accurate; using it involves just about as much work as computation by basic formulas. Let us consider in detail the present tables of Professor A. Yushchenko.

Yushchenko's "Tables for Computing Altitude and Azimuth" (TBA-52, TBA-57 or Table 27, MT-43)

The formulas used to compile the TBA tables were given above (Sec. 104, III) under the name of "tangent formulas".

Yushchenko reduced the tangent formulas to the functions tan α and sec α and to the following form:

$$\left.\begin{aligned} \tan x &= \tan\delta\cdot\sec t \\ \tan A_c &= \frac{\tan t\cdot\sec\,[90^\circ+(x\approx\varphi_c)]}{\sec x} \\ \tan h_c &= \frac{\tan\,[90^\circ+(x\approx\varphi_c)}{\sec A_c} \end{aligned}\right\} \quad (19.10)$$

The meaning of the sign $\sim$ is explained in the derivation of the tangent formulas.

To simplify computations by these formulas, Yushchenko compiled special tables of modified logarithms of tangents and secants.

Here, the values of four-place logarithms are multiplied by 10^4 to obtain whole numbers, and then again by 2 in order to improve

accuracy. For angles less than 45°, the tangents will be less than unity, and their logarithms less than 0. So as to avoid negative characteristics, a constant quantity 70725 is added to the logarithm of tan α; this number is $10^4 \times 2 \log \frac{1}{\tan 1'}$, or the largest negative value of log tan α. We thus have the artificial quantities

$$\left.\begin{aligned} S(\alpha) &= 2 \times 10^4 \log \sec \alpha \\ \text{and} \qquad T(\alpha) &= 2 \times 10^4 \log \tan \alpha + 70725 \end{aligned}\right\} \qquad (19.11)$$

With these functions the formulas (19.10) will take the form

$$\left.\begin{aligned} T(x) &= T(\delta) + S(t) \\ T(A) &= T(t) - S(x) + S[90° + (x \approx \varphi_c)] \\ T(h) &= T[90° + (x \approx \varphi_c)] - S(A_c) \end{aligned}\right\} \qquad (19.12)$$

These formulas are used to compile schemes (forms) which yield solutions by mechanically filling in the values of the functions from the tables. The tables give the values of $T(\alpha)$ for 1′, (TBA-52) and 0′.1 (TBA-57) intervals of the arguments. For angles from 75° to 105°, the value of $S(\alpha)$ is also given in TBA-57 at 0′.1 intervals, so that the TBA-57 tables are in the main noninterpolation tables. Since the values of t, x and A may lie between 0° and 180°, the arguments in the TBA tables are given from 0° to 180°, but due to the fact that these functions vary symmetrically, angles from 0° to 90° are given at the top and from 90° to 180° at the bottom. In the latter case, minutes are taken from the right side of the page.

The user of TBA tables should be guided by the rules which follow from the construction of a spherical perpendicular and the triangle CDZ (see Fig. 167 and Sec. 104) in various cases:

(1) the arc x is always of the same name as the declination (N or S);

(2) if $t > 90°$, then $x > 90°$;

(3) for x of the *same name* as latitude, we have the difference $\varphi - x$ or $x - \varphi$ (if $x > \varphi$). For x of *contrary name* to φ, we take the sum $\varphi + x$, since x will be negative. This rule is shown in formulas (19.11) and (19.12) by the symbol $\varphi \sim x$;

(4) the first letter in the azimuth designation is of contrary name to φ in all cases except when x and φ are of the same name and $x > \varphi$; then the first letter is the same as φ, and the second always coincides with the hour angle. Instead, we can apply the general rule shown above.

For azimuths close to 90° the tabulated differences of the quantities T and S become very large and it is rather difficult to interpolate them (in TBA-52). However, taking account of the fact that

here S and T vary in about the same way, we can add to S (A) the difference that was formed for (A_c) with the nearest integral value of A. In the TBA-57 tables, S and T are given at 0′.1 intervals, thus obviating any interpolation. A disadvantage of these tables is their inaccuracy for t or A close to 90° and a departure from the ordinary scheme for t or A equal to 90°.

The following example illustrates the computation form.

Example 3. Find h_c and A_c, given $\varphi_c = 60°2'.5$N; $\delta = 28°44'.9$N; $t = 108°44'.2$E.

$\delta = 28°44'.9$N	T +	65510
$t = 108°44'.2$E	S	9864
$x = 120°21'.0$N	T	75374

T −	80116
S	5929
P +	74187
S	1222
T	75409

T −	65846
S	5955
T	59891

$\varphi = 60°\ 2'.5$N

$90° + (x - \varphi) = 150°18'.5$

$A_c = 59°45'.0$NE

$h_c = 16°\ 1'.7$

II. "ARTIFICIAL" TABLES FOR COMPUTING h_c AND A_c

Tables of this type are varied and common in many countries outside the Soviet Union. Their advantage is simplicity, uninvolved solutions, and small size (not more than 200 pages). But as a rule, accuracy is low, while the fact that the solutions are obtained mechanically makes for blunders that are hard to locate since there is no intermediate check.

The compilation of these tables involves either basic formulas for solving the astronomical triangle, or the triangle is divided into two right triangles, or, finally, adjacent triangles are built on the horizon, equator, vertical circle, and so forth. The formulas obtained are artificially transformed to obtain the most elementary computational operations, which are used to compile tables (ordinarily of a mixed type) for logarithms, the natural values of functions, and the numerical values of angles.

To illustrate, let us take V. Akhmatov's tables of "Altitude and Azimuth in Three-Minute Intervals", in which the formula $\sin^2 \frac{z}{2}$ is transformed and reduced to the quantities А (Б), В, Г, Д which represent the logarithms or natural values of the functions sec α, $\operatorname{cosec}^2 \frac{\alpha}{2}$, $\sin^2 \frac{\alpha}{2}$, for which the tables I, II, III, and IV are compiled.

These tables are small in size and simple to handle, but the accuracy of the altitude obtained is low, and they are no longer in use.

The British merchant marine makes use of Hughes' tables*; the formulas for which these tables are compiled are obtained by dividing the triangle into two right triangles by a perpendicular dropped from the zenith onto the meridian of the celestial body. Hughes' tables yield a mixed numerical-logarithmic solution for a chosen position (integral φ and t_{loc}), which, on the whole, is inconvenient. These tables have no particular advantages over others.

Of recent tables, mention should be made of Kotlaric's tables (Yugoslavia, 1957), which were compiled from formulas for triangles adjacent to the polar triangles in vertical circle and meridian and yield a mixed solution that is simplified by the introduction of differential corrections.

III. "NUMERICAL" TABLES

If altitudes and azimuths are computed for whole values of φ_T, δ_T, t_T at small intervals (0°.5 to 1°) and tabulated, we get what are called numerical tables. When computing altitude for intermediate values of φ_c, δ, t_{loc}, we must introduce corrections into the chosen h_T to account for increments in altitude due to change of arguments. These corrections are obtainable if we expand the function h in a Taylor's series in three variables. Confining ourselves to the second terms of the expansion, we obtain

$$\begin{aligned} h_c &= f(\varphi_T + \Delta\varphi;\ \delta_T + \Delta\delta;\ t_T + \Delta t) = f(\varphi_T,\ \delta_T,\ t_T) + \\ &+ \left[\left(\frac{\partial f}{\partial \varphi}\right)_T \Delta\varphi + \left(\frac{\partial f}{\partial \delta}\right)_T \Delta\delta + \left(\frac{\partial f}{\partial t}\right)_T \Delta t\right] \\ &+ \frac{1}{2}\left[\left(\frac{\partial^2 f}{\partial \varphi^2}\right)_T \Delta\varphi^2 + \left(\frac{\partial^2 f}{\partial \delta^2}\right)_T \Delta\delta^2 + \left(\frac{\partial^2 f}{\partial t^2}\right)_T \Delta t^2\right. \\ &+ 2\left(\frac{\partial^2 f}{\partial \varphi\,\partial \delta}\right)_T \Delta\varphi\Delta\delta + 2\left(\frac{\partial^2 f}{\partial \varphi\,\partial t}\right)_T \Delta\varphi\,\Delta t \\ &+ 2\left(\frac{\partial^2 f}{\partial \varphi\,\partial t}\right)_T \Delta\varphi\,\Delta t + 2\left(\frac{\partial^2 f}{\partial \delta\,\partial t}\right)_T \Delta\delta\Delta t\bigg] + \ldots \end{aligned} \tag{19.13}$$

In this series, second powers may be neglected only when the intervals of the arguments φ, δ, t are sufficiently small. However, the size of the tables becomes prohibitive. If, on the other hand, we increase the intervals of the arguments, we will have to introduce corrections for all six squared terms; this will complicate the work to such an extent that the computations will be more involved than through the use of formulas.

* Hughes' Tables for Sea and Air Navigation (126 p).

For this reason, second-power terms are neglected in all existing numerical tables with the exception of the latest Soviet BAC-58 tables. The intervals of the arguments φ, δ, t are chosen so that for altitudes up to 60°-70° the errors due to such an assumption are sufficiently small. Despite every possible increase in the intervals, the size of modern numerical tables is very great, on the order of 2,100 to 2,700 pages.

To interpolate altitude for given values of arguments, it is usual to apply three linear terms of the series (19.13). This is the underlying principle of the interpolational tables in Soviet TBA tables for aviation, the H.O. No. 214 tables of the United States, H.D. No. 486 tables of Great Britain, and others. The second terms of the expansion (19.13) are taken into account only in BAC-58.

Confining ourselves in (19.13) to the first terms of the series and taking the increments $\Delta\varphi$, $\Delta\delta$, Δt as corrections, we get

$$h_c = h_T + \frac{\partial h}{\partial \delta}\,\Delta\delta + \frac{\partial h}{\partial t}\,\Delta t + \frac{\partial h}{\partial \varphi}\,\Delta\varphi \tag{19.14}$$

The values of the last two partial derivatives have been obtained (Sec. 98, Ch. 18) in the form

$$\frac{\partial h}{\partial \varphi} = \cos A \quad \text{and} \quad \frac{\partial h}{\partial t} = -\cos\varphi\cdot\sin A \tag{19.15}$$

The derivative $\frac{\partial h}{\partial \delta}$ is obtained by differentiation of the formula $\sin h$ with respect to h and δ, that is,

$$\cos h\,\partial h = (\cos\delta\cdot\sin\varphi - \sin\delta\cdot\cos\varphi\cdot\cos t)\,\partial\delta$$

We obtain the quantity in parentheses from the astronomical triangle by the formula of five parts:

$$\cos q\cdot\cos h = \cos\delta\cdot\sin\varphi - \sin\delta\cdot\cos\varphi\cdot\cos t$$

whence

$$\frac{\partial h}{\partial \delta} = \frac{\cos q\cdot\cos h}{\cos h} = \cos q \tag{19.16}$$

Substituting the values of the derivatives, we get

$$h_c = h_T + \cos q\,(\delta - \delta_T) - \cos\varphi\cdot\sin A\,(t_{loc} - t_T) + \cos A\,(\varphi_c - \varphi_T) \tag{19.17}$$

where δ_T, t_T and φ_T are the accepted tabulated values of the arguments.

This is the formula that is used to compute corrections to the tabulated value of altitude in most tables. In compiling numerical tables, the leading argument is the latitude, which means that all tables are constructed according to a band of latitude. Let us examine the construction of the H.O. No. 214 and BAC-58 tables.

1. H.O. No. 214. "Tables of Computed Altitude and Azimuth"

The set of these tables comprises 9 volumes, each of which embraces a 10° band of latitude (0°-9°, 10°-19°, etc.). Each volume is divided into sections of 1° of latitude that are marked by tabs on the pages of the book. In each section, values of h_T and A_T are given at 30′ intervals of declination and 1° intervals of hour angles, and altitude corrections every 1′ of declination and hour angle; that is, the coefficients $\overline{\Delta d} = 100 \cos q$ and $\overline{\Delta t} = 100 \cos \varphi \cdot \sin A$, where the multipliers 100 are introduced so as to obtain whole numbers. Declinations are given continuously up to 29°; after 29° they are given in separate bands for the navigational stars. The usual practice is to have the declinations of same name as latitude on the left hand side of the page, and those of contrary name on the right. For this reason, the tables are equally suitable for northern and southern latitudes. When working with the tables, the quantities h_T and A_T are taken from the *nearest* values of the arguments φ_T, δ_T, t_T. Interpolation via declination and hour angle is performed with $\overline{\Delta d}$ and $\overline{\Delta t}$ and the differences $\delta - \delta_T$ and $t_{loc} - t_T$ on the basis of formula (19.17) with the aid of "multiplication tables" on the last page of each volume. The signs of corrections are determined by inspection of adjacent tabulated values of altitude (to see whether the altitude is increasing or decreasing).

Interpolation of latitude is performed on the basis of the formula

$$\Delta h_\varphi = (\varphi_c - \varphi_T) \cdot \cos A$$

by means of a special table on the second to the last page of each volume, which is entered with A and $\varphi_c - \varphi_T$ obtained in semicircular reckoning. The sign of the correction Δh_φ is found from formula (*), in which $\cos A$ is negative if $A > 90°$, and $\cos A$ is positive if $A < 90°$; in the first term, φ_T is always subtracted from φ_c. This rule of signs is given at the bottom of the interpolation table.

The corrections obtained are applied to the chosen value of h_T, that is,

$$h_c = h_T + \Delta h_\delta + \Delta h_t + \Delta h_\varphi \tag{19.18}$$

The computation form is given in Example 4. The azimuth is interpolated between adjacent values by "eye".

Example 4. Find h_c, A_c and $h_0 - h_c$ from $\varphi_c = 44°35'$N; $t^{\odot}_{loc} = 34°44'.5$W; $\delta_{\odot} = 7°13'.8$S; $h_0 = 29°34'.3$.

Open section $\varphi = 45°$ and in column $\delta = 7°$ (contrary name) and row $t = 35°$, take out h_T and A_T.

Table	Differences	Δh	$h_T + \Delta h$	Δd Δt ΔA
$\varphi_T = 45°N$	$\varphi_c - \varphi_T = 25'.0$	Δh_φ	$29°15'.5$	89 47 139.6
$\delta_T = 7$ S	$\delta - \delta_T = 13\ .8$	Δh_δ	$+19\ .1$	
			$-12\ .3$	
$t_T = 35W$	$t_{loc} - t_T = 15\ .5$	Δh_t	$+7\ .3$	
		h_c	$29°29'.6$	A_c
		h_0	$29°34'.3$	$=$N 139.6W
		$h_0 - h_c$	$+4'.7$	$=$ 220°.4

Note. The exact value of $h_c = 29°29'.4$, $A_c = 220°.5$.

The H.O. No. 214 tables are convenient to use and permit obtaining h_c and A_c very quickly; for altitudes up to 50°-60° they ensure an accuracy up to $\pm 0'.3$-$0'.4$ (see Sec. 110). But for altitudes in excess of 60° the errors due to neglect of the second terms of the series (19.13) may prove to be considerable, making it impossible to use these tables.

In addition to the basic tables, the end of each section has tables for star identification, more precisely, for the transformation of coordinates h_* and A_* into t and δ, and at the beginning of each volume there are tables for altitude correction.

A drawback of the H.O. No. 214 tables is the lack of altitudes less than 5°, inaccurate results for high altitudes, inexact extraction of azimuth, and large size of the tables as such (a set for latitude from 0° to 80° embraces 2,100 pages). These tables were published between 1936 and 1946 by the Hydrographic Office, U.S.A.

The British numerical tables H.D. No. 486 are analogous to the H.O. No. 214 tables, but are published in 6 volumes (360 pages each), one volume embracing 15° of latitude.

2. Tables of Altitude and Azimuth of Celestial Bodies (BAC-58)

In 1957, the Soviet Navy developed an original variant of numerical tables that yield both altitude and azimuth of celestial bodies with a rather high accuracy. Publication of these tables began in 1959 under the title "Altitudes and Azimuths of Celestial Bodies" (BAC-58).

Basis for designing the BAC-58 tables. Unlike nearly all existing numerical tables, the BAC-58 tables take almost complete account of the second terms in the expansion (19.13) of the function sin h

in a Taylor's series. As a result, the intervals of all arguments may be taken equal to 1°, and the accuracy of the tables will be sufficient for all altitudes for which interpolation tables are given, that is, up to 73°. When computing with chosen position, the tables are applicable up to $h = 88°$. The underlying principle for partial account of second terms of the series (19.13) during interpolation is *joint account of the first and second terms in the form of general corrections.*

These corrections are obtained by a joint inspection of the series expansions, the altitude h_c and the azimuth A_c; certain second terms Δh are accounted for by introducing the correction ΔA into the argument A_T.

Indeed, write down the values of six second terms of the series (19.13) alongside the formulas (17.12) and (17.13) for ΔA (Table 10).

Table 10

	Second terms Δh_{II}	First terms ΔA
$\Delta h_{\varphi\varphi}$	$= -\sin^2 A \cdot \tan h \cdot \frac{\Delta\varphi^2}{2}$	$\Delta A_\varphi = \sin A \cdot \tan h \cdot \Delta\varphi$
$\Delta h_{\delta\delta}$	$= -\sin^2 q \cdot \tan h \cdot \frac{\Delta\delta^2}{2}$	
Δh_{tt}	$= \cos q \cdot \cos\delta \cdot \sec h \cdot \cos A \cdot \cos\varphi \cdot \frac{\Delta t^2}{2}$	$\Delta A_t = -\cos q \cdot \cos\delta \cdot \sec h \cdot \Delta t$
$\Delta h_{\varphi t}$	$= \cos q \cdot \cos\delta \cdot \sec h \cdot \sin A \cdot \Delta\varphi \cdot \Delta t$	$\Delta A_\delta = -\sin q \cdot \sec h \cdot \Delta\delta$
$\Delta h_{\varphi\delta}$	$= \sin q \cdot \sec h \cdot \sin A \cdot \Delta\varphi \cdot \Delta\delta$	
$\Delta h_{\delta t}$	$= \sin q \cdot \sec h \cdot \cos A \cdot \cos\varphi \cdot \Delta\delta \cdot \Delta t$	

The similarity of certain terms is seen from a comparison of the values of these series. We can thus write the first column for Δh_{II} in the form

$$\Delta h_{II} = -\Big(\sin^2 A \cdot \tan h \frac{\Delta\varphi^2}{2} + \sin^2 q \cdot \tan h \frac{\Delta\delta^2}{2}$$

$$+ \cos\varphi \cdot \cos A \frac{\Delta A_t}{2} \Delta t + \sin A \cdot \Delta A_t \cdot \Delta\varphi$$

$$+ \cos\varphi \cdot \cos A \cdot \Delta A_\delta \cdot \Delta t + \sin A \cdot \Delta A_\delta \cdot \Delta\varphi\Big) \tag{19.19}$$

To take joint account of the first and second terms of the series (19.13), affix to (19.19) the values (19.17) of the first terms of the

series, arranging them near the terms of Δh_{II} that are similar in form

$$\Delta h = \Delta h_I + \Delta h_{II} = \left(\Delta\varphi \cdot \cos A - \sin^2 A \cdot \tan h \frac{\Delta\varphi^2}{2}\right)$$

$$+ \left(\Delta\delta \cdot \cos q - \sin^2 q \cdot \tan h \frac{\Delta\delta^2}{2}\right) - \Delta t \cdot \cos\varphi \cdot \sin A$$

$$\underbrace{- \Delta t \cdot \cos\varphi \cdot \cos A \frac{\Delta A_t}{2} - \Delta\varphi \cdot \sin A \cdot \Delta A_t - \cos\varphi \cdot \cos A \cdot \Delta A_\delta \cdot \Delta t}_{} - [\sin A \cdot \Delta A_\delta \cdot \Delta\varphi] \quad (19.20)$$

The formulas in parentheses were used to compile interpolation Table 1, the arguments with which to enter the table being: $\Delta\varphi$ (or $\Delta\delta$), A (or q) and h of the celestial body. The corrections $\Delta h_\varphi'$ and $\Delta h_\delta'$ are taken from Table 1. Table 1 also gives the values of ΔA_φ and ΔA_δ which have similar arguments: A, h, $\Delta\varphi$ and q, h, $\Delta\delta$, as may be seen from Table 10 given above.

The terms of formula (19.20) underlined with a wavy line are, after transformations, assumptions and simplifications, reduced to the form

$$\Delta h_t' = -\Delta t \cdot \cos\left(\varphi_T + \frac{\Delta\varphi}{2}\right) \cdot \sin\left(A_c - \frac{\Delta A_t}{2}\right) \quad (19.21)$$

From expression (19.21) for $\Delta h_t'$, which takes joint account of the expansion terms Δh_t, Δh_{tt}, $\Delta h_{\varphi t}$ and $\Delta h_{\delta t}$ and certain III terms of the series (19.13), we have interpolation Table 2, whose arguments for entry are φ_c, $A_c - \frac{\Delta A_t}{2}$ and Δt.

The last term of formula (19.20), in square brackets, is given in the small interpolation Table 3 with arguments A, $\Delta\varphi$ and ΔA_δ and yields Δh_{ad}.

As we see, in Tables 2 and 3, the second terms are taken into account via azimuth corrections.

Thus, four entries into three tables take into account nearly all 9 terms of the series (19.13); however, the interpolation tables here are much more complicated than in other numerical tables, and this slows up working of sights. According to the findings of V. Gubanov, the accuracy of h_c will be about $\pm 0'.2$.

Description of tables. The BAC-58 tables consist of basic tables designed on the principle of bands of latitude and three tables of corrections to the tabulated altitudes and azimuths.

The BAC-58 tables have been published in four volumes, each of which embraces 20° of latitude (for all latitudes from 0° to 80°). Each volume is divided into sections of 1-degree bands of latitude; inside the section we have columns (Fig. 172) for values of declination from 0° to 29° at 1° intervals and above 29° for 18 selected values

Declination of Same Name

t \ δ	8°			11°			12°			15°			
	h	A	q	h	A	q	h	A	q	h	A	q	
90°	$5^{\circ}\ 07_9$	83°_9	50°	$7^{\circ}\ 02_7$	81°_5	51°	$7^{\circ}\ 40_8$	80°_8	51°	$9^{\circ}\ 34_6$	78°_4	51°	
1	$4\ 22_3$	83_2	50	$6\ 17_3$	80_9	50	$6\ 55_5$	80_1	50	$8\ 49_6$	77_8	51	
2	$3\ 36_7$	82_6	50	$5\ 31_9$	80_3	50	$6\ 10_2$	79_5	50	$8\ 04_8$	77_2	51	
3	$2\ 51_1$	81_9	50	$4\ 46_7$	79_6	50	$5\ 25_1$	78_9	50	$7\ 20_0$	76_5	50	
4	$2\ 05_6$	81_3	50	$4\ 01_5$	79_0	50	$4\ 40_1$	78_2	50	$6\ 35_4$	75_9	50	
95	$1\ 20_3$	80_7	50	$3\ 16_4$	78_4	50	$3\ 55_1$	77_6	50	$5\ 50_8$	75_3	50	
6	$0\ 34_9$	80_0	60	$2\ 31_5$	77_7	50	$3\ 10_3$	77_0	50	$5\ 06_4$	74_7	50	
7	$0\ 10_3$	100_6	131	$1\ 46_6$	77_1	50	$2\ 25_3$	76_3	50	$4\ 22_2$	74_1	50	83
	$42\ 00_0$	180_0	180	$39\ 00_0$	180_0	180	$38\ 00_0$	180_0	180	$35\ 00_0$	180_0	180	00
	h	A	q	h	A	q	h	A	q	h	A	q	t
	8°			11°			12°			15°			δ

Declination of Contrary Name

Fig. 172

Example 5. $\varphi = 57°38'.6\text{N}$; $\delta = 23°24'.2\text{S}$; $t = 25°24'.5\text{W}$. Find h_c and A_c.

Arguments	Given	Tabulated	Difference $G-T$	Correction	h_T	A_T	q
					6°21′.4	157°.0	167°
φ_c	57°38′.6N	58°	−21′.4	for φ	+19′.7	0.0	
δ	23 24 .2S	23	+24 .2	for δ	−23 .6	+0.1	
t_{loc}	25 24 .5W	25	+24 .5	for t	−5 .1	−0.4	
				ad	0 .0	156 .7	$-\frac{\Delta A_t}{2} =$ =156° .9
					$h_c = 6°12'.4$	156°.7NW	

which in part were suitable for nautical astronomy too. For instance the Soviet TBA3 tables, the American H.O. No. 249, the British AP No. 3270, Japanese No. 603, and others.

They are largely of similar construction: the values of h_c (up to 1′) and A_c (up to 1°) are given in the first part of the tables via the arguments φ and t_{loc}^{Υ} at 1° intervals and the name of seven stars. Thus, in place of declination we have the names of seven bright stars visible at a given time: in addition the tables are designed for plotting from a chosen position (integral φ and t_{loc}), thus dispensing with interpolation; working a sight occupies only about one minute.

The second parts of the tables resemble ordinary numerical tables of the H.O. No. 214 type.

SEC. 108. SPECIAL INSTRUMENTS FOR COMPUTING ALTITUDE AND AZIMUTH (FUNDAMENTALS)

A number of instruments and machines have been constructed to simplify the computation of h and A.

The problem of transformation of coordinates φ, δ, t to h and A may be solved, as has already been mentioned (Sec. 5, Ch. 2):

(a) by means of a model of the celestial sphere;

(b) graphically by means of grids;

(c) analytically by solving the astronomical triangle with the formulas of spherical trigonometry.

Instruments built on the first principle were in the form of systems of precisely divided moving circles and indices, appropriate setting of which solved the problems. Such, for example, were the Willis "machine", the Hegner computer and a number of others. This

solution turned out to be unwieldy and inaccurate due to mechanical errors of the instruments.

For a graphic solution by means of grids there are two instruments: the Loth computer and a similar (in principle but improved) German instrument called ARG-3, which we shall consider below.

Mechanical, electric, and (recently) electronic computing machines have been designed on a wide variety of principles for the analytic solution of the triangle. None of them have come into common use due to complexity, inconvenience and inaccuracy and also high prices. A cylinder type of slide rule has also been offered and we shall consider it later on. Of all these numerous instruments and machines, only two of the simplest (the ARG-3 and the cylindrical slide rule) have found any application in the fleet, especially during and following World War II.

1. THE ARG-3 (ASTRONOMISCHE RECHEN GERÄT) ASTRONOMICAL RESOLVER

This instrument is designed on the principle of Kavraisky's planisphere* (see Sec. 6, Ch. 1) and consists of a coordinate grid

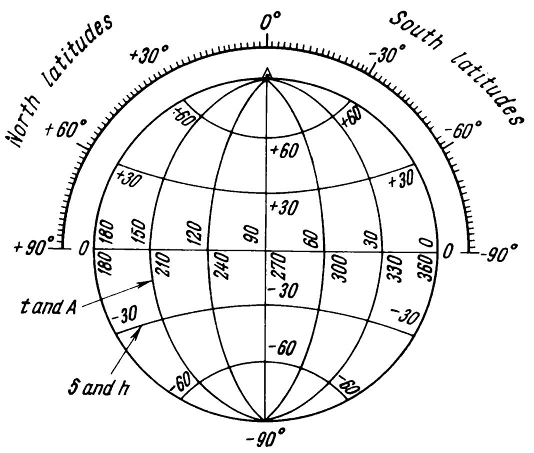

Fig. 173

that rotates about the centre and microscopes for taking readings. The grid is constructed as follows (Fig. 173): the image of the celestial sphere with meridians and parallels is projected on a glass in transverse stereographic projection. The circles of parallels

* In the ARG-3 the grid is constructed in a transverse stereographic projection.

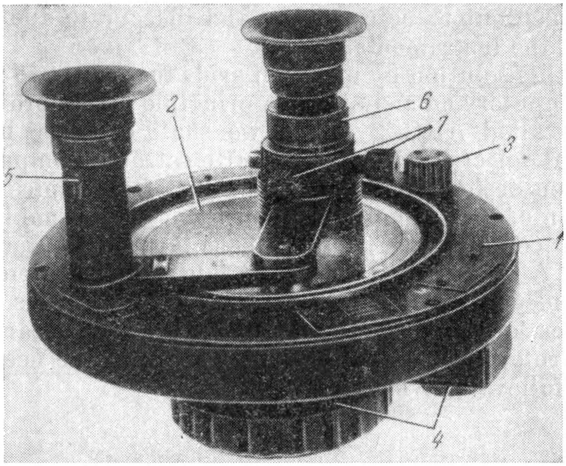

Fig. 174

are plotted every 10′ (in Fig. 173, every 30°) over the entire grid, the meridians are plotted every 10′ in the operative part of the grid (from 60° N to 60°S) and at larger intervals in areas close to the poles.

The resultant grid is labelled for declinations from 0° to ±90 every 2°, a plus sign affixed to north declinations and a minus sign to south declinations; for hour angles from 0° to 180° and back from 180° to 360° every 2 degrees. As will be seen from the figure, the hour angles on the grid are always measured in circular reckoning from one (the lower) branch of the observer's meridian. For this reason, to obtain a "reference angle" t_{ref}, add 180° to the *west* hour angle (in circular reckoning), that is,

$$t_{set} = t_{loc}^{W} + 180^\circ$$

Round the circumference of the grid is a scale of latitudes (every 10′) with the plus sign affixed to north latitudes and the minus sign to south latitudes.

The glass and grid are held in a metallic frame with teeth around the circumference and a circular groove underneath to allow for turning the grid precisely round its centre. Handle *3* (Fig. 174) of the turning mechanism is used to rotate the grid. On top, the grid is covered with glass *2*, rigidly connected with the body *1* of the instrument; underneath is a lighting system *4*. Due to the small diameter of the grid (100 mm), the scale is very small and can be read only by microscope (26 ×). The fixed microscope *5* is designed for setting the grid in latitude; the moving microscope *6*, for setting and reading coordinates on the grid itself.

Like the Kavraisky planisphere, coordinates are transformed by means of the grid alone, which depicts the equatorial system in one position and the horizon system in the other.

By putting the cross hairs (index) of the moving microscope at the reading t_{set} and δ and by turning the grid through an angle of $90° - \varphi$, which is done by bringing the reading φ (with "+" for φ_N and "—" for φ_S) to the cross hairs of the fixed microscope, we get the readings of h_c and A_c on the index of the moving microscope. Azimuth is always in circular reckoning (0° to 360°).

The order of work with the ARG-3 is as follows:

1. Using the MAE, obtain t_{loc} and δ of the celestial body and compute $t_{set} = t_{loc}^{W} + 180°$.

2. Prepare focus and illuminate the instrument.

3. Turn handle *3* of the turning mechanism to align the index of the fixed microscope *5* with the reading of the latitude scale "+90°" that corresponds to the initial position of the grid.

4. Set intersection of hairs (index) of the moving microscope *6* at the values of δ and t_{set}; to do this:

(a) by hand turn microscope to desired quadrant to approximate values of t_{set} and δ;

(b) using micrometer screws *7*, set index at precise values of t_{set} and δ.

5 Rotate handle of turning mechanism, thus moving grid until fixed microscope *5* gives a reading equal to φ_c to within 1′.

6. Read the values of h_{set} and A_c in the moving microscope *6* (at the intersection of the hairs).

7. To check, turn grid to initial position and take the readings of t_{set} and δ.

The ARG-3 instrument computes h_c with an accuracy of the order of ±0′.5-1′.0 for declinations up to ±60°, that is, in the operative part of the grid. The instrument is very easy to operate and it takes only about 2 minutes to make a computation, which is less than with any kind of tables. The ARG-3 can also be used in place of azimuth tables, and the accuracy will be greater and the computation time less than with tables. The instrument has the following disadvantages: it is difficult to set the hairs of the microscope at the reading of δ and t_{set}, particularly at sea; considerable eye fatigue due to bright background; and, finally, accuracy of h_c is not always sufficient, especially if the instrument has an eccentricity.

Example 6. Given $\varphi_c = 44°35'N$; $t_{loc}^{\odot} = 34°44'.5W$; $\delta = 7°13'8S$. Find h_c and A_c.

(1) $t_{set} = t_{loc}^{W} + 180 = 214°44'.5$.

(2) From the instrument we get $h_c = 29°30'$, $A_c = 220°30'$.

II. CYLINDRICAL RULE*

This instrument is based on the principle of the common slide rule, but is made in the form of cylinders with helical scales of log sec α and log tan α.

These scales permit solving the astronomical triangle using the tangent formulas (19.5), (19.7), and (19.8) converted to the form

$$\left.\begin{aligned} \tan x &= \tan \delta \cdot \sec t \\ \tan (180^\circ - A) &= \frac{\tan t \cdot \operatorname{cosec} (\varphi - x)}{\sec x} \end{aligned}\right\} \tag{19.23}$$

or, putting $\varphi - x = 90^\circ - y$,

$$\tan (180^\circ - A) = \frac{\tan t \cdot \sec y}{\sec x} \tag{19.24}$$

and

$$\tan h = \frac{\tan y}{\sec (180^\circ - A)} \tag{19.25}$$

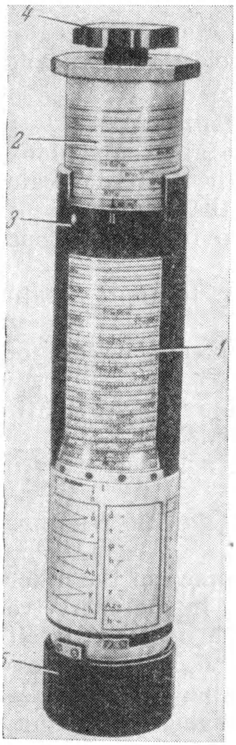

Fig. 175

The outer cylinder *1* (Fig. 175) has a scale of log sec α, the readings of which are set by the lower index *I*. The inner cylinder *2* has a scale of log tan α, whose readings are set by the upper index *II*.

Both indices are given on a third, auxiliary, outer cylinder *3* and represent a continuation of a single line parallel to the axis of the cylinders. The two inner cylinders can be fixed in a definite position by means of a locking device situated in the inner cylinder. At the top of the rule is the head of screw *4*. The instrument is held by hande *5*. The outer cylinder also has the setting scheme of the cylinders, the rules of signs, the names of the quantities, and a table for recording computations. The rule itself is small in size (length 28 cm, diameter about 6 cm) but the scales plotted along a screw line with a pitch of 4.5 mm are much longer: the log sec α scale is 4,100 mm in length, with a range from 0° to 89°40′, the log tan α scale is 7,660 mm long and has a range of 0°20′ to 89°40′. It is impossible to take the scale to the end because log tan 90° $= +\infty$ and the log tan 0° is $-\infty$. The smallest divisions of the log sec α scale are at the beginning of the scale (near 0°), of the log tan α scale,

* In some manuals this instrument is called "Bigreave's rule" after the name of the inventor.

at the middle (near 45°). Thus, the accuracy of index setting differs in various portions of the scale, but on the average corresponds to four-place tables of logarithms. The scales are graduated directly in degrees and minutes of arc.

Scale setting will be explained by an analysis of a problem based on the first of the formulas in (19.23). Taking logs, we get

$$\log \tan x = \log \tan \delta + \log \sec t$$

To obtain log tan x, that is, the quantity x, the segments of the two scales should obviously be combined (Fig. 176a). To do this,

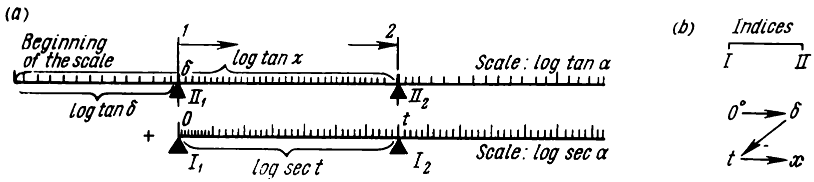

Fig. 176

put the first index I_1 on 0°; the second II_1 gives the reading of the inner cylinder (log tan δ) equal to δ. Fix the cylinders with the locking device and transfer the first index I_2 to t on the outer cylinder (log sec t); the second index II_2 will indicate the value of x on the scale log tan x. This solution is shown on the rule by the scheme (Fig. 176b). Formulas (19.24) and (19.25) are solved in similar fashion.

The rules of signs and other operations that must be observed when working with this rule follow from an investigation of formulas (19.23) and (19.25):

(1) For $t > 90°$ take $x > 90°$.

(2) In the formula $y = (90° - \varphi) \pm x$, the plus sign is taken for φ and δ of the same name, the minus sign for contrary names. The quantity 90°— φ is designated as "b" on the rule.

(3) For $y > 90°$ the azimuth is taken greater than 90°, for $y <$ 90° the azimuth is taken less than 90°. On the cylindrical rule, the azimuth is always reckoned less than 90° and then converted to semicircular reckoning by this law.

The cylindrical rule gives an accuracy of solution from $\pm 0'.5$ to $\pm 1'.5$ with an average of about $\pm 1'.0$. Determining h and A by means of this rule occupies somewhat more time than by the H.O. No. 214 tables, but the work is a little less tiring; the cylindrical rule is particularly convenient for checking computations of h and A. A disadvantage is that problems are unsolvable for values of t and A within $\pm 20'$ from 90°.

Example 7. Determine A_c and h_c from the data of Example 1. Write down the knowns and the solution in the same order as that on the cylindrical rule:

$$\delta = 28°44'.9\text{N}$$
$$t = 108°44'.2\text{E}$$
$$\varphi = 60°2'.5\text{N}$$
$$b = (90° - \varphi) = 29°57'.5$$
$$x = 120°21'\text{N}$$
$$(b \pm x) = 150°18'.5$$
$$A_c = 59°45'\text{NE}$$
$$h_c = 16°2'.0$$

SEC. 109. ERRORS IN CONSTRUCTING ALTITUDE LINES OF POSITION ON A CHART, AND ERRORS THAT FOLLOW FROM THE METHOD PROPER OF POSITION LINES. LIMITATIONS OF THE METHOD

From the foregoing consideration of the theory of the method of lines of position (the Saint-Hilaire method, in particular) it follows that this is an approximate method which holds only for small segments of circles of equal altitudes. In addition, supplementary errors in the position of an altitude line arise when performing the graphical constructions on a Mercator map, that is, in plotting.

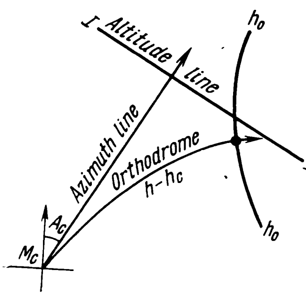

Fig. 177

We recall that the vertical circle of a celestial body is a great circle of the sphere and, hence, the azimuth line in a Mercator projection should be depicted as a curved line (Fig. 177). When plotting it in the form of a straight line (loxodrome), two types of errors occur in the plotted line of position:

(a) errors in the distance due to plotting the altitude difference $h - h_c$ along the straight line of the azimuth instead of plotting $h - h_c$ along the curve, or orthodrome;

(b) errors in the position and direction of the line I-I due to construction of the position line perpendicular to the straight line of the azimuth instead of constructing it normally to the orthodrome.

Thus, due to peculiarities in the Mercator projection, the line of position plotted on a chart is in error both as to distance and direction. Other cartographic projections may yield other errors; in other words, the errors noted are the result of distortions of cartographic projections.

Irrespective of these errors, the above-mentioned *error of the method itself* (that is, errors due to replacement of a circle of equal altitudes $h_0 - h_0$ by a straight line *I-I* tangent to it, or a chord in other methods of line plotting) is always operative. In a Mercator projection, the arc of a small circle is depicted as a cyclic curve, and the error of the method remains in force as before. This error will continue to operate even if plotting is substituted by calculating corrections to the computed coordinates by formulas (18.32), which correspond to equations of straight lines and not circles. In this instance, the error is equal to the sum of the higher-order of the series (18.30). Due to the generality of the substitution error, it is called the "error of the method of position lines". Let us examine these two sources of errors separately.

I. ERRORS DUE TO PLOTTING AZIMUTH LINES AS STRAIGHT LINES ON A MERCATOR CHART (ERROR IN DIRECTION OF POSITION LINE)

Let us consider Fig. 178 where the action of the errors is greatly exaggerated for clarity. For a D.R. position C (φ_c, λ_c), let the azimuth A_c of the body and the difference $h - h_c = \Delta h$ be computed. Performing the conventional plotting on a chart with straight lines, we get the determining point K_1 and the altitude line $I—I$. Actually, however, the azimuth line is an arc of an orthodrome shown on the chart as a certain curve CK_0, which, due to smallness of Δh, may be taken as an arc of a circle of radius R. If we lay off $\Delta h = CK_0$ on this arc, we get another determining point K_0. Drawing a *normal* at the point K_0 to the arc CK_0, we get the *proper position* of the altitude line I_0-I_0. Joining C and K_0 with a straight line, we see that the angle between this line and the azimuth line CK_1, plotted in the normal way, is equal to the orthodromic correction ψ, which is a correction for curvature of the orthodrome image on a Mercator chart. In navigation, an approximate formula is derived for $\psi : \psi = \frac{1}{2}\,\Delta\lambda \cdot \sin\varphi$, or, as applied to our case*,

$$\psi = \frac{1}{2} s \cdot \sin A_c \cdot \tan \varphi_c \tag{19.26}$$

where s is the distance CK_0 in minutes along a straight line.

Constructing the line K_0L tangent to the arc of the circle at the point K_0, we see that it also makes an angle ψ with the line CK_0, and an angle equal to 2ψ with the azimuth line A_c. The angle at the centre of the circle will be exactly the same.

Thus, the error in direction of an altitude line of position or, what is the same thing, in the azimuth line, is equal to a double

* $\Delta\lambda = \text{Dep.} \sec\varphi = S \cdot \sin TC \cdot \sec\varphi$, $TC = A_c$; $S = s$.

orthodromic correction, 2ψ. For the altitude line I-I to occupy the correct position I_0-I_0, A_c has to be corrected by twice the ψ correction, or,

$$A_1 = A_c + 2\psi \tag{19.27}$$

Here, the azimuth line A_1 is plotted as before in the form of a straight line CK_2, and the altitude line I_0-I_0 perpendicular to it. The error in distance Δh is small (for $\Delta h < 100'$).

From a theoretical analysis we can draw the following conclusions:

(1) In low latitudes (up to 30°), the correction 2ψ may be disregarded.

(2) In middle latitudes, the correction should be taken into account for $h - h_c$ exceeding 15′.

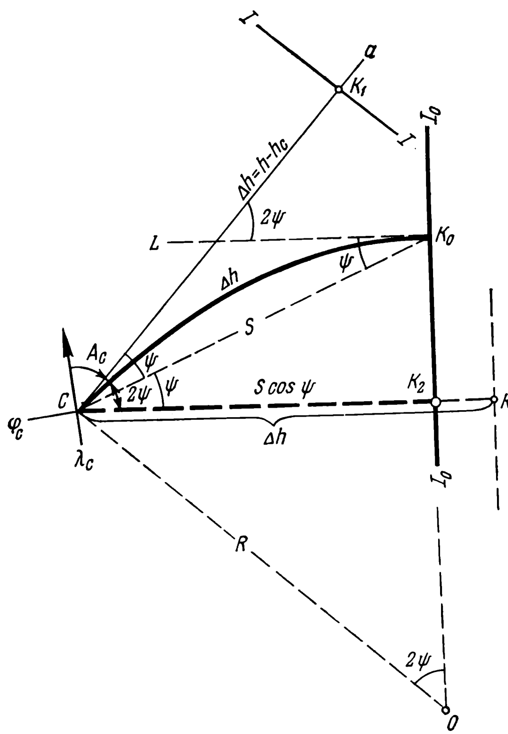

Fig. 178

(3) In high latitudes (> 70°) with $h - h_c \geqslant 7'$, check the value of the correction 2ψ in tables for the orthodromic corrections, and if it is in excess of 0°.3-0°.5, use it to correct the computed azimuth: $A_1 = A_c + 2\psi$. Bear in mind the sign of the correction, which is

determined by the rule: if in northern latitude the determining point K is east (west) of the D.R. position, the correction should be added to (subtracted from) A_c expressed in circular reckoning (0° to 360°). In southern latitude the signs of the correction ψ are reversed.

II. ERROR DUE TO SUBSTITUTING THE CURVE OF EQUAL ALTITUDES BY A STRAIGHT LINE (ERROR OF THE METHOD PROPER OF POSITION LINES)

We have pointed out (Sec. 99) that a circle of equal altitudes is depicted in Mercator projection in the form of one of three types of cyclic curve.

To find the error due to replacement of such a curve, tangent to it at points close to the point of tangency, it is necessary to find

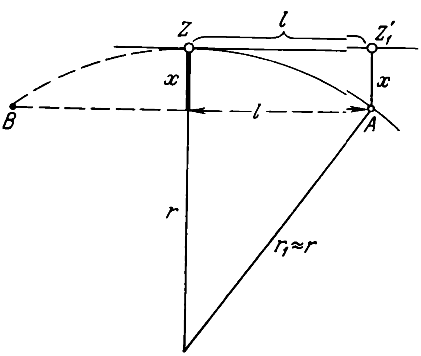

Fig. 179

the curvature or the radius of curvature of the cyclic curve r, which (in Mercator miles) will be

$$r' = \frac{1}{(\tan h - \tan \varphi \cdot \cos A) \operatorname{arc} 1'} \tag{19.28}$$

The magnitude of the correction x (Fig. 179) is found from the formula

$$x = \frac{l^2 \operatorname{arc} 1'}{2} (\tan h - \tan \varphi \cdot \cos A) \tag{19.29}$$

The error in the observed position y is found from the formula

$$y = \frac{1}{\sin \theta} \sqrt{x_1^2 + x_2^2 - 2x_1x_2 \cdot \cos \theta} \tag{19.30}$$

where θ is the angle between the lines of position ($\theta = A_2 - A_1$).

An investigation of formula (19.30) shows that for $\Delta h < 25'$ and $h \leqslant 60°\text{-}70°$ the error y in any latitude will practically be less than the accuracy of nautical observations.

When sailing in low latitudes and the tropics, it is sometimes necessary to measure and work altitudes of the sun greater than 80° (up to 88°). For this case we have the following approximate formula for low latitudes:

$$y = \frac{\Delta h^2}{6{,}875 \cdot \sin \Delta A} \sqrt{\tan h_1^2 + \tan h_2^2 - 2 \tan h_1 \cdot \tan h_2 . \cos \Delta A} \quad (19.31)$$

Example 8. Compute y.

(a) For $\Delta A = 90°$ and $h_1 \approx h_2 = 85°$; for $\Delta h = 20'$ we have $y = 0'.94$; for $\Delta h = 10'$, $y = 0'.23$.

(b) For $\Delta A = 30°$ and the same conditions we have: for $\Delta h = 20'$, $y = 0'.70$ and for $\Delta h = 10'$, $y = 0'.17$.

An investigation of (19.31) shows that if $\Delta h \leqslant 10'$, it is possible to plot position lines in the ordinary way without appreciable errors in low latitudes for sun altitudes up to 85°.

When observing stars, do not measure altitudes exceeding 60° especially in high latitudes, all the more so since it is difficult to take such sights.

CONCLUSIONS

1. The method of altitude lines of position has the following limitations:

(a) The altitude differences (intercepts) $h - h_c$ must not exceed 20′-25′. If $h - h_c > 25'$, the problem should be solved a second time taking the obtained φ_0, λ_0 for the computed ones.

(b) It is not advisable to observe altitudes of celestial bodies in excess of 60°-70° under ordinary conditions. In the tropics, sun altitudes up to 85° may be observed, but the intercept $h - h_c$ should not exceed 10′ in this case.

(c) If in middle latitudes $h - h_c > 15'$ and in high latitudes greater than 7′, introduce the orthodromic correction 2ψ into the computed azimuth.

2. When applying the method of lines of position in ordinary sailing conditions (with small errors in D.R. position), the errors due to all assumptions are slight and will have practically no effect on the accuracy of an observed position.

SEC. 110. THE EFFECT OF ERRORS IN THE ALTITUDE DIFFERENCE (INTERCEPT) $h - h_c$ ON THE POSITION OF AN ALTITUDE LINE

The correct position of a position line on a chart depends (in addition to the geometrical errors we have examined) also on the effect of errors in the accepted elements of the position line, that is, $h - h_c$ and A_c.

Whereas the geometrical errors of this method may be eliminated or avoided by adhering to the above-indicated restrictions, errors

in altitude are always operative; it is only the magnitude that changes. For this reason, accuracy and reliability in determining position from lines of position depend mainly on how accurately one obtains the difference $h - h_c = \Delta h$ in each of the lines.

This difference Δh includes random and systematic errors in the observed altitude h and random errors in the computed altitude h_c. Let us examine them separately.

(1) *Errors in observed altitude* were considered in Chapter 15. We recall that experiments and analysis established that systematic errors more often than not exceed random errors (in magnitude). Considerable systematic errors arise mainly due to taking inexact values of dip of the horizon and also due to sextant errors.

Random errors of observed altitude depend mainly on the state of the horizon and the skill of the observer, as has been established, and are characterized for the average observer, by mean errors of measurement of a single altitude:

$$\text{for the sun, } \varepsilon_h = \pm 0'.7$$

$$\text{for stars, } \varepsilon_h = \pm 0'.9 - 1'.0$$

Since it is advisable to observe no less than three altitudes, these values may be somewhat reduced; for the sun to $\varepsilon_0 = \frac{\pm 0'.7}{\sqrt{3}} = \pm 0'.4$, for stars, to $\varepsilon_0 = \frac{\pm 1'.0}{\sqrt{3}} = \pm 0'.6$. Under the most favourable conditions for the sun an experienced observer may expect $\varepsilon_h = \pm 0'.3 - 0'.4$.

(2) *Errors in computed altitude* h_c. As we know, computed altitudes are obtained in one of the following ways:

(a) computation by general formulas;

(b) by special tables;

(c) by means of special instruments.

The altitude h_c and azimuth A_c thus computed invariably have errors of calculation. Since the plotting accuracy of azimuth A_c will be of the order of $\pm 0°.3$, and the limiting error of computation by tables after rounding off, $\pm 0°.05$, we can consider A_c computed from any tables (with the exception of numerical tables) to be absolutely correct.

Errors in computed altitude are of a random nature. They arise from rounding off the last figures in logarithms, from errors of interpolation and performed operations, and their magnitude depends on changes in the functions used to select the desired quantities, and on the number of decimals in the logarithmic tables.

The slower the function changes (the less "acute" it is), the stronger the effect on the result of errors that accumulate in computa-

tion. For example, an error in a logarithm of two units in the final decimal, when picking out h_c from log sin h_c and four-place tables, will yield (for $h \approx 5°$) an error in altitude of $\Delta h \approx 0'.14$ and near 70° it will yield an error Δh of 5′. It is obvious that in the first case the function will be "acute" and advantageous to use, while in the latter case it cannot be used.

Soviet scientists M. Musselius, N. Preipich and, recently, A. Demin have studied accuracy in computations of h_c. Demin has derived the following approximate formulas for mean-square errors ε_{hc} in computed altitude:

(a) When computing h_c from log sin h with n-place tables

$$\varepsilon_{hc} = \pm \frac{2,285}{10^n} \sqrt{1+5\tan^2 h_c} \qquad (19.32)$$

(b) When computing h_c from $\log \sin^2 \frac{z_c}{2}$ with n-place tables

$$\varepsilon_{hc} = \pm \frac{5,405}{10^n} \cdot \tan\left(45° - \frac{h_c}{2}\right) \qquad (19.33)$$

(c) When computing h_c from TBA-52

$$\varepsilon_{hc} = \pm 0'.08 \sqrt{1+3.38\sin^2 2h_c} \qquad (19.34)$$

In these formulas, the errors are obtained on the assumption that interpolation and rounding are performed according to the general rules. Table 11 contains the values of errors computed from these formulas.

Errors in computing h_c from instruments are somewhat higher and for the ARG-3 come out to an average of $\varepsilon_h = \pm 0'.7\text{-}1'.0$, for the cylindrical rule $\varepsilon_h = \pm 0'.5\text{-}1'.5$.

Table 11

No.	Method of computation	Altitudes					
		0°	10°	20°	30°	40°	45°
1	Formula log sin h_c and MT-63	0′.023	0′.024	0′.029	0′.037	0′.048	0′.056
2	Formula log $\sin^2 \frac{z_c}{2}$ and MT-63	0 .054	0 .045	0 .038	0 .031	0 .025	0 .022
3	TBA-52	0 .08	0 .10	0 .12	0 .14	0 .16	0 .17
4	Table H. O. No. 214*	0 .3	0 .3	0 .3	0 .4	0 .5	0 .5

No.	Method of computation	Altitudes				
		50°	60°	70°	80°	90°
1	Formula log sin h_c and MT-63	0'.069	0'.092	0'.143	0'.292	[illegible]
2	Formula log $\sin^2 \frac{z_c}{2}$ and MT-63	0 .020	0 .014	0 .010	0'.005	0
3	TBA-52	0 .16	0 .14	0 .12	0 .10	[illegible]
4	Table H. O. No. 214*	0 .6	0 .8	1 .1	2 .1	[illegible]

* For the H. O. 214 tables, the limiting errors are given.

It is important to remember the following fundamentals of computation: (a) computation accuracy depends on the precision of our measurements; (b) it is impossible to obtain a more accurate result than the initial data of observations.

Starting out from these propositions we can state that accuracy in computing h_c should be somewhat higher than the accuracy of observation h, so that the errors of computation should not reduce perceptibly the accuracy of determination. At the same time we should not strive for too great an accuracy in computations, for this increases the work of the calculator without materially increasing the accuracy of the final result. We stated above that the highest observational accuracy would be $\varepsilon_h = \pm 0'.3$-$0'.4$. The mean error of ordinary navigational computations should not exceed this most favourable magnitude, that is, $\varepsilon_{hc} \leqslant \pm 0'.3$.

From Table 11 it is evident that all tables mentioned yield this accuracy with the exception of the H.O. No. 214 tables, which give such accuracy only up to altitudes of 50°-60°. Hence, the latter tables should not be used for greater altitudes.

An examination of the table and an analysis of the formulas will show us that four-place tables of logarithms (they yield a computational accuracy 10 times lower than that given in Table 11 for the MT-63) permit computing h_c from log sin h with the desired accuracy up to altitudes of 30° and from log $\sin^2 \frac{z_c}{2}$ for altitudes exceeding 30°. The same columns of the table show that five-place tables yield, under ordinary conditions, a somewhat excessive accuracy in computing h_c by the logarithmic method. Now in some cases of navigational practice and in certain methods (see Sec. 132)

the requirement is that the process of computation should not have any practical effect on the final result. In such cases, use should be made of five-place tables and most favourable formulas, in addition to all other techniques for increasing the accuracy of determinations. In some fleets, a mixed method (without tables of α and β; see Sec. 104) is used in computing from formulas $\sin^2 \frac{z}{2}$ and $\sin h$. Here, four-place tables do not ensure sufficient accuracy, and five-place tables must be used.

(3) *Total random error in the position line* ($\varepsilon_{\Delta h}$). The resultant error in the $h - h_c$ difference will include errors of observation and computation and will represent a random error in the position of the position line. Taking the above-mentioned values for ε_h (under average conditions for a good observer) and $\varepsilon_{hc} = \pm 0'.3$ (for four-place tables), we get from formula $\varepsilon_{\Delta h} = \pm \sqrt{\varepsilon_h^2 + \varepsilon_{hc}^2}$ for a single altitude

$$\text{for the sun:} \quad \varepsilon_{\Delta h} = \pm 0'.8$$

$$\text{for stars:} \quad \varepsilon_{\Delta h} = \pm 1'.0$$

Thus, due to random errors, the position line may be displaced relative to its actual position, and should thus be represented in the form of a *belt of positions*, within which lies the position of the ship with a probability of about 68%. To increase the probability of finding the ship in this belt, increase the magnitude $\pm\varepsilon_{\Delta h}$, that is, the width of the belt, according to the rules of error theory, as shown in Table 12.

Table 12

Probability of finding ship	68.3%	90%	95%	99%
width of "belt of position" in $\varepsilon_{\Delta h}$	$\pm\varepsilon_{\Delta h}$	$\pm 1.6\varepsilon_{\Delta h}$	$\pm 2\varepsilon_{\Delta h}$	$\pm 2.5\varepsilon_{\Delta h}$
Average for sun	$\pm 0'.7$	$\pm 1'.1$	$\pm 1'.4$	$\pm 1'.8$
Average for stars	$\pm 1'.0$	$\pm 1'.6$	$\pm 2'.0$	$\pm 2'.5$

These figures represent the mean values for good observers, so somewhat larger errors may appear in practical work.

In addition to the foregoing errors, an altitude line may include also blunders in correcting h or in computing h_c. Only through exceedingly careful work and constant practice can one avoid blunders when working sights. Certain approximate methods for checking blunders and checking computations are given below where determination of position from several lines is considered.

From the foregoing it follows that in the general case an observed altitude (position) line I_0-I_0 (Fig. 180) does not coincide with the "actual" line of position I-I. Due to the operation of random errors, a "belt of position" of width $\pm\varepsilon_{\Delta h}$ is formed about the line I_0; the actual line of position will be found inside this belt with a probability that increases as the width of the belt gets closer to $3\varepsilon_{\Delta h}$. Now

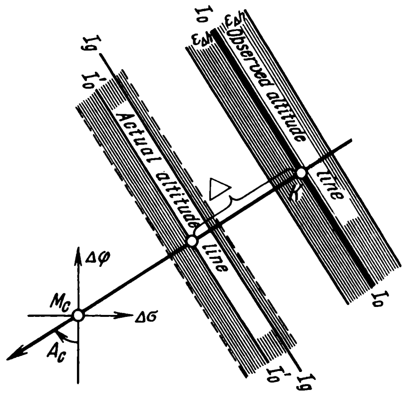

Fig. 180

due to the effect of the total systematic error Δ, this entire belt is displaced along the azimuth line by a magnitude Δ in a direction that depends on the sign of the systematic error. If we eliminate the systematic error and in plotting take the "belt of position", then the actual position line I will be in this belt and will be taken into account in subsequent plotting.

If there is a blunder, the belt is shifted along the azimuth line as well, and if a blunder is committed in A_c, the belt will also turn by the magnitude of the blunder.

If the coordinate origin is placed in the D.R. position and we take the same axes as in Sec. 102, then equation (18.31) of the position line will take on the following form (errors allowed for)

$$\Delta\varphi \cdot \cos A + \Delta\sigma \cdot \sin A = \Delta h + \Delta \pm \delta \qquad (19.35)$$

where $\Delta h = h - h_c$ is the "true" altitude difference (intercept) free from errors of observation and computation

Δ and $\pm\delta$ are total (systematic and random) errors.

To reduce the effect of all these errors in observations and computations, take the measures indicated in Sec. 85 for increasing accuracy of observed altitude and note the remarks in this section pertaining to computation procedures.

CHAPTER 20

DETERMINING POSITION OF SHIP BY THE METHOD OF ALTITUDE LINES OF POSITION

SEC. 111. PECULIARITIES IN DETERMINING POSITION FROM SIMULTANEOUS OBSERVATIONS OF TWO BODIES

As was shown in Sec. 97, to determine the position of a ship on a chart or to obtain the coordinates φ_0 and λ_0, we have to get at least two lines of position, which means we must measure at least two altitudes of celestial bodies having an azimuth difference close to 90°. If the altitudes are measured of several bodies seen at the same time, these observations are called "simultaneous observations of bodies". If altitudes of a single body are observed, an appreciable interval of time (from 1h to 3-4h) is required for sufficient change in azimuth (40°-45°); the observations of the body will then be called "double sights or running fix". Sometimes, for simplicity, determination from simultaneous sights of two (three and so forth) bodies is called "star sights".

Not only will stars, planets and the moon be visible at the same time at night, but certain bodies are visible in the daytime as well; the sun and moon, the sun and a bright planet (Venus, for example), the moon and a planet. This allows for a choice of different bodies in simultaneous sights, provided their mutual positions satisfy the most favourable conditions of observations.

Simultaneous sights of bodies obviously require two observers, but since one person usually takes sights on board ship, these observations are actually taken from different places on the earth and at different times.

For both altitudes and, hence, both lines to be considered as obtained from a single place on earth, we have to note the distance run by the vessel during the time between observations, i.e., to reduce the altitudes to a single zenith. The fact that observations are taken at different times is of no importance since the computed altitude and azimuth of every body are found from the appropriate instants T_{gr} and, therefore, with their values of δ and t of the celestial body.

Reducing to a single zenith from tables was considered in Sec. 81, where we derived the formulas (15.6) and (15.7). In an analytical

reduction to a single zenith, use is made of Table 16, MT-53, compiled from formula (15.8) for the interval $\Delta T = 1\text{m}$, therefore

$$\Delta h_z = \Delta h_{1\mathrm{min}} (T_2 - T_1) \text{ min}$$

where $T_2 - T_1$ is the time interval between observations
$\Delta h_{1\mathrm{min}}$ is a correction taken from Table 16 via $(A - TC)$ [or $(A - K)$] and V of the vessel.

For speeds less than 6 knots and time intervals between observations up to 3 to 5 minutes, the correction for reducing altitudes to a single zenith may be ignored.

Graphical reduction to a single zenith. Reduction to a single zenith may be executed graphically too—when plotting position lines on a chart or paper.

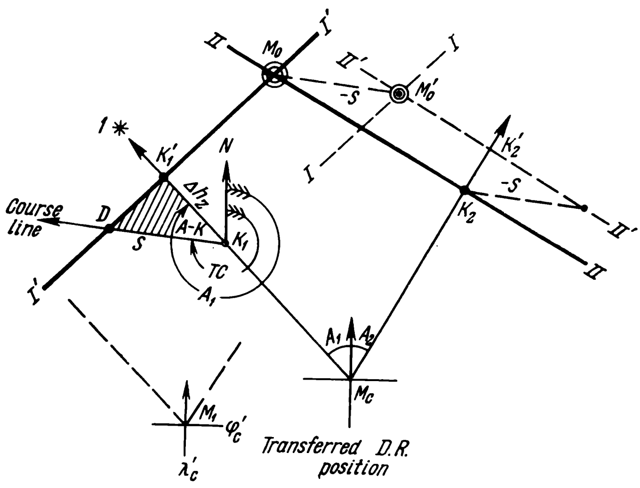

Fig. 181

Computing the elements of position lines for both bodies and constructing the line of azimuths (Fig. 181), we obtain the determining points K_1 and K_2 as shown in Sec. 105. Then, computing the run S between observations from the formula $S = V(T_2 - T_1)$, Table 27a, MT-63, or from a calculating chart, we find out to which zenith the altitude should be reduced; ordinarily, the first altitude is reduced to the second zenith for which T_{sh} and lr are noted. In this case, we draw the course line track from the determining point K_1 and lay off on it the sailing distance (run) S. Through the point obtained D draw the position line perpendicular to the azimuth line A_1. This line I'-I' will be reduced to the second zenith. Indeed,

from the triangle K_1DK_1', $K_1K_1' = S \cdot \cos(A - TC)$, that is, it is equal to Δh_Z. The observed position M_0 is obtained at the point of intersection of the lines I'-I' and II-II.

If the second altitude were reduced to the first zenith, then we would have to lay down the course line (track) from the determining point K_2 and plot the run S in the reverse direction (see Fig. 181), since the first altitude was observed before the second. The observed position would then be in M_0', which is a distance S from M_0 backwards along the course line.

To avoid reducing the altitude of the first body to the zenith of the second set of observations, one could observe the first body abeam; then $A - TC = 90°$ and $\Delta h_Z = 0$.

Another way to avoid reducing to a single zenith is by means of symmetric observations of the first celestial body executed according to the scheme: first body—second body—first body. If the intervals between observations of the bodies are roughly the same, the mean altitudes will refer to a single place on the earth.

However, observations in the foregoing cases are complicated. Complications should be avoided because stellar observations are involved as it is.

The order of computations, the forms and process of plotting lines will be shown later on.

From plotting two altitude lines on a chart we obtain, at the point of their intersection, the so-called *observed position* of the vessel (shown in Fig. 181 by double circles) for the instant T_{sh} to which the altitudes are reduced. If the altitudes are reduced to the second zenith, the exact value of ship time may be had from the formula $T_{sh} = T_{gr2} \pm ZD_W^E$, where T_{gr2} is the time of the second set of observations.

In some countries a method is used for plotting from a transferred position, chosen (C.P.) or assumed position (A.P.). In this case, computations from special tables are simplified by taking values of φ'_c rounded off to a whole degree and t^0_{loc}. To obtain a whole value of t_{loc} appropriate values of λ_c' are chosen in the formula

$$t^{\circ}_{loc} = t_{gr} \pm \lambda'^{E}_{cW}$$

No interpolation is required in the tables, but plotting should be done from the chosen position M_1 (φ_c', λ_c', see Fig. 181). This method in ordinary cases is not suitable due to the fact that the slight simplification in computations complicates plotting and, in addition, disconnects the position line from dead reckoning, thus complicating the analysis. But in methods of precomputation and in some tables it is very convenient.

SEC. 112. THE EFFECT OF ERRORS OF OBSERVATION AND COMPUTATION ON THE OBSERVED POSITION OBTAINED FROM "TWO STARS"

The observed position thus obtained is subject to the effects of a number of errors: errors due to plotting on a Mercator chart, errors of the method itself of altitude lines of position, systematic and random errors in the intercepts $h - h_c$ and, finally, blunders in observations and computations.

We can disregard the first two categories of errors if we abide by the recommendations given in Sec. 109, that is, if the differences $h - h_c$ do not exceed 20′ and the altitudes are less than 60°-70°, otherwise this problem should be solved a second time with the coordinates obtained and regarded as computed. For this reason, we shall from now on consider only errors of observation and computation and also blunders.

When considering errors that affect one line of position, it was found that due to the action of random errors the position line is actually converted into a "belt of position", and due to systematic errors this belt can be shifted in either direction along the azimuth line. What is more, a blunder made in observations or working sights can produce an additional shift in the "belt of position".

When determining a position from two lines, these errors are possible in both lines. If the coordinate origin is put in the D.R. position, the equations of these lines (19.35) will take the form

$$\left.\begin{aligned} \Delta\varphi\cdot\cos A_1 + \Delta\sigma\cdot\sin A_1 &= \Delta h_1 + \Delta_1 \pm \delta_1 \\ \Delta\varphi\cdot\cos A_2 + \Delta\sigma\cdot\sin A_2 &= \Delta h_2 + \Delta_2 \pm \delta_2 \end{aligned}\right\} \quad (20.1)$$

where Δh_1 and Δh_2 are the "true" intercepts $h - h_c$ free from errors of observation and computation

Δ_1 and Δ_2 are systematic errors in the intercept $h - h_c$

δ_1 and δ_2 are individual values of random errors in the first and second intercepts $h - h_c$

$\Delta\varphi$ and $\Delta\sigma = \Delta\lambda\cdot\cos\varphi$ are the desired corrections of the D.R. coordinates.

If there were no errors or blunders, a graphical or analytical solution of equations (20.1) would yield the actual position of the ship or its *real* coordinates φ and λ (via the increments $\Delta\varphi$ and $\Delta\sigma$). However, there are *always* errors in observations and computations; and sometimes even blunders. As a result, for $h - h_c$, in plotting, we apply the sum of the terms on the right-hand side and get the above-mentioned observed position of the ship or its observed coordinates φ_0, λ_0 instead of the actual position.

From equations (20.1), which contain many unknowns, it follows that for two lines it is impossible to eliminate systematic errors from the observed position by means of a direct analytical or graphical solution. We can reduce them or judge of their absence only indirectly. And the same goes for blunders.

Let us consider the effect of errors and establish the most favourable conditions of observation for reducing them.

1. THE EFFECT OF SYSTEMATIC ERRORS

Put the origin of coordinates $\Delta\varphi$, $\Delta\lambda$ in the actual position of the ship M_1 (see Fig. 152). Then in equations (20.1) $\Delta h_1 = \Delta h_2 = 0$ and the increments $\Delta\varphi$ and $\Delta\sigma$ will express errors in the coordinates of the observed position relative to the actual position. Equations (20.1) will take on the form

$$\left.\begin{aligned}\Delta\varphi\cdot\cos A_1+\Delta\sigma\cdot\sin A_1&=\Delta_1\pm\delta_1\\ \Delta\varphi\cdot\cos A_2+\Delta\sigma\cdot\sin A_2&=\Delta_2\mp\delta_2\end{aligned}\right\}\qquad(20.2)$$

Let us assume that there are no blunders, and systematic errors greatly exceed random errors so that for the sake of simplicity we can take $\delta_1=\delta_2=0$; then

$$\left.\begin{aligned}\Delta\varphi\cdot\cos A_1+\Delta\sigma\cdot\sin A_1&=\Delta_1\\ \Delta\varphi\cdot\cos A_2+\Delta\sigma\cdot\sin A_2&=\Delta_2\end{aligned}\right\}\qquad(20.3)$$

Errors in the coordinates $\Delta\varphi$ and $\Delta\sigma$ are obtained by solving 20.3) as in Sec. 98.

$$\Delta\varphi=\frac{\begin{vmatrix}\Delta_1\cdot\sin A_1\\ \Delta_2\cdot\sin A_2\end{vmatrix}}{\begin{vmatrix}\cos A_1\cdot\sin A_1\\ \cos A_2\cdot\sin A_2\end{vmatrix}};\qquad \Delta\sigma=\frac{\begin{vmatrix}\cos A_1\cdot\Delta_1\\ \cos A_2\cdot\Delta_2\end{vmatrix}}{\begin{vmatrix}\cos A_1\cdot\sin A_1\\ \cos A_2\cdot\sin A_2\end{vmatrix}}$$

whence

$$\Delta\varphi=\frac{\Delta_1\cdot\sin A_2-\Delta_2\cdot\sin A_1}{\sin(A_2-A_1)}$$

and

$$\Delta\sigma=\frac{\Delta_2\cdot\cos A_1-\Delta_1\cdot\cos A_2}{\sin(A_2-A_1)}\qquad(20.4)$$

The linear error in the ship's position (M_1M_2 in Fig. 152) is determined from the triangle M_1M_2D

$$(\Delta M)^2=(\Delta\varphi)^2+(\Delta\sigma)^2$$

After substituting the values of $\Delta\varphi$ and $\Delta\sigma$ and some manipulation we have

$$\Delta M=\frac{\pm\sqrt{\Delta_1^2+\Delta_2^2-2\Delta_1\Delta_2\cdot\cos\Delta A}}{\sin\Delta A}\qquad(20.5)$$

where $\Delta A = (A_2 - A_1)$ is the difference in azimuth of the celestial bodies.

In the general case, systematic errors Δ_1 and Δ_2 differ. If for the sake of simplicity we take $\Delta_1 = \Delta_2 = \Delta$, then (20.5) will take the form

$$\Delta M = \frac{\Delta \sqrt{2(1-\cos \Delta A)}}{\sin \Delta A} \tag{20.6}$$

An investigation of (20.6) shows that

(a) for $\Delta A = 0$, $\Delta M = \frac{0}{0}$, and for $\Delta A = 180°$, $\Delta M = \infty$, which means that for azimuth differences of 0° and 180° the position cannot be obtained, since the two lines become one;

(b) for $\Delta A < 90°$ there will be a difference in the radicand, while for $\Delta A > 90°$ there will be a sum. Thus, the systematic error in the

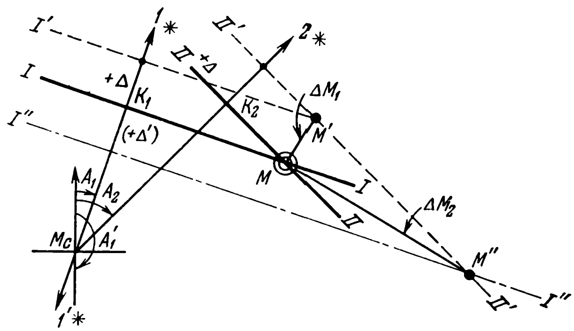

Fig. 182

position will be less for a difference of azimuths less than 90° and greater for a difference greater than 90° (in the second quadrant).

Indeed, putting $\Delta A = 30°$ and 150°, we get $\Delta M_1 = \pm 1{\cdot}03\Delta$ and $\Delta M_2 = \pm 3{\cdot}86\Delta$, respectively.

This same regularity follows from graphic constructions as well. Referring to Fig. 182, let M be the actual position obtained from two lines I and II. Both stars are situated so that $\Delta A < 90°$. If both lines have the same systematic error $+\Delta$, they will be shifted towards the bodies and the erroneous position will be at M'. The quantity ΔM_1 will represent the shift of the observed position relative to the actual position.

However, if the stars ($1'_*$, and 2_* in Fig. 182) are situated so that $\Delta A' > 90°$, then the very same error $+\Delta$ will yield a greater shift ΔM_2 of the observed position M''.

"Astronomical" bisector. Let us see how we can eliminate the effects of these errors. From Fig. 183 it is seen that if both lines are affected by the same systematic errors, the position obtained at the point of intersection of the displaced lines will always be located on the bisector of the angle between the lines of position.

Indeed, when the first and second lines are displaced $+\Delta$, $+2\Delta$, $+3\Delta$, . . ., etc., the observed positions are at points M_1, M_2, M_3, M_4, etc., which lie on the bisector M_0M_3 of the angle between the

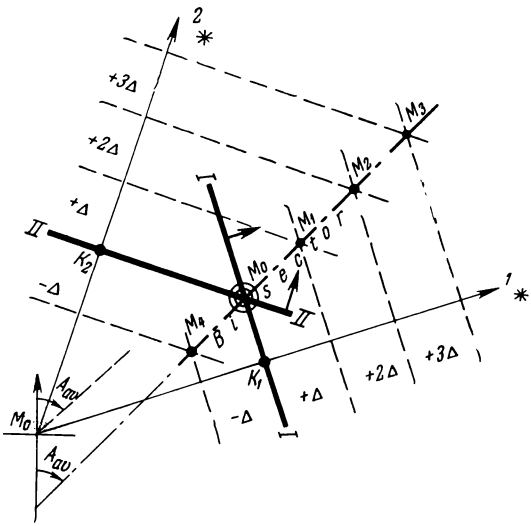

Fig. 183

lines of position. If at the observed position we construct arrows parallel to the azimuth lines and directed at the celestial bodies, the angle between these arrows will show which angle between the lines should be divided by the bisector. This bisector will be a line of position of the ship and one that is free from systematic errors. As may be seen (see Fig. 183), two lines of position yield one bisector free from systematic errors. To obtain two independent bisectors, it is obviously necessary to have four lines of position.

The fact that the bisector is a line of position free from errors Δ also follows from a solution of the equations (20.3).

Subtracting the first equation from the second, we see that the systematic errors are eliminated:

$$\Delta\varphi\,(\cos A_2 - \cos A_1) + \Delta\sigma\,(\sin A_2 - \sin A_1) = 0$$

or

$$-\Delta\varphi \cdot 2\sin A_{av} \cdot \sin\frac{\Delta A}{2} + \Delta\sigma \cdot 2\cos A_{av} \cdot \sin\frac{\Delta A}{2} = 0 \qquad (20.7)$$

whence, after simplifications,

$$\Delta\varphi = \Delta\sigma_0 \cdot \cot A_{av} = \Delta\sigma \cdot \tan(90^\circ - A_{av}) \qquad (*)$$

The expression (20.7) or (*) is the equation of a straight line (of form $y = x \cdot \tan\alpha$) passing through the coordinate origin: the position M_0 at an angle A_{av} to the meridian of the position. This line may be called the difference line of position or the astronomical bisector, and it is free from systematic errors.

From the foregoing analysis it follows that if only the same systematic (recurring) errors were operative in lines of position, then to reduce their effects the azimuth difference of the stars would have to be taken considerably less than 90°. But in actual conditions, systematic errors always operate jointly with random errors and, in addition, they are not always the same in the two lines.

Now let us consider the effects of a systematic error in the hour angle.

If the origin of the coordinate system $\Delta\varphi$ and $\Delta\lambda$ is put in the actual position, and the systematic error in the altitude is considered due to an identical error in the chronometer correction, recorded instants, or hour angles, then $\Delta h = \Delta = \cos\varphi \cdot \sin A \cdot \Delta t$, and the equations (20.3) will take the form

$$\Delta t \cdot \cos\varphi \cdot \sin A_1 = \Delta\varphi \cdot \cos A_1 + \Delta\lambda \cdot \sin A_1 \cdot \cos\varphi$$
$$\Delta t \cdot \cos\varphi \cdot \sin A_2 = \Delta\varphi \cdot \cos A_2 + \Delta\lambda \cdot \sin A_2 \cdot \cos\varphi$$

Solving these equations like (20.4), we get

$$\Delta\varphi = 0$$
$$\Delta\lambda = \pm\,\Delta t$$

which means that if in both lines we have committed the same error in hour angles (chronometer correction), then the position will be shifted in longitude alone and by the amount of the error Δt.

II. THE EFFECT OF RANDOM ERRORS. CIRCLES AND ELLIPSES OF ERRORS

(1) The Mean Square Error in an Observed Position

Assuming in equations (20.2) that systematic errors are eliminated, that is, that $\Delta = 0$, and solving them for $\Delta\varphi$ and $\Delta\sigma$ by means of determinants, we get (like in the preceding case)

$$\Delta\varphi = \frac{(\pm\,\delta_1)\sin A_2 - (\pm\,\delta_2)\sin A_1}{\sin(A_2 - A_1)}$$

and

$$\Delta\sigma = \frac{(\pm\,\delta_2)\cdot\cos A_1 - (\pm\,\delta_1)\cos A_2}{\sin(A_2 - A_1)}$$

Passing to mean square errors on the basis of the formulas of the theory of errors (see Appendix V, formula 5), we get

$$\left.\begin{aligned} \varepsilon_\varphi^2 &= \frac{\varepsilon_{h1}^2\cdot\sin^2 A_2}{\sin^2(A_2 - A_1)} + \frac{\varepsilon_{h2}^2\cdot\sin^2 A_1}{\sin^2(A_2 - A_1)} \\ &\text{and} \\ \varepsilon_\sigma^2 &= \frac{\varepsilon_{h2}^2\cdot\cos^2 A_1}{\sin^2(A_2 - A_1)} + \frac{\varepsilon_{h1}^2\cdot\cos^2 A_2}{\sin^2(A_2 - A_1)} \end{aligned}\right\} \tag{20.8}$$

The error in position will be

$$\varepsilon_{loc}^2 = \varepsilon_\varphi^2 + \varepsilon_\sigma^2$$

or, after substitution and simplifications,

$$\varepsilon_{loc} = \pm\,\frac{\sqrt{\varepsilon_{h1}^2 + \varepsilon_{h2}^2}}{\sin(A_2 - A_1)} \tag{20.9}$$

If the mean errors in both intercepts be taken equal, that is, $\varepsilon_{h1} = \varepsilon_{h2} = \varepsilon_h$, which is quite realistic, then

$$\varepsilon_{loc} = \pm\,\varepsilon_h\,\frac{\sqrt{2}}{\sin\Delta A} \tag{20.10}$$

From (20.9), (20.10) and (20.8) it will be seen that:

(a) ε_{loc} is directly proportional to the error ε_h in the intercept $h - h_c$ and inversely proportional to $\sin(A_2 - A_1)$; hence, the error ε_{loc} does not depend on the azimuths themselves of the celestial bodies, but solely on their difference;

(b) for a difference $A_2 - A_1 = 90°$ the error ε_{loc} will be minimal

$$\varepsilon_{loc} = \pm\,\varepsilon_h\sqrt{2}$$

(c) for $\Delta A = 0°$ or $180°$ the position error ε_{loc} increases to infinity, which means that the position is unobtainable;

(d) the errors ε_φ and ε_σ depend on the azimuths of celestial bodies: the closer the observed bodies are to the meridian, the less ε_φ and the greater ε_σ; if the bodies are closer to the prime vertical, then $\varepsilon_\varphi > \varepsilon_\sigma$.

(2) Circle of Errors

The quantity ε_{loc} of the mean square error in position of an observed point as expressed by formulas (20.9) or (20.10) is geometrically represented by the radius (ρ) of a **circle of errors** drawn about the

observed position (Fig. 184). This circle gives an approximate description of the spread of observed positions due to random errors. If only random errors of observation and computation are operative, the actual position of the ship lies somewhere inside this circle with a probability of about 63-68%. As ΔA departs from 90°, the radius of the circle and its area increase, which means that the accuracy of the observed position diminishes.

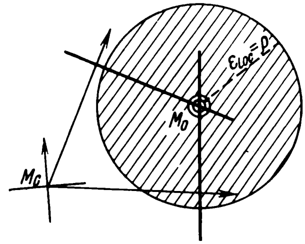

Fig. 184

(3) Rhombus of Errors

It is inconvenient to evaluate accuracy of position by means of a circle of errors in that a circle does not yield directions in which the position is determined with more or less reliability. This circumstance is clearly evident from the construction of a rhombus of errors or, still better, from an ellipse of errors.

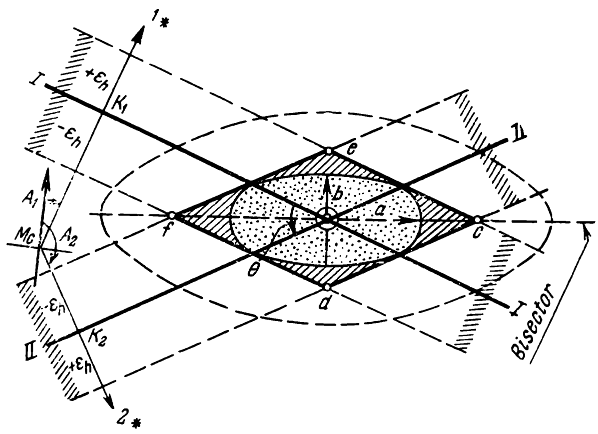

Fig. 185

Around each line of position we can construct a "belt of random errors" characterized by the quantity ε_h.

The intersection of two such "belts of errors" (Fig. 185) yields a **rhombus of random errors** *cdfe*. We can approximately say that the actual position of the ship lies within this rhombus with a probability of 46.6%. However, the rhombus is not a figure with the same

probability density of random errors. What this means is that a rhombus does not give an accurate picture of the possible distribution of ship positions due to the effects of random errors ε_h.

(4) Ellipse of Errors

Error theory offers proof that of all the figures, in which the probabilities of finding positions are the same, the **ellipse of random errors** has the smallest area. The ellipse is a curve along which the probabilities are equally distributed.

An ellipse constructed for mean square errors is called a "mean square ellipse of errors" or simply a **mean ellipse of errors**. There is a probability of 39.3% that the position of a vessel lies within the mean ellipse; a 1.5-fold increase in the axes raises the probability to 67.5%, a twofold increase brings the probability up to 86.5%.

Constructing an ellipse and rhombus (see Fig. 185), we see that the mean ellipse lies inside the rhombus and is tangent to its sides near the lines of position. Comparing the ellipse and rhombus, we see that the areas near the vertices f, c, e, d of the rhombus are "superfluous"; they increase considerably with diminishing ΔA. That is why the rhombus of errors is not suitable for evaluating the area of a probable position of a vessel. Indeed, if the semiaxes of the ellipse are doubled, then, as the laws of random errors state, the probability of finding the position in the ellipse will be 86.5%. Now the size of such an ellipse is only slightly greater than a rhombus in which the probability of finding the position is only 46.6%.

Error theory provides formulas for computing the semimajor axis a and the semiminor axis b of an ellipse of errors. As applied to the observation of altitudes, provided that $\varepsilon_{h1} = \varepsilon_{h2} = \varepsilon_h$, that is, for observations of equal accuracy, we get

$$a = \frac{\varepsilon_h}{\sqrt{2}\cdot\sin\frac{\theta}{2}}; \quad b = \frac{\varepsilon_h}{\sqrt{2}\cdot\cos\frac{\theta}{2}} \tag{20.11}$$

where θ is an acute angle between the lines of position equal to ΔA if $\Delta A < 90°$, and $180° - \Delta A$, if $\Delta A > 90°$

ε_h is the mean square error in the intercept $h - h_c$.

The semiaxis a is always directed along the bisector of the acute angle θ, and the semiaxis b along the bisector of the obtuse angle.

An investigation of formulas (20.11) for the semiaxes of an ellipse shows that:

(a) as the difference in azimuths $\Delta A = \theta$ decreases, the semiaxis a increases, the semiaxis b decreases;

(b) when $\Delta A = 0$ the semiaxis a increases to ∞ and the position cannot be determined; analogously, for $\Delta A = 180°$ the semiaxis a

becomes infinite and again the position is unobtainable. In these two cases we have one "belt of position";

(c) for $\Delta A = 90^\circ$ we have $a = \pm\, \varepsilon_h$, $b = \pm\, \varepsilon_h$ and the ellipse becomes a circle with radius $\rho = \varepsilon_{loc} = \sqrt{a^2 + b^2} = \pm\, \varepsilon_h \sqrt{2}$, which is a "circle of errors" with a probability of 39.3%.

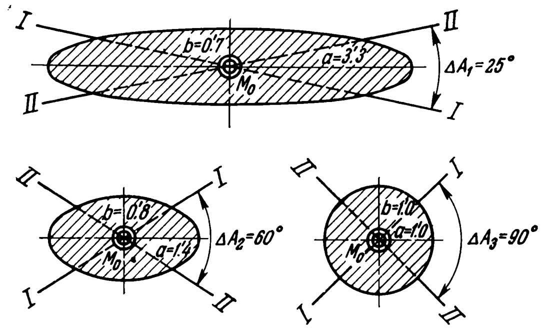

Fig. 186

By way of illustration let us construct ellipses of errors for $\varepsilon_h = \pm\, 1'0$ and $\Delta A_1 = 25^\circ$, $\Delta A_2 = 60^\circ$, $\Delta A_3 = 90^\circ$ to the same scale (Fig. 186).

From the investigation of formulas (20.11) and the figures it will be seen that we obtain the least spread in positions for an azimuth difference of celestial bodies of 90°, which is in agreement with the preceding analysis.

(5) Constructing an Ellipse of Errors

For an approximate construction of an ellipse of errors in analyzing determination of position, we can apply the following simple graphical procedure (Fig. 187):

(a) construct lines of position *I-I*, *II-II*;

(b) draw bisectors of the angles between them (by eye);

(c) specify (from personal experience) the quantity ε_h;

(d) compute the quantity $y = 0.7\varepsilon_h$ and shift, by this amount, one of the lines of positions parallel to itself in either direction; we get segments equal to the semiaxes a and b on the bisectors;

(e) take the semiaxes obtained and lay them off on the bisectors in both directions from the observed position and construct an ellipse by hand. This will be area in which there is a 39.3% probability of containing the observed position. To increase the ellipse, increase the semiaxes accordingly.

If we compare the evaluation of area of probable ship position by the three methods: circle, rhombus and ellipse of errors, it will be seen that the ellipse yields the smallest area and orients it correctly, and there are no difficulties in construction. Incidentally, for ordinary differences, $\Delta A > 60°$, there is no essential discrepancy in the areas of ellipse, circle and even rhombus. Here the important

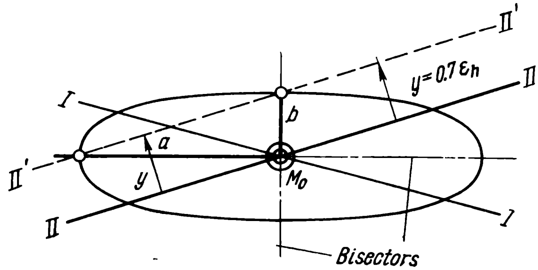

Fig. 187

thing is that when analyzing the position, it is the area containing the observed position and not the point that should be taken into account. It is also important that this area be properly calculated and oriented.

In future we shall use the area of an ellipse of errors with axes increased 1.5- to 2-fold.

(6) The Effect of Random Errors on the "Astronomical" Bisector

The formula for errors of bisector is similar to (20.11) for a, that is

$$\varepsilon_b = \pm \frac{\varepsilon_h}{\sqrt{2} \cdot \sin \frac{\Delta A}{2}} \qquad (20.12)$$

From (20.12) it is seen that the error in the bisector depends on the azimuth difference of the altitude lines.

For example, when $\Delta A = 180°$, $\varepsilon_b = \frac{\varepsilon_h}{\sqrt{2}} = 0.71\ \varepsilon_h$; when $\Delta A = 90°$; $\varepsilon_b = \varepsilon_h$; when $\Delta A = 30°$, $\varepsilon_b = \sqrt{2}\ \varepsilon_h \approx 1.4\varepsilon_h$. Consequently, the bisector is least in error when $\Delta A = 180°$, that is, when observing bodies in opposite azimuths. For angles ΔA less than 90°, errors in the bisector exceed errors in the lines (this is particularly evident for $\Delta A < 30°$) and it is not advisable to construct a bisector in these cases because it becomes unreliable.

III. THE JOINT EFFECT OF SYSTEMATIC AND RANDOM ERRORS

It has already been stated that a systematic (recurring) error shifts the observed position by an amount ΔM expressed by formula (20.6) along the line of mean azimuth to either side, depending on the sign of the error Δ. Now random errors lead to a spread of positions that is characterized by an ellipse or a circle of errors. The most favourable difference of azimuths of celestial bodies for systematic

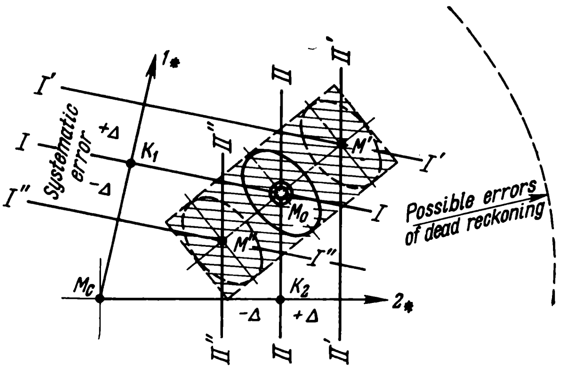

Fig. 188

errors will be less than 90°, for random errors, equal to 90°. Thus the most favourable conditions for observations and for taking these errors into account differ.

When determining a position at sea, systematic and random errors always operate *jointly*; in most cases it is difficult to separate them and indicate their specific magnitudes, and only in certain cases is it possible to gauge their probable values for given circumstances.

These are all complicating factors in any joint analysis of such errors and in any attempt to indicate the most favourable difference of azimuths in a general case.

Recent investigations into this problem have not yielded a consistent conclusion due to unlike approaches to the principle of combining systematic and random errors.

Without going into specifics, let us examine the most important practical conclusions from these investigations.

1. If the relationship of the magnitudes of random and systematic errors is quite unknown or may be presumed equal, then the most favourable azimuth difference of celestial bodies is 90°.

2. If systematic errors are assumed to be greater than random errors, then the most favourable azimuth difference will be less than 90° (of the order of 60° to 70°).

3. To take into account the entire area which might contain the actual position of the vessel in the case of both systematic and random errors, construct a rectangle of errors about the observed position. To do this, lay off on the line of mean azimuth the amount of shift ΔM (20.6) of position due to systematic errors and construct (about the two positions obtained) ellipses of errors that characterize the spread of positions due to random errors. Then, constructing a rectangle about the ellipses, we get the total area of the possible position of the vessel (see Fig. 188).

A more detailed analysis of the effects of all errors in practical work is given below.

SEC. 113. METHODS OF PRACTICAL ANALYSIS OF AN OBSERVED POSITION

Due to the fact that the observed position does not in practice coincide with the actual position of the ship, it must be subjected to analysis, which consists in the following:

(a) an evaluation of area that might contain the ship's position and choice of the observed position;

(b) detection of possible blunders;

(c) establishing reliability of observation and the possibility of advancing the dead reckoning to the observed position.

I. CONSTRUCTING AN AREA OF THE POSSIBLE POSITION OF A SHIP

Suppose the navigator has fulfilled all requirements, in observations, to ensure a reliable fix; that is, he has checked instruments and their corrections, has taken 3 to 5 altitude sights of each celestial body, and so forth (See Sec. 85). However, for various reasons he was unable to measure the dip of the horizon or apply special methods of observations for eliminating systematic errors.

Taking into account existing conditions and previous experience, we accept the possible systematic error at, say, $\pm 2'$. On the same grounds, for the given conditions, we estimate the random error (ε_h) at, say $\pm 1'.0$. To obtain the position shift due to a systematic error, we shift the lines $2'$ to either side and get two more observed positions M' and M'' (Fig. 188). Of the three positions, the observed one is that which is closest to danger in subsequent sailing. If there is no danger, then M_0 is taken as the observed position.

About the observed position M_0 we construct an ellipse of errors on the basis of the bisectors of angles and the displacement of one line by $0.7\ \varepsilon_h$ as indicated above. Doubling its semiaxes and drawing an ellipse by hand, we get the area of the probable position due

to the effects of random errors. Transferring this ellipse to position M' and M'' and encircling the entire area about the 3 ellipses with a dashed line, we get the *area of the possible position of the ship.* If the systematic error is presumed to be small, for instance in the case of a measured dip of the horizon and an adjusted sextant, the probable position may be taken to be in the area of the appropriate ellipse near position M_0.

II. DETECTION OF BLUNDERS

A peculiarity of astronomical observations at sea is that they are taken quickly, by one person, and usually are not checked by anybody else. This goes for computations as well.

An inexperienced observer lacking practice makes more blunders and spends more time than an experienced observer. For this reason, it is wise to practice in taking sights, working the sights and plotting the sights even when there is no real necessity. To avoid blunders, take and work a sight with *concentrated attention.*

Checking observations for blunders is examined in detail in Sec. 84. We shall therefore consider only a general check of an observed position and computation checks.

Checking computations and observed positions for blunders can be handled by a number of methods, many of which are used in practice but are not considered in manuals. These methods rely on the properties of errors and on taking account of dead reckoning, and so the Marcq Saint-Hilaire method is more convenient for analysis than any others.

Given a large number of lines of position (4 or more), it is possible to effect a direct check of observations for blunders. For a smaller number, only obvious blunders can be found that exceed the dead-reckoning error.

III. TRANSFERRING DEAD RECKONING TO OBSERVED POSITION

From the foregoing it will be seen that a position determined from two position lines may include systematic errors and blunders, but it is not always possible to detect them. For this reason, a fix obtained from two lines is to be regarded as unreliable.

To increase the reliability when observing only two bodies, it is advisable to invite a *second observer* (even a third observer) who would take sights in reverse azimuth, work and plot all the sights at the same time but quite *independently* (the time being recorded from a different chronometer). Only the final results are compared and analyzed.

Coincidence of the two positions to within $\pm 1'$-$1'.2$ means an absence of blunders at all stages (with the exception of the chronometer correction; when using one instrument the first thing to note is the shift of position in longitude). Both positions are analyzed for systematic errors, which for both observers are the same to a first approximation, and random errors. These then yield the final observed position: the mid-point of these two positions or nearer one of them.

As a rule, one should not advance the dead reckoning position to an observed position obtained from two position lines. Only if 2 or 3 positions are obtained at the same time or in succession from two lines and if they are in good agreement should dead reckoning be advanced to an observed position.

If dead reckoning is transferred to an observed position, then in plotting the position the area of the possible ship's position should always be taken into account, particularly in sailing near danger.

Failure to analyze astronomical observations and an incorrect decision to transfer dead reckoning have time and again resulted in accidents.

From the foregoing it is also abundantly clear that dead reckoning is extremely important, because even though observations of two (or more) lines of position are not associated with dead reckoning, an analysis of the results of a fix is complicated without dead reckoning.

SEC. 114. ORDER OF OBTAINING AN OBSERVED POSITION FROM TWO STARS SIGHTS (INSTRUCTIONS)

Obtaining an observed position requires the following instruments and manuals: (1) sextant CH or ИМС, (2) deck watch or stop watch, (3) chronometer, (4) star globe or similar device, (5) almanac (MAE), (6) nautical tables or special tables, (7) special sight notebook or notebook, (8) plotting instrument and course chart. Also desirable is a dipmeter.

The work of obtaining a position (fix) breaks down into a number of stages and operations that we recommend to be fulfilled in the following order.

I. PREPARATION FOR OBSERVATIONS IN TWILIGHT

(1) For the presumed T_{sh} of the onset of twilight, take φ_c and λ_c from a chart.

(2) Compute T_{sh} of twilight (to within 2-3 minutes) with the aid of the MAE from φ_c and the date and obtain t_{loc}^{Υ} for this time.

(3) Using φ_c and t^{Υ}_{loc}, set the star globe and choose several pairs of stars or planets. Note the visibility of the horizon and the celestial body and abide by the conditions $\Delta A \approx 70° - 90°$ and $h < 60°$. Record h and A of the stars according to the following form.

No.	Name of body	h	A	CB

II. TAKING SIGHTS

(1) Before commencing observations: (a) prepare sextant for day or star observations and make a rapid check of its errors; (b) if a deck watch or other timepiece is available and suitable for observations, compare it with the chronometer; otherwise check the stop watch; (c) compute u_{ch} and, if necessary, u_{wat} for the instant of observations.

(2) Between 10 and 15 minutes prior to commencement of observations, take the sextant to the site of observations.

(3) Determine the index correction of the sextant. These observations may be taken after the basic ones.

(4) Measure a round of 3 (or 5) altitudes of each body and note the time by watch, stop watch or chronometer. This operation is described for a single observer in Sec. 56; it is more convenient to take those observations with an assistant. The brighter star or planet is observed in the evening at the very beginning of twilight; the fainter star, in the morning. In addition to the two basic stars, one backup star is observed to eliminate dubious results or to resolve any doubt.

(5) When observing the second body, note and record T_{sh}, the log reading, the path (or true course), speed of vessel, air temperature and pressure, and height of eye.

If stars are observed other than specified, then take their bearings after measuring the altitude.

(6) If possible measure the dip of the horizon.

III. WORKING SIGHTS

A. Computations

(1) Using the recorded time T_{sh} and lr, take φ_c and λ_c from a map to within $0'.1$.

(2) For each of the two bodies compute the mean reading and the mean instant.

Note. Do not compute each altitude and instant separately, as is sometimes done. This is no guarantee against blunders and the work is much more involved. It is better to observe more stars (4-5) or repeat the observations; and work each sight on the basis of the arithmetic mean of sextant readings (altitudes) and the instants.

(3) Compute $i + s$ for each body and correct the measured altitudes by MT.

(4) Give an approximate computation of $T_{gr} = T_{sh} \pm \mathrm{ZD}_E^W$ and the exact instants of T_{gr} for each body and date.

(5) For each body, compute t_{loc} and δ from the MAE. If t_{loc} (W) > > 180°, then convert it to t_E. If you are not sure that you have observed the specified stars, check them on a globe, utilizing t_{loc}^{Υ} obtained during the computations and the φ_c.

(6) Perform the computations of the computed altitudes h_{1c} and h_{2c}, the azimuths A_1 and A_2 and the intercepts $h - h_c$.

Check your computations; compare names of azimuth with compass bearings of the bodies or with the azimuths obtained from the globe.

(7) Reduce the first line to the zenith (position) of the second round of sights; to do this, compute Δh_Z and apply it to the first intercept, that is, $(h - h_c) + (\pm \Delta h_Z)$. The sign of the correction Δh_Z is indicated in Table 16, MT-53. The reduction may also be performed graphically, which is useful for checking.

B. Plotting sights

(1) When plotting on a chart from a computed point at the instant of the second round of sights, plot both lines as indicated above. Note the signs of $h - h_c$ ("+" towards the body, and "—" away from the body).

(2) Take the point of intersection of the lines of position (see Sec. 113) for the observed position, give it an appropriate label and write T_{sh} and lr. Enter in the log book φ_0, λ_0 and the leeway (C).

(3) When plotting on paper, obtain φ_0 and λ_0, and from them indicate the position on the course chart.

C. Analysis of an Observed Position

(1) Evaluate the possible magnitude of systematic error (Δ) and indicate the shift it may cause in the observed position along the line of mean azimuth.

(2) Evaluate or obtain the magnitude of random (mean square) error in each intercept $h - h_c$.

(3) Construct an ellipse (or rhombus) of random errors about the observed point and about the points M' and M'' of the presumed shift of this point due to systematic errors.

(4) On the chart, sketch the area of possible ship position and see if it is clear of dangers and fits into the area of presumable drift due to errors of dead reckoning.

(5) From an analysis of possible errors of dead reckoning and the obtained area of possible ship position, check for blunders or unknown errors. Note the shift in longitude.

(6) Judge about the possibility of advancing the dead reckoning to the observed point obtained.

Example 1. On 28 July, 1968, on course in Pacific Ocean with course $= 210°$ ($-1°$) at a speed of 12 knots; planned to observe stars during morning twilight; at this time, $\varphi_c \approx 32°.0$N, $\lambda_c \approx 143°0'$E (latitude taken to within 0°.1, longitude to 5′ to simplify computations of t^{Υ}_{loc}; subsequently, with globe set, all quantities are rounded to $\pm 0°.3$).

(1) Determine time of observations (see Sec. 139)

Sunrise ⊙	T_T	5h 16m (−22)
	$\Delta T_{\varphi\lambda}$	− 5
	T_{loc}	5 11
	$+$ ZD $-\lambda$	27
Sunrise ⊙	T_{sh}	5 38
Duration of nautical twilight	ΔT	1 00
Onset of nautical twilight	T_{sh}	4h 38m
Set observation time at	$T_{sh} = 5$h	

(2) Choice of stars for observations

T_{sh}	5h 28.07
$-$ ZD	10
T_{gr}	19h 27.07

t^{Υ}_{gr}	230°30′
$+\lambda$	143 00
t^{Υ}_{loc}	373°30′ ≈ 13°.5

Using star globe, we choose:

No.	Name	h	A	CB
1	Aldebaran (α Tauri)	37°.5	85°SE	96°
2	Fomalhaut (α Piscis Aust.)	21	28SW	208
3	Capella (α Aurigae)	40°	53°NE	54°
4	Rigel (β Orionis)	17 .5	69SE	112

(3) Taking sights (with a deck watch)

Chronometer comparison

$-\begin{matrix}T_{ch}\\T_{wat}\end{matrix}$	6h 15m 30s 6 3m 18
$ch-wat$	– 3m 18s

Chronometer correction was determined:

on 27.07 at $T_{gr}=12$h; $u_{ch}=+6$m 42s: $\omega=-3$s.
Reduce u_{ch} to $T_{gr}=19$h.
$\Delta T=T_2-T_1=19-12=7\text{h}=0\text{d}{:}\ 29.$
$u''_{ch}=u'_{ch}+\omega\Delta T=+64\text{m}\ 2\text{s}+(-3\text{s})\,0.29=+6\text{m}\ 41\text{s}.$

We determine the watch correction.

$$u_{wat}=u_{ch}+ch-wat=+6\text{m}\ 41\text{s}-3\text{m}\ 18\text{s}=+3\text{m}\ 23\text{s}.$$

(4) Only the first pair of stars was observed (due to visibility conditions).

Aldebaran (α Tauri)

sr	T_{wat}
37°53′.5	6h 54m 00s
38 8 .5	55 08
38 20 .5	56 07
Av. 38° 7′.5	6h 55m 05s

Fomalhaut (α Piscis Aust.)

sr	T_{wat}
22° 6′.4	6h 57m 46s
21 58.5	6– 59 –04
21 57.8	7– 00 –10
Av. 21°58′.9	6h 59m 00s

At $T_{sh}=5$h 5m; $lr=53.0$, taken from chart $\varphi_c=31°51'.5$N; $\lambda_c=143°13'.6$E. Determined $i=+2'.2$. Selected $s_1=+0'.8''$; $s_2=+0'.4''$; $e=13$ metres; $t=+20°$; $B=765$ mm.

(5) Working of sights.

I. Aldebaran

sr	38° 7′.5
$i+s$	+3 .0
h'	38°10′.5
Δ_{tot}	−7 .6
h	38° 2′.9

II. Fomalhaut

sr	21°58′.9
$i+s$	+2 .6
h'	21°61′.5
Δ_{tot}	−8 .8
Table 14a	+0 .1
h	21°52′.8

T_{wat}	6h 55m 05s	6h 59m 00s
u_{wat}	+3m 23s	+3m 23s
T_{gr}	18h 58m 28s	19h 02m 23s
t^{Υ}_{T}	215°28′.5	230°30′.9
Δt	14 39 .4	0 35 .8
t^{Υ}_{gr}	230 07 .9	231 06 .7
λ_{W}	143 13 .6	143 13 .6
t^{Υ}_{loc}	13 21 .5	14 20 .3
τ_{*}	291 28 .6	16 00 .9
t^{*}_{loc}	304 50 .1	30°21′.2W
δ_{*}	=55°09′.9E	
	16°27′.0N	29°47′.2s

I. Aldebaran

δ=16°27′.0N t=55 09 .9E	T + S	60 129 4 864
x=27°20′.0N φ=31 51 .5N	T	64 993

T − S	73 874 1 028
P + S	72 846 22 059
T	94 905

T − S	92 757 24 196
T	68 561

y = 90° + 4°31′.5 = 94°32′.5

A_0 = 86°27′.8SE = 86°.5SE

h_c	37°56′.2
h	38 02 .9
$h-h_c$	+6 .7
Δh_z	−0 .4
Intercept	+6′.3

A_c = 86°.5SE

Reduction to II zenith

A	93°.5
TC	209 .0
$A-TC$	244 .5
h_1	−0 .09′1/4 m
T_2-T_1	3/m9
h_z	−0′.4

II. Fomalhaut

$\delta = 29°47'.2S$	T +	65 879
$t = 30\ 21\ .2W$	S	1 280
$x = 33\ 33\ .4S$	T	67 159

$\varphi = 31\ 51\ .5$

T −	66 077
S	1 584
P +	64 493
S	825
T	65 318

T −	63 933
S	1 099
T	62 834

$90° + 65°24'.9 = 155°24'.9$

$A_c = 28°13'.0SW$

h_c	$21°57'.4$
h	$21\ 52\ .8$
$h - h_c$	$-4'.6$

$A_c = 28°.2SW$

(6) Plotting sights (on paper, Fig. 189).
Construct scale angle on a blank space of the drawing.

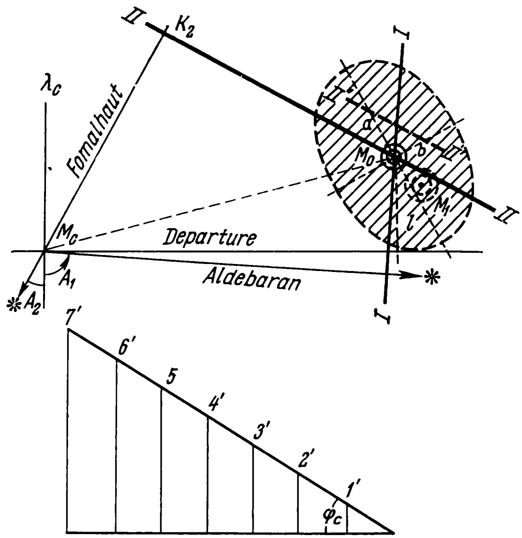

Fig. 189

(7) Observed coordinates and leeway (c) at $T_{sh} = 5h02m$, $lr = 52.8$ (corrected for $T_{sh} = 5h\,02m$).

φ_c	31°51′.5N	λ_c	143°13′.6E	
l	1′.8N	DLo	7.6E	$C = 75° - 6'.7$
φ_0	31°53′.3N	λ_0	143°21′.2E	

(8) Analysis of observed position:

From the conditions of observations and previous experience we assume that the systematic error is small. For an average out of 3 altitudes we take the random error equal to $\varepsilon_h = \pm 0'.7$. Compute $y = 0.7\varepsilon_h = \pm 0'.5$. Referring to Fig. 189 and constructing the semiaxes of the ellipse and doubling them, we obtain an area which contains the ship's position with a probability of 86.5% (roughly in the region 3.5 × 2.5 miles). The drift in direction 75° is due to inaccurate account of current after emerging from the Tsugaru Kay Kio straight. Since a leeway of about 10 miles was expected, no blunders are found.

The position was determined at the same time by a second observer who obtained it at point M_1 (see Fig. 189, dashed line), which turned out to be within one mile of M_0. We consider the observation reliable and transfer the dead reckoning to M_0.

SEC. 115. DETERMINING A SHIP'S POSITION FROM SIMULTANEOUS OBSERVATIONS OF THE ALTITUDES OF THREE BODIES (THREE STARS)

1. TRIANGLE OF ERRORS ("COCKED HAT")

Let us suppose that we have observed three bodies (instead of two) simultaneously and after working the sights have plotted three altitude lines on a chart. If the observations and their analysis are free from errors and no blunders were made, then all three lines would intersect at the point M_0 (Fig. 190) and the observed position would coincide with the actual position M of the ship. Actual conditions of work of a navigator always involve errors of observation and working sights; therefore, after plotting three lines, there will always be a so-called **triangle of errors** (or "**cocked hat**").

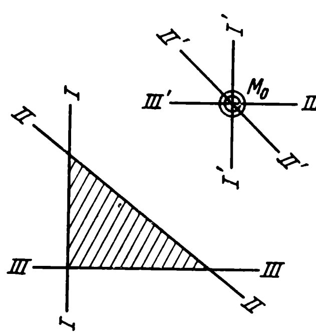

Fig. 190

The dimensions and shape of a triangle of errors differ and depend on the relationship of the errors in the position lines, possible blunders, and the azimuths of the observed bodies. Indeed, if we write the equations of three position lines subjected to the effects of systematic (Δ) and random ($\pm\delta_i$) errors and take the origin of the coordinate system in

the D.R. position, we will have (using previous designations)

$$\left.\begin{aligned}\Delta\varphi\cdot\cos A_1+\Delta\sigma\cdot\sin A_1&=\Delta h_1+\Delta\pm\delta_1\\ \Delta\varphi\cdot\cos A_2+\Delta\sigma\cdot\sin A_2&=\Delta h_2+\Delta\pm\delta_2\\ \Delta\varphi\cdot\cos A_3+\Delta\sigma\cdot\sin A_3&=\Delta h_3+\Delta\pm\delta_3\end{aligned}\right\}\qquad(20.13)$$

where Δh_1, Δh_2, Δh_3 are correct values of intercepts. The quantities on the right side operate jointly, that is, they are represented by a single number.

The fact that this system of three equations is not simultaneous is an analytical manifestation of the triangle of errors. In other words, when solving these equations in pairs we get different coordinates $\Delta\varphi$ and $\Delta\sigma$ corresponding to the vertices of the triangle. But this would not occur if all three lines had a common point of intersection.

After obtaining a triangle of errors it is of practical importance to resolve the following problems:

(a) Where should one select an observed position of the ship?

(b) How are we to evaluate with maximum assurance the area containing the actual position of the ship?

(c) Is it possible to detect blunders committed in one or several position lines?

(d) What azimuth difference between lines will be the most favourable?

To answer these questions, let us first consider separately and then jointly the effects of systematic and random errors in position lines.

II. THE EFFECTS OF SYSTEMATIC ERRORS

Let us assume that in equations (20.13) the random errors are very small, that is, $\delta_i \approx 0$, and there are no blunders. Then, from the three equations we can determine three unknowns: $\Delta\varphi$, $\Delta\sigma$ and the systematic error Δ, which we assume to be the same in all three lines. However, an analytic solution of the system of equations (20.13) is intricate and is ordinarily replaced by a graphic solution.

Above, in Sec. 112, it was established that if we bisect the angle between two position lines (in the direction of the mean azimuth), then the "difference" line obtained, or the astronomical bisector, will be free of the effects of systematic errors.

In (20.13) we can take the equations two at a time. Via manipulations similar to those performed in Sec. 112 (1), we get two equations of "difference" lines, that is, of the bisectors of the angles between the lines of position, which equations will be free of the effects of systematic errors. These readily yield the quantities $\Delta\varphi$ and $\Delta\sigma$. Two bisectors are sufficient for the construction.

The graphic solution of equations (20.13) is based on the bisector method and consists in constructing the bisectors of angles between the lines of position. Bisectors are constructed between each pair of lines; the observed position is obtained at the point of their intersection. This operation is sometimes called finding the observed point in the triangle of errors. Three equivalent procedures may be applied for constructing the bisectors of those angles that must be divided:

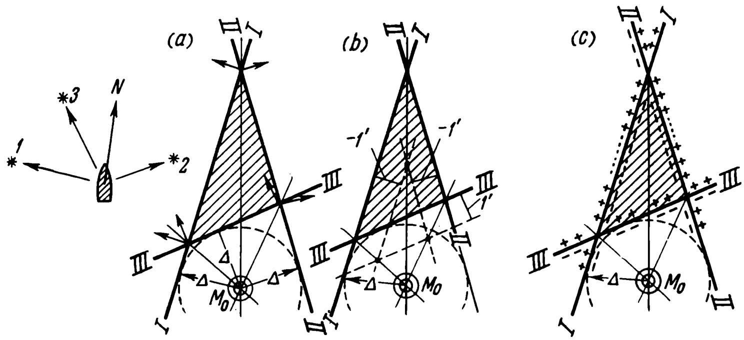

Fig. 191

(a) The construction of arrows of azimuths (at the vertices of the triangle) perpendicular to the lines (Fig. 191*a*) with subsequent division of the smaller angle between them.

(b) Displacement of all lines by an equal magnitude, say, +2', or —1', and so forth, with subsequent joining of like vertices of the triangles by straight lines (Fig. 191*b*).

(c) Labelling signs near the lines according to the principle of "plus" for "towards the star" and "minus" for "away from the star" (Fig. 191*c*).

The first procedure is recommended for construction of arrows of azimuths at the vertices of the triangle as being the more pictorial and convenient for subsequent analysis.

Irrespective of the type of triangle of errors, two cases may be encountered in the construction of bisectors and in locating the observed position:

(1) Celestial bodies located in one half of the horizon, which means that ΔA is less than 90° in each pair. In this case, the point of intersection of the bisectors (taken as the observed position M_0) will lie *outside the triangle* of errors (see Fig. 191).

Due to the fact that in real conditions random errors are superimposed on systematic errors, this procedure may lead to crude errors

in the observed position, which can be greater than in a determination based on two lines; it should therefore be used with great caution and only in cases indicated below.

(2) Celestial bodies located in different parts of the horizon, that is, ΔA in each pair is greater than 90°. In this case, the point M_0 of intersection of the bisectors lies *inside the triangle* of errors (Fig. 192).

Underlying these constructions are elementary geometrical procedures: in the first case, the construction of point M_0 amounts to finding the centre of a circle tangent to three given lines; in the

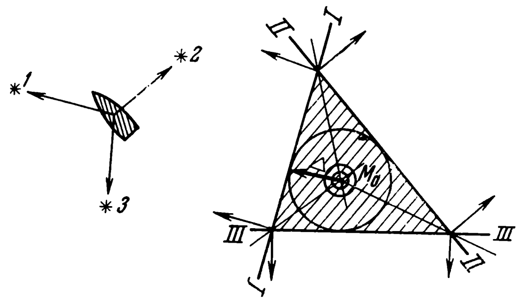

Fig. 192

second case, finding the centre of an inscribed circle. The radii of these circles are equal to systematic errors Δ. Consequently, to obtain the systematic error Δ, drop a perpendicular from the position M_0 found onto any one of the lines of position; its magnitude in the scale of intercepts (that is, distances) is equal to the error Δ. The sign of the error is determined by the direction of transfer of the line, for instance, it will be $+\Delta$ in Fig. 191.

Everything that has been said about the elimination of systematic errors holds only for the assumptions made, that is, if we consider the random errors equal to zero. For this reason, we cannot say that our constructions *exhibit* the systematic error Δ; on the contrary, the constructions themselves are a *corollary of the supposition* that only equal systematic errors are operative.

III. THE EFFECTS OF RANDOM ERRORS

The most probable position of a ship. If we assume that only random errors are operative in position lines, that is, in equations (20.13), we put $\Delta = 0$, then the values $\Delta\varphi$ and $\Delta\sigma$ obtained in the solution of these three equations may be more precise than from two equations. As we know, given superfluous observations, equations (20.13) are solved by the method of least squares. In navigation,

their analytical solution is not used and is replaced by a graphic solution, which is simpler.

The graphic solution consists in the following: inside the triangle of errors (which in this case is considered the result solely of random errors in the lines) there is a point, the sum of the squares of the distances of which from the sides of the triangle will be a minimum. This point will represent the most probable observed position of the ship (MPP). If all three lines are of equal accuracy, that

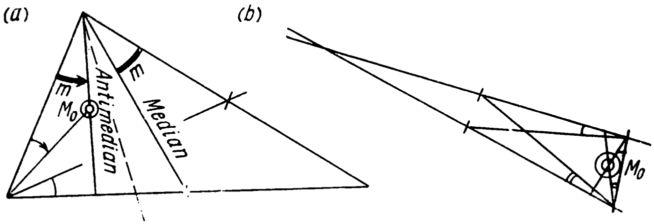

Fig. 193

is, $\varepsilon_{h1} = \varepsilon_{h2} = \varepsilon_{h3} = \varepsilon_h$, the most probable position will be located at the point of intersection of the antimedians (countermedians) of the triangle.

The antimedian is a mirror image of the median in the bisector and is constructed as follows. Joining a vertex of the triangle with the midpoint of the opposite side, that is, constructing a median (Fig. 193), we mark the smaller angle m between the median obtained and the side of the triangle; taking this angle on the other side and constructing a straight line, we get the antimedian. Only two antimedians are needed to find the most probable position.

From Fig. 193*b* (after construction of the antimedians) it will be seen that as the triangle of errors is increased in length the position is not displaced towards the middle, but towards the right angle and the short side (there is proof in error theory that the closer angle ΔA of the intersection of two lines is to 90°, the greater the weight, or reliability, of the point of their intersection).

If all three lines intersect at an angle of 60°, the most probable position will be in the centre of the triangle.

Evaluating the area of the probable ship position; the most favourable conditions of observation. An ellipse of errors should also be used to evaluate the area of the probable position of the ship in the case of three lines of position, but a rigorous construction is impracticable. We shall therefore confine ourselves to constructing a circle of errors with radius $\rho = \varepsilon_{loc} = \pm 1.5\varepsilon_h$. The most favourable conditions of the locations of celestial bodies will be: $\Delta A = 120°$ in each pair of lines.

IV. A PRACTICAL ANALYSIS OF THE DETERMINATION OF POSITION FROM "THREE STARS"

(a) Choice of Observed Position

On the assumption that the triangle of errors is a consequence of systematic errors, the position is obtained at the point M_0' of intersection of bisectors of the appropriate angles (see Fig. 194). But if we assume that the triangle is obtained due to random errors, the probable position will be at point M_0'' of the intersection of antimedians. These points coincide only when $\Delta A = 120°$ in each pair, but in the general case, points M_0' and M_0'' do not coincide.

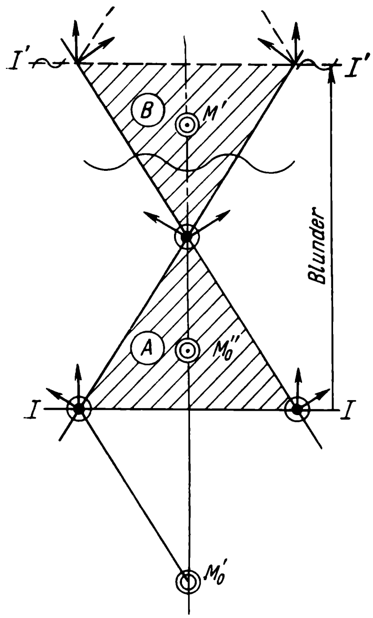

Fig. 194

For both positions to be inside the triangle, choose bodies in different parts of the horizon. Then the observed position may be taken between the points M_0' and M_0'' and will approximately satisfy both conditions.

But if stars are observed in one part of the horizon ($\Delta A < 90°$) then the point M_0'' (due to random errors) will be inside the triangle, and the point M_0' (due to systematic errors) will be outside it.

In this case, choice of place is difficult. An indirect and extremely unreliable criterion is the size of the triangle of errors. If the sides of the triangle exceed 3 to 5 miles, it is believed that constant errors are operative and the position is taken at M_0'. But if the sides are less than this quantity, the assumption is that random errors are operative and the position is taken at M_0''. It is risky to apply this criterion.

In this case, it is best to take into account both points and the corresponding area about them.

In addition to the indicated errors, blunders are also possible (for example, the line $I' - I'$ in Fig. 194). For this reason, when sailing towards dangerous regions, in addition to the positions M_0' and M_0'', *all vertices of the triangle* are taken into account (with the exception of very acute angles), and the *observed position is chosen at the point that lies closest to danger.* At this point we construct an area of the probable position of the ship.

(b) Constructing the Area of Probable Position of the Ship

As indicated above, we can roughly take $\rho = \pm\ 1.5\ \varepsilon_h$ for the radius of a circle of errors due to random errors.

(c) Detecting Blunders

For three lines, large blunders are revealed in a manner similar to that for two lines: from the magnitudes of the intercepts.

Only in certain cases can we judge of blunders of smaller magnitude. If from two lines the position was obtained within expected limits, and the third line goes sharply to the side, then it may have a blunder. If the dimensions of the triangle of errors substantially exceed the expected errors of observation, a blunder may have been made in one of the lines.

(d) Transferring the Dead Reckoning

From the foregoing it follows that a determination from three stars is not fully reliable, particularly as concerns blunders, although it is more accurate than from two stars. Therefore, to improve reliability, it is best to have a second observer. If both observers obtain positions, when observing stars in different parts of the horizon ($\Delta A \approx 120°$), that agree within the limits of random errors, the observation may be considered reliable and the dead reckoning may be advanced to a position inside the triangle with account taken of the appropriate area. But if there is only one observation from three lines, it is best to choose a position closer to danger with account taken of the same area, if it is necessary to transfer dead reckoning.

(e) Remarks on Taking Observations

In the general case, three celestial bodies may be observed simultaneously only in twilight or sometimes at night. For night sights of stars (under ordinary conditions) only the ИАС-1 sextant is suitable. The CH sextant may be used above the visible horizon on very clear bright nights. In some cases, soon after sunrise or just before sunset it is possible to observe Venus, the moon and the sun, or we can add to two position lines a navigational line of position. Most common is the general case of observing stars and planets in twilight.

Observations of three stars are carried out in the same way as of two stars; the only thing is that the differences of azimuths between each pair of bodies must, as we know, be taken roughly equal to 120°.

It is also advisable to take three altitudes of each star, the T_{sh} and log being noted at the last observation; in addition to the basic stars, it is best to observe one or two additional stars in case of doubt when working the sights. So as to have time to take the observations during twilight, it is absolutely necessary to select the stars and planets with the aid of a star globe and to record their A and h in accord with the scheme shown for two stars. Begin observations immediately after sunset as stated for two stars. When working sights, the first and second altitudes are reduced to the third zenith analytically, since the graphic procedure is not convenient. However, in plotting, it is advisable to check by eye the signs and magnitudes of Δh_Z on the drawing to avoid slight blunders.

Example 2. On 12 September, 1968, a ship on course in the Atlantic Ocean, heading 241° (+2°) with a speed of 13 knots. The plan is to observe three stars in evening twilight.

(1) Determining time of observations.

At $T_{sh} \approx 18$h; $\varphi_c \approx 30°.0$N; $\lambda_c = 62°4'$ W.

Sunset $\odot$ T_T $\Delta T_{\varphi,\lambda}$	18h 10m 00
T_{loc} $(-ZD+\lambda)$	18 10 + 9
Sunset $\odot$ T_{sh} Duration of nautical twilight ΔT	18 19 52
End of nautical twilight T_{sh}	19h 11m

(2) Choice of stars.
We set time of observations at $T_{sh} = 18$h 25m.

T_{sh} $+$ ZD_W	18h 25m 12.09 4
T_{gr} t^{Υ}_{T} Δt	22h 25m 12.09 321°.9 6 .3
t_{gr} $-\lambda$	328°.2 62 .4
t^{Υ}_{loc}	265°.8

From the globe we get:

Celestial body	h	A	CB
α Scorpii	30°	18°SW	196°
α Cygni	52	53 NE	51
η Ursae Majoris	41	50 NW	308

(3) Taking sights (three altitudes and instants each).

	Antares (α Scorpii)	Deneb (α Cygni)	Benetnasch (η Ursae Majoris)
Av. T_{ch}	10h 24m 18s	10h 28m 41s	10h 32m 30s
Av. h_r	30°3′.8	54°30′.7	39°51′.5

At T_{sh}=18h 40m; lr=38.5; φ_{sh}=30°17′N; λ_{sh}=62°15′.5W; u_{ch} = +6m 57s; i=+1′.6; s_1 = −0′.5; s_2= −0′.7; s_3= −0′.3; e=13 metres.

(4) Working the sights

T_{sh}	18h 40m
ZD_W	4
T_{gr}	22h 40m 12.09

	I α Scorpii	II α Cygni	III η Ursae Majoris
T_{ch}	10h 23m 18s	10h 28m 41s	10h 32m 30s
u_{ch}	+6 57	+6 57	+6 57
T_{gr}	22h 31m 15s	22h 35m 38s	22h 39m 27s
t_T^{Υ}	321°57′.8	321°57′.8	321°57′.8
Δt	7 50 .0	8 56 .0	9 53 .4
t_{gr}^{Υ}	329°47′.8	330°53′.8	331°51′.2
— λ_W	62 15 .5	62 15 .5	62 15 .5
t_{loc}^{Υ}	267°32′.3	268°38′.3	269°35′.7
τ_{*}	113 07 .9	49 53 .4	153 25 .6
t_{loc}^{*}	380°40′.2	318°31′.7	423°01′.3
$t_{loc\ practic}$	20°40′.2W	41°28′.3E	63°01′.3W
δ	269°22′.0S	45°10′.2N	49°28′.3N

Perform computations by tables BAC-58

I

	h_T	30°29'.8	A_T	158°.0	
$\varphi = 30° + 17.0$N $\delta = 26° + 22.0$S $t = 21° - 19.8$W	Δh_φ Δh_δ Δh_t	—15'.8 —20 .5 +6 .3	ΔA_φ ΔA_δ ΔA_t	+0°.1 +0 .2 +0 .3	$q = 159$
	h_c	29°59'.8	ΔA_c	158°.6NW = 201°.4	

II

	h_T	54°39'.5	A_T	53°.3	
$\varphi = 30° + 17'.0$N $\delta = 45° + 10'.2$N $t = 41° + 28'.3$E	Δh_φ Δh_δ Δh_t	+10'.1 —1 .9 —19 .6	ΔA_φ ΔA_δ ΔA_t	+0°.3 —0 .3 +0 .1	$q = 101$
	h_c	54°28'.1	A_c	53°.4NE	

III

	h_T	39°27'.6	A_T	48°.5	
$\varphi = 30° + 17'.0$N $\delta = 49°30' - 1'.7$N $t = 63' + 1'.3$W	Δh_φ Δh_δ Δh_t h_c	+11'.2 0 .0 —0 .8 39°38'.0	ΔA_φ ΔA_δ ΔA_t A_c	+0°.2 +0 .0 0 .0 48°.7NW	$q = 88$

Reduction to III zenith

A_1 TC	201°.4 243		A_2 TC	53°.4 243 .0
$A - TC$ h_1 $T_3 - T_1$	318°.4 +0 .16'/min 8 .2 min		$A - TC$ Δh_1 $T_3 - T_2$	170°.4 —0 .22'/min 3 .8 min
Δh_Z	+1'.3		Δh_Z	—0'.8

	I	II	III
av. *sr*	30°03′.8	54°30′.7	39°51′.5
$i+s$	+1 .1	+0 .9	+1 .3
h_1	30 04 .9	54 31 .6	39 52 .78
Δ_{tot}	−8 .1	−7 .1	−7 .5
h	29 56 .8	54 24 .5	39 45 .3
Δh_Z	+1 .3	−0 .8	—
h_0	29 58 .1	54 23 .7	39 45 .3
h_c	27 59 .8	54 28 .1	39 38 .0
h_0-h_c	−1′.7	−4′.4	+7′.3

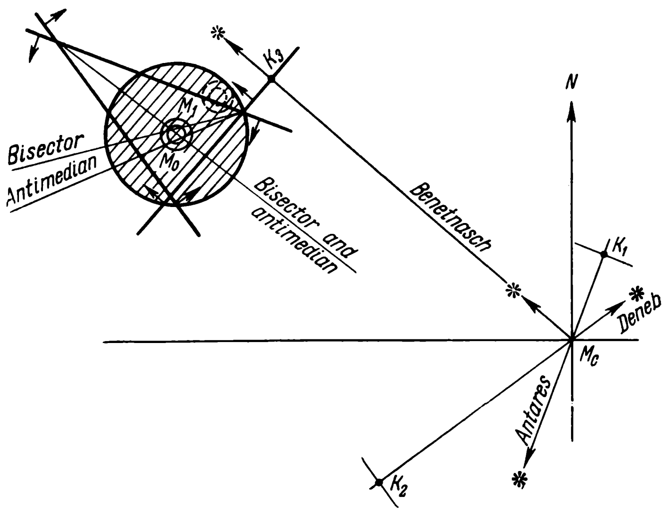

Fig. 195

(5) Plotting on chart (Fig. 195)

$T_{sh}=18\text{h}\ 41\text{m}$, $lr=38.7$

φ_c	30°17′.0N	λ_c	62°15′.5W	(*Dep.*=7′.3W)
l	3 .8N	*DLo*	8 .4W	
φ_0	30°20′.8N	λ_0	62°23′.9W	

$C=298°-8'.3$

(6) Analysis of determination.

The observed position is chosen at point M_0; the intersection of the antimedians. Constructing bisectors for a check, we see that their intersection is close to M_0 in the given case. We compute the radius of the circle of errors. The conditions were good: clear horizon, the dip measured before sunset showed it to be close to the tabular value. On the basis of previous experience we take $\varepsilon_{h1} = \pm 1'.2$, on the average $\varepsilon_h = \frac{\varepsilon_{h1}}{\sqrt{3}} = \pm 0'.7$.

In the given case, the azimuth differences are close to $\Delta A \approx 120°$, therefore $\rho = 1.3 \cdot \varepsilon_h = \pm 0'.9$. The real position of the ship lies within the circle of this radius. At the same time, the second observer obtained M_1 from two stars. No blunders were detected. The leeway of 8 miles is due to an incorrect account of the favourable current and drift. Taking these factors into consideration we transfer dead reckoning to the point M_0, $T_{sh} = 18\text{h}\,41\text{m}$, $lr = 38.7$.

SEC. 116. DETERMINING A POSITION FROM FOUR STARS

I. A "QUADRANGLE" OF ERRORS

The intersection of four altitude (position) lines plotted on a chart always yields a figure of errors which we shall term a **quadrangle of errors**. When the celestial bodies are located in different parts of the horizon, this figure is indeed a quadrangle; but if all the four bodies are located in one part of the horizon, the figure of errors may not be a quadrangle. A quadrangle of errors is due to the effects on position lines of systematic and random errors, and sometimes of blunders.

II. ELIMINATING SYSTEMATIC ERRORS

From four equations of position lines, similar to (20.13), we can get two independent bisectors free of systematic errors Δ.

At their point of intersection we have the observed position of the ship, which is also free from systematic errors. Since these errors operate jointly with random errors, the position of the bisectors does not correspond to their position $\Delta = 0$, but errors due to displacement of bisectors will be taken into account in subsequent analysis of random errors.

Every bisector expressed by the equation (20.7) will be as close to the position $\Delta = 0$ as the azimuth difference in the given pair of lines is to 180°. This is evident from equations (20.12). For this reason, the bisectors are drawn between lines with ΔA closest to 180°. To do this, construct at the vertices of the quadrangle arrows parallel to the azimuth directions of the appropriate lines, and choose arrows directed "out" or "into" the quadrangle.

The direction of the arrows in pairs depends on the sign of the systematic (recurring) error and should be the same in both pairs if there is such an error.

The closer the angle between the bisectors is to 90°, the better the observed position. Choice of lines for drawing bisectors and locating observed position are shown in Fig. 196 for two quadrangles

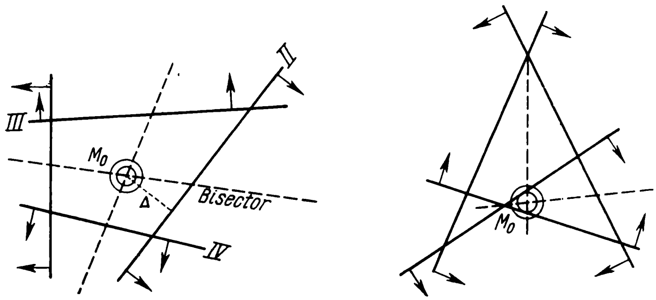

Fig. 196

of errors. The magnitude of the systematic error Δ and its sign may be obtained by dropping a perpendicular onto one of the lines of position or by taking the mean of the lengths of four perpendiculars.

III. THE EFFECT OF RANDOM ERRORS

If all lines are considered of equal accuracy, the random errors of bisectors are expressed by the formula (20.12). The mean error of their point of intersection may be obtained from (20.9) for two lines if we take ε_{h1} and ε_{h2} as the bisector errors (ε_b), that is, if we reduce the problem to two position lines:

$$\varepsilon_{loc} = \frac{\sqrt{\varepsilon_{b1}^2+\varepsilon_{b2}^2}}{\sin\gamma} = \pm\frac{\varepsilon_h}{\sqrt{2}\sin\gamma}\sqrt{\frac{1}{\sin^2\frac{\Delta A_1}{2}}+\frac{1}{\sin^2\frac{\Delta A_2}{2}}} \quad (20.14)$$

where γ is an acute angle of intersection of the bisectors ΔA_1 and ΔA_2 are the azimuth differences of two pairs of stars.

For $\Delta A_1 = \Delta A_2 = 180°$ and $\gamma = 90°$ we have $\varepsilon_{loc} = \varepsilon_h$; which means that for the most favourable conditions the ship's position will be most accurately determined, the accuracy being equal to the error in a single line.

If all four stars were observed in one part of the horizon, the bisectors will be found with much lower accuracy and their angle of intersection will be much less than 90°, thus resulting in a less accurate observed position. It is therefore not advisable to observe four stars in one part of the horizon and take such sights.

To estimate the area of the probable ship's position, one can apply an ellipse of errors constructed for bisectors as for two lines (see

Sec. 112). However, for practical purposes it will be sufficient to construct a circle of errors of radius $\rho = \varepsilon_{loc}$, taking $\varepsilon_{loc} = \pm\varepsilon_h$ or, under worse conditions of intersection of bisectors, to take $\rho = \varepsilon_{loc} \approx \pm 1.2\varepsilon_h$.

IV. DETECTING BLUNDERS

As shown above for two lines of position, big blunders in each line are found from an excessive intercept.

Smaller blunders, in the case of four lines, may be detected via the following reasoning:

1. If the same blunder is made in all four lines, for instance, when correcting altitudes, it will be eliminated as a repeating error in the construction of the bisectors.

An exception is an identical error in the chronometer correction or identical blunders in times or hour angles, as a result of which

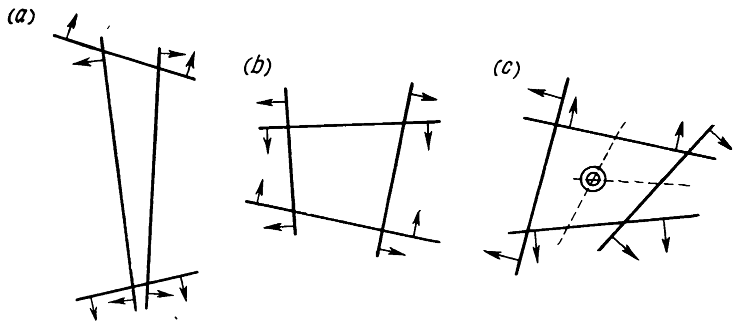

Fig. 197

the observed position will be displaced in longitude by the amount of the error. It is this displacement that will indicate an error in hour angle or chronometer correction.

2. As a rule, blunders are admissible in one or two lines but not in all lines. In such cases, it is necessary to examine the magnitude and form of the quadrangle of errors and the direction of the azimuth arrows constructed at its vertices (Fig. 197).

(a) If the size of the quadrangle in the direction of one pair of lines is excessive, one of these lines might contain a blunder (Fig. 197*a*).

(b) If the sides of the quadrangle are approximately (to within random errors) equal, and the azimuth arrows are in the same direction relative to the observed position (that is, all "in" or all "out", see Fig. 197*c*), then there will obviously be no blunder in any of the lines.

In the given case the arrows indicate whether a general systematic shift has occurred in all lines "towards the celestial body" or "away from the body" by approximately the same amount Δ.

(c) If the direction of arrows is correct and the form of the quadrangle is incorrect, then there is either a predominance of random errors, or there is a small blunder in one of the lines.

(d) But if the arrows in one pair of lines go in, while in another pair they go out, and the quadrangle is not small, it may be that there is a blunder in one of the lines (Fig. 197*b*).

To determine in which of the lines a blunder has been made in cases (*c*) and (*d*), check computations or work a fifth (check) line and plot it. This line will indicate which line is in error.

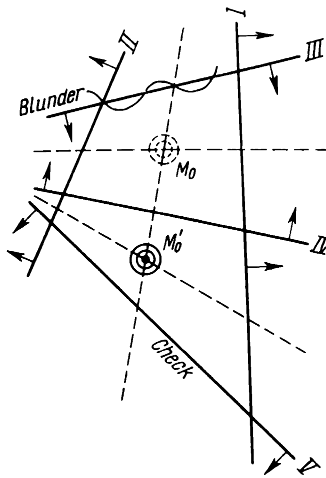

Fig. 198

Indeed, in Fig. 198 the arrows near lines *I* and *II* indicate a shift towards the celestial body (out), and near lines *III* and *IV*, away from the body (in). Which means that a blunder has been made somewhere. Plotting the fifth line, observed at an intermediate azimuth, we find that it likewise yields a shift outwards along with the fourth line. We conclude that a slight blunder has been made in line *III*; either disregard this line or check the calculation. The observed position will be at M_0' instead of M_0.

V. TRANSFERRING THE DEAD RECKONING

If there are no discrepancies after plotting the lines and analyzing for blunders, then the observed position is, for all practical purposes, taken at the point of intersection of the lines connecting the midpoints of the opposite sides of the quadrangle*. The position obtained is quite reliable, and the dead reckoning may be transferred to the observed point. However, as before, the ship's position should be regarded as being located in the area—for example, of a circle of errors circumscribed about the observed position with radius $\varepsilon_{loc} \approx$ 1-1.2 ε_h.

* A more exact probable position is obtained if we take into account the weights of all points of intersection of the four lines (there are six in all).

VI. PRACTICAL SUGGESTIONS FOR MAKING FOUR STAR SIGHTS

To obtain a fix from four position lines, first choose four stars on a star globe in pairs with an azimuth difference of about 180° in each pair and 90° between pairs (Fig. 199). An additional "reserve" star is selected for a "check" at intermediate azimuth (its designation is explained in the analysis of the fix). It is best to take sights with an assistant so as to have time to measure three altitudes each

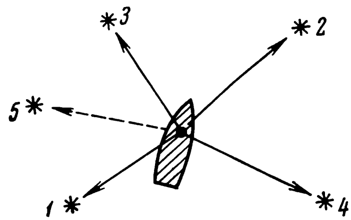

Fig. 199

of the four basic stars and of the check in twilight. It is advisable to observe one or two of the brightest stars or planets during civil twilight, when they are visible only in the sextant telescope. This extends convenient observation time and increases the accuracy of altitude measurements. When working the sights of four stars (the fifth is not worked), reduce the lines to the zenith of the last observations analytically, using Table 16, MT-53.

Determining a ship's position from four position lines is the most reliable and precise method of nautical astronomy. Five and more lines are never used in navigation.

Example 3. On 5 April, 1968, in the Indian Ocean, we desire to observe four stars during morning twilight; $\varphi_c \approx 32°.5S$; $\lambda = 35°15'E$; ZD = 2E.

(1) Nautical twilight found to begin at $T_{sh} = 5h$; sunrise at 5h 53m. We set observation time at $T_{sh} = 5h\ 15m$. For this time, $t^{\Upsilon}_{loc} = 277°$. Select four stars.

	α Pisc. Austra.	α Aquilae	α Crucis	α Virginis
h	34°	44°.5	27°	17°
A	73°SE	27°NE	30°SW	88°SW

(2) Observations

av. T_{ch} av. sr	3h 16m 15s 33°46′.1	3h 19m 05s 44°45′.6	3h 22m 31s 27°37′.5	3h 26m 12s 15°25′.18
Corrections	$i+s_1=2'.1$	$i+s_2=+2'.3$	$i+s_3=+1'.9$	$i+s_4=+1'.6$

$u_{ch}=-3$m 42s; $e=10$ metres; $t°=+25°$C; $B=758$ mm; $T_{sh}=5$h 25m; $\varphi_c=32°31'$S; $\lambda_c=35°23'$E; $TC=210°$; velocity $V=15$ knots.

(3) Working of sights from numerical tables yields:

I		II		III		IV
$A=73°.6$SE		$A=27°.7$NE		$A=30°.7SW$		$A=86°\ 7$SW
$h-h_c$	$4'.7$	$h-h_c$	$+0'.6$	$h-h_c$	$-2'.6$	
Δh_Z	$-0\ .7$	Δh_Z	$-1\ .8$	Δh_Z	$+0\ .9$	
$(h-h_c)_{red}$	$+4'.0$	$(h-h_c)_{red}$	$-1'.2$	$(h-h_c)_{red}$	$-1\ .7$	$h-h_c=-7'.0$

(4) The plotting is shown in Fig. 200.

(5) *Analysis of the observed position.* The original plotting yielded the cross-hatched quadrangle A. An analysis revealed that the azimuth arrows on lines II'

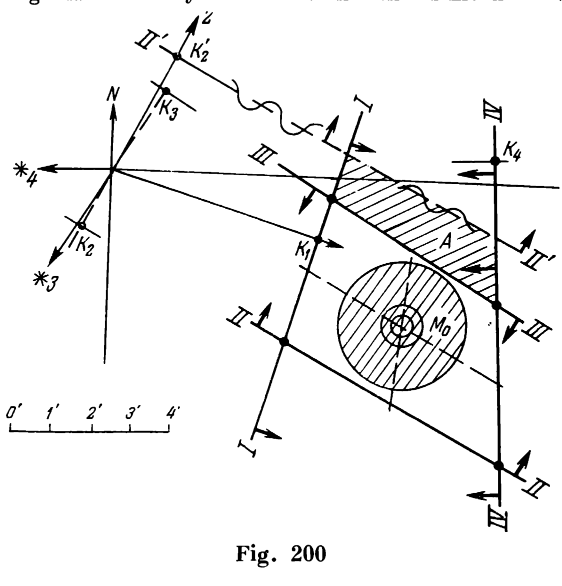

Fig. 200

and III are directed outwards from the quadrangle, and inwards on lines I and IV. Judging from the shape of the quadrangle, we assume that a blunder has

been made in line II' or III. A check reveals that a blunder was made when reducing to a single zenith: $+1'.8$ was taken instead of $-1'.8$. Rectifying the blunder, we get $h - h_c = -1'.2$ (instead of $+2'.4$) and construct line II. Now all arrows point in, which indicates, in addition to random errors, a systematic error with minus sign of magnitude about $1'.6$. Connecting the midpoints of opposite sides of the quadrangle, we obtain the observed position M_0. To construct the area of the probable ship's position we take $\varepsilon_h = \pm 1'.0$ as the arithmetic mean of three altitudes; on the basis of formula (20.14) we have $\rho = \varepsilon_{loc} = \pm 1.2 \cdot \varepsilon_h = \pm 1'.2$; we use this radius to construct an error circle with centre at M_0. We transfer dead reckoning to M_0. The results of the fix are: $T_{sh} = 5\text{h}\ 22\text{m}$, $lr = 73.5$, $l = 2'.7\text{S}$, *departure* $= 5'.4\text{E}$; $DLo = 6.5\text{E}$

$$\varphi_0 = 33°33'.7\text{S}$$
$$\lambda_0 = 35°29'.5\text{E}$$
$$C = 116° - 6'$$

SEC. 117. MAKING A RUNNING FIX FROM TWO ALTITUDES OF ONE CELESTIAL BODY (SUN) MEASURED AT DIFFERENT TIMES

It often happens at sea that there is no possibility of taking the altitudes of two bodies at the same time. For instance, in daylight only the sun is ordinarily visible, and only occasionally together with the moon or Venus.

This brings up the necessity of determining a ship's position from a single body, mainly the sun. But one altitude of a body yields only one line of position, whereas a fix requires at least two lines with an azimuth difference of 60° to 90°. To obtain two lines in this case the body is sighted at different times (double sights). Indeed, the place of a body (sun) changes relative to the meridian and zenith of the observer (that is, its altitude and azimuth change) due to diurnal motion. For this reason, if after obtaining the first line I-I (Fig. 201) drawn on a globe with a radius $90° - h_1$ from the subsolar point s_1 we wait some time, the subsolar point will move along the arc $s_1 s_2$, thus changing the azimuth of the sun and the radius of the circle of equal altitudes ($90° - h_2$). A second line II-II appears on the globe or chart that does not coincide with the first. If the vessel has not changed position (at anchor, for example), the position will be at M_0, the point of intersection of these two lines.

But if in the general case we regard observations on a moving ship, the latter will, during this time, move over the earth a distance of, say, $M_{c1}, M_{c2} = S$ (Fig. 202). The altitudes obtained, h_1 and h_2, are then measured from different places on the earth, and the lines I-I and II-II will also refer to different D.R. positions M_{c1} and M_{c2}. The first coordinates h'_c and A'_c will be obtained from the computed astronomical triangle constructed for the zenith of M_{c1}

and the place of the sun s_1 (from coordinates φ_{c1}, λ_{c1}, δ_1, t_1); now the second set of quantities h_c'' and A_c'' will be obtained from the second astronomical triangle constructed for the zenith of M_{c2} and the place of the sun s_2 (from the coordinates φ_{c2}, λ_{c2}, δ_2 and t_2).

It is obvious that lines *I* and *II* cannot be combined until they have been reduced to a single place (zenith). In practice, this reduction is executed by a graphical technique that is based on the principle of cross-bearing and that stems from the following: during the first set of sights the ship was located somewhere on the first

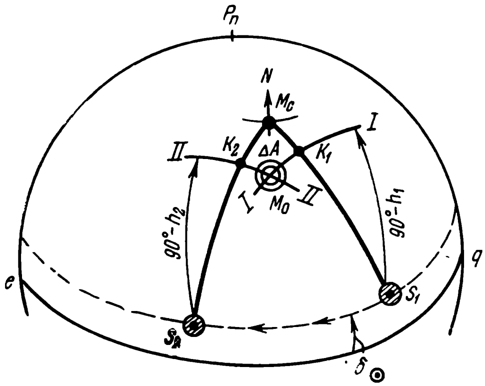

Fig. 201

line *I*; during the second set of sights, on the second line *II*. Between sights, the ship's run was *S* (Fig. 202), which means that it passed from line *I* to line *II*. Obviously, to comply with these conditions, the run *S* should fit in between lines *I* and *II* as shown in Fig. 203 for plotting on a chart. Then point M_0 on the second line will satisfy both conditions and will correspond to the point of intersection of the two position lines, which means that it will represent the observed position of the ship at the second instant. Instead of fitting segment *S* between lines *I* and *II*, we can move the line *I-I* by the distance run, parallel to the original position into the new position *I′-I′*. Finally, as will be seen from Fig. 203, we can simply lay down the first line from the second D.R. position, using its elements $h - h_c$ and A_c obtained from the first position, which is the same as moving line *I* the distance run *S*.

The observed position of the ship will be obtained at point M_0, at the intersection of lines *II-II* and *I′-I′*. Since the first line moved both in course and distance run, it will obviously include dead-reckoning errors. For this reason, the method of observations at different times actually yields a computed-observed position, simi-

lar to the cross-bearing and called a running fix, and not an observed position. However, this position is usually called an observed position and is designated by a double circle like any other astronomical observation. Carrying forward the first line by the distance run is analogous to reducing it to a second zenith and could be solved analytically using formula (15.6) or Table 16, MT-53. In this

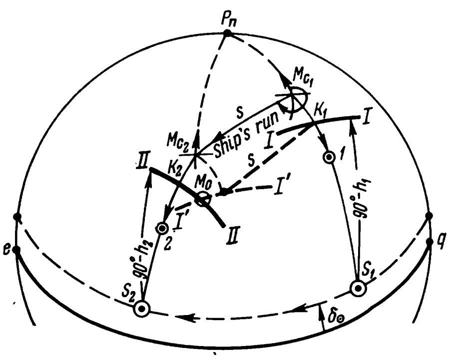

Fig. 202

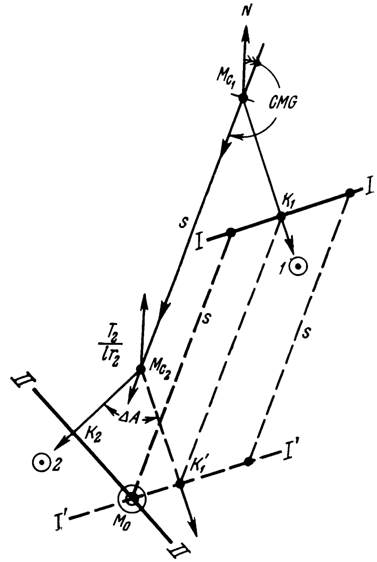

Fig. 203

case, we could work the sights with the second set of computed coordinates and introduce into the first observed altitude the correction Δh_Z for reduction to the second zenith, in exactly the same way as for two-star fixes. However, when the distance run is considerable, the first term of the series expansion of Δh_Z, that is $S \cos (A - K)$, may prove insufficient, as was shown in Sec. 81, and the reduction will be erroneous; what is more, the first line is usually worked and laid as soon as it is obtained so as to speed up working and checking the computation. As a result, in this case the analytical method of reduction to the zenith is not applicable.

Any celestial bodies are suitable for obtaining a running fix via observations at different times—the moon, planets or stars. But under practical conditions it is done only in the daytime using the sun.

The correctness of the graphical transfer of the first line to the zenith of the second set of observations in course and sailing (distance run) may be proved analytically from equations (20.1), which in this case (errors disregarded) will take on the form

$$\left.\begin{aligned}\Delta\varphi_1\cdot\cos A_1+\Delta\sigma_1\cdot\sin A_1&=\Delta h_1+\Delta h_Z=\Delta h_1+S\cos(A-K)\\ \Delta\varphi_2\cdot\cos A_2+\Delta\sigma_2\cdot\sin A_2&=\Delta h_2\end{aligned}\right\}\quad(20.15)$$

The quantity $\Delta h_Z = S\cdot\cos(A_1 - K) = S\cdot\cos K\cdot\cos A_1 + S\cdot\sin K\cdot\sin A_1$. But by the formulas of short-distance sailing $S\cdot\cos K = l$, $S\cdot\sin K = Dep.$ and so $\Delta h_Z = l\cdot\cos A_1 + Dep.\cdot\sin A_1$. Substituting (20.15) in the first equation, we get

$$(\Delta\varphi_1 \pm l)\cos A_1 + (\Delta\sigma_1 \mp Dep.)\sin A_1 = \Delta h_1$$

or

$$\Delta\varphi_2\cdot\cos A_1+\Delta\sigma_2\cdot\sin A_1=\Delta h_1$$

Adding the second equation of (20.15), we get a system, the solution of which yields $\Delta\varphi_2$ and $\Delta\sigma_2$, which are corrections to the second set of computed coordinates.

SEC. 118. THE EFFECT OF ERRORS ON A POSITION OBTAINED FROM DIFFERENT-TIME SUN SIGHTS (RUNNING FIX)

When obtaining a fix from two lines obtained at different times, examination of errors is complicated by the fact that the lines obtained have different errors. Indeed, the first line contains systematic and random errors of observation and their working. Besides it contains errors due to displacement in course by the distance run, that is, due to reduction to the second zenith (Δh_Z). Differentiating formula $\Delta h_Z = S\cdot\cos(A - K)$ with respect to S and K and equating the differentials to the increments, that is, $d\Delta h_Z = \Delta_Z$; $dS = \Delta S$ and $dK = \Delta K$, we get

$$\Delta_Z=\Delta S\cdot\cos(A-K)+\Delta K\cdot S\sin(A-K)\qquad(20.16)$$

From this formula it is evident that errors ΔS and ΔK in distance run and in course will cause an additional error Δ_Z of the first line, thus increasing the errors in its position. But the second line will have only the customary errors of observation and working.

I. RANDOM ERRORS

Error in the first line. The mean square error ε_1 in the first line, after its advance, will depend on the error ε_{h1} in the first difference of altitudes and dead-reckoning errors between the first and second

sights. The mean square error in the first line due to dead reckoning errors may be obtained from formula (20.16), taking the errors Δ_Z, ΔS and ΔK to be random and describing them by the mean square values $\varepsilon_{\Delta Z}$, ε_S and ε_K (or $\varepsilon_{C\ MG}$), i.e.,

$$\varepsilon_{\Delta Z}^2 = \varepsilon_S^2 \cdot \cos^2 q + \varepsilon_K^2 \cdot S^2 \cdot \sin^2 q \tag{20.17}$$

where $q = A_1 - K$ is the course angle towards the sun.

From (20.17) it is seen that the displacement $\varepsilon_{\Delta Z}$ of the first line depends on errors in the run and course and on the position of the first line relative to the ship's course. For instance, if q is about 90°, that is, the first line is roughly parallel to the course, then the effect of errors in the run is reduced ($\cos q \approx 0$).

The overall error ε_1 in the first line is obtained by the laws of random errors

$$\varepsilon_1 = \pm \sqrt{\varepsilon_{h1}^2 + \varepsilon_{\Delta Z}^2} = \pm \sqrt{\varepsilon_{h1}^2 + \varepsilon_S^2 \cdot \cos^2 q + \varepsilon_K^2 \cdot S^2 \cdot \sin^2 q} \tag{20.18}$$

From (20.18) it is evident that the mean error ε_1 in the advanced line I' has increased due to errors of dead reckoning as a result of which the "band of the position" about line I', which contains the probable position of the ship, has been substantially expanded.

The mean error of an observed position. The second position line is also affected by random errors in the difference $h - h_c$, which are characterized by the quantity ε_{h2}. This gives rise to its own "band of position" for the ship.

The mean square error in the observed position is obtained from the above-derived formula (20.9) for two lines of position in which ε_1 obtained from (20.18) is taken in place of ε_{h1}. Then

$$\varepsilon_{loc} = \frac{\sqrt{\varepsilon_1^2 + \varepsilon_{h2}^2}}{\sin \Delta A} = \frac{\sqrt{\varepsilon_{h1}^2 + \varepsilon_{h2}^2 + \varepsilon_{\Delta Z}^2}}{\sin \Delta A} \tag{20.19}$$

If we take it that the errors of observation and working are about the same in both lines, that is, $\varepsilon_{h1} \approx \varepsilon_{h2} = \varepsilon_h$, we finally get

$$\varepsilon_{loc} = \frac{\sqrt{2\varepsilon_h^2 + \varepsilon_{\Delta Z}^2}}{\sin \Delta A} \tag{20.20}$$

where ΔA is the difference of azimuths of the first and second lines.

Geometrically, the mean square error ε_{loc} in the observed position is the radius ρ of a circle of errors inside which (given the absence of systematic errors) we should find the actual ship's position with a probability of about 63-68% (see Fig. 206).

If in formula (20.20) we take $\varepsilon_h = 0$, then we have

$$\varepsilon_{loc} = \pm \varepsilon_{\Delta Z} \cdot \frac{1}{\sin \Delta A} = \sqrt{\varepsilon_S^2 \cdot \cos^2 q + \varepsilon_K^2 \cdot S^2 \cdot \sin^2 q} \cdot \frac{1}{\sin \Delta A} \tag{20.21}$$

Expression (20.21), which is usually given in courses of nautical astronomy in a somewhat different form (without account taken of the course angle of line *I*), indicates possible random displacements of the observed position along the second line of position due to errors in dead-reckoning advancement of line *I*, without account taken of the errors of the lines themselves.

From (20.20) it is seen that position errors depend on:

(a) random errors in the position lines;

(b) errors of dead reckoning;

(c) the position of line *I* relative to the course;

(d) the difference of azimuths of the position lines.

The quantity $\frac{1}{\sin \Delta A}$ will be least for $\Delta A = 90°$; however, from 4 to 6 hours is needed to change the sun's azimuth by this amount in middle latitudes. During which time the ship will cover a distance of 50 to 100 miles and the dead-reckoning errors may become appreciable (from 2 to 7 miles under average conditions). This will make the advanced first line extremely inexact.

For this reason, it is desirable to have the time interval between observations as small as possible. In the general case, the most favourable conditions will obviously be for greatest difference of azimuths during the shortest possible time interval.

To comply with this condition, it is necessary to find sections of the diurnal circle of the celestial body in which the rate of change of azimuth $\frac{dA}{dt}$ will be greatest for the given conditions (φ, δ). In Sec. 11 it was found, when investigating formula (3.19), that the greatest rate $\frac{dA}{dt}$ in the general case will be for azimuths of the sun from SE to SW in middle north latitudes and from NE to NW in middle south latitudes, which is near the upper transit of the sun.

In low latitudes, the highest rate $\frac{dA}{dt}$ occurs for azimuths still closer to upper transit. For this reason, the most favourable time interval for observations of the sun in medium latitudes is from 2 to 2.5 hours prior to and following upper transit; in low latitudes (for high altitudes of the sun) between 40 minutes and 1.5 hours prior to and following its upper transit.

The effect of dead-reckoning errors will also depend on the time interval and the ship's speed; therefore, to find the most favourable interval between observations, establish which differences $\Delta A = A_2 - A_1$ will be the most favourable for various values of $\frac{\Delta A}{\Delta t}$, the ship's speeds and for various values of dead-reckoning errors. Table 13 (after B. Maltsev) gives the most favourable values of the azimuth difference ΔA_{mf} for various conditions.

Table 13

Expected errors in route and in run	Ship's speed in knots	Rate of change of azimuth $\left(\frac{dA}{dt}\right)$					
		$\frac{dA}{dt} = 10°/\text{hour}$		$\frac{dA}{dt} = 15°/\text{hour}$		$\frac{dA}{dt} = 20°/\text{hour}$	
		$\varepsilon_h = 0'.5$	$\varepsilon_h = 1'.0$	$\varepsilon_h = 0'.5$	$\varepsilon_h = 1'.0$	$\varepsilon_h = 0'.5$	$\varepsilon_h = 1'.0$
		ΔA_{mf}	ΔA_{mf}	ΔA_{mf}	ΔA_{mf}	ΔA_{mf}	ΔA_{mf}
Slight $\varepsilon_{CMG} \approx 1°$ $\varepsilon_s \approx 2\%$	10	49°	63°	57°	71°	63°	76°
	20	35	49	42	57	49	63
Medium $\varepsilon_{CMG} \approx 1°.5$ $\varepsilon_s \approx 3\%$	10	40	54	49	63	54	69
	20	28	39	35	49	40	54
Large $\varepsilon_{CMG} \geqslant 2°$ $\varepsilon_s \geqslant 4\%$	10	35	49	42	57	49	63
	20	25	35	30	43	35	49

From the table it is evident that for mean conditions and sun altitudes less than 65°-70°, the most favourable azimuth difference will be 35° to 60°, which means the interval between observations should be from 1.5 to 4 hours. Just about the same difference in azimuth (40°-60°) has been established in practice on the basis of extensive experience. The usual time interval between observations is taken as 2 to 3 hours.

From Table 13 it is possible to select, approximately an interval of time for the second set of observations (under given conditions). To do this, take the values of the errors, and determine $\frac{dA}{dt}$ every 10-15 minutes via compass bearings of the sun or from Table 15 MT-63. Then $(CB_2 - CB_1)\,\frac{60\text{ min}}{\Delta T\text{ min}}$ will give $\frac{\Delta A}{\Delta t}$ for one hour. The quantity $\frac{\Delta A}{\Delta t}$ is most simply obtained from any numerical tables of azimuths or altitudes and azimuths (A. Yushchenko's tables, BAC-58, H.O. No. 214, and others).

In order to reduce random errors of observations and in working of sights, it is advisable to take 3 to 5 sun sights instead of one and to take other measures as well (see Secs. 85, 110).

II. SYSTEMATIC ERRORS

In plotting lines *II* and *I′* from the second D.R. position, the observed position obtained may be shifted relative to the actual position due to systematic errors in the differences $h - h_c$; operative in the first line, in addition to errors of observations and in working of sights, is the error Δ_Z, the result of systematic errors in dead reckoning. The systematic error Δ_Z in the first line is expressed by the formula (20.16), from which we see that errors in the first line depend on its position relative to the course line, that is, on $A - K$ and on the magnitude of errors in the run and the course. The general systematic error in position may be expressed by the formula (20.5), in which one should take $\Delta_1 + \Delta_Z$ in place of Δ_1.

All systematic errors operate to shift the observed position, not along the line of mean azimuth, as in the case of fixes via two stars, but at an angle to this direction, which angle depends on the relationship of the systematic error in $h - h_c$ and the error Δ_Z.

It is impossible to detect systematic errors with two lines; for this reason take measures to eliminate them from daytime observations. These measures are mainly:

(a) to measure the dip of the horizon with an instrument;

(b) to reduce dead-reckoning errors.

For high sun altitudes (exceeding 50°-55°), one can observe for each line the altitudes above the opposite parts of the horizon (normally, and back sights "via the zenith"). Then after plotting two lines, the systematic errors in $h - h_c$ in their bisector are partially eliminated; in this case the dead reckoning errors Δ_Z operate as usual.

III. REDUCING THE EFFECTS OF DEAD-RECKONING ERRORS

From the very idea of determining position from observations made at different times it follows that an error in dead reckoning prior to the first set of observations is taken into account in the first line and is of no importance, whereas errors in dead reckoning *between observations* will enter the transferred first line in toto. For this reason, the second computed coordinates should be obtained as accurately as possible relative to the first ones. These errors will naturally not enter the second line.

In the general case, to reduce dead-reckoning errors it is necessary to take measures to specify dead reckoning more accurately during the time between observations. To check the compass correction, it is best to take the *CB* of the sun during the first sun sights and determine ΔK, if the altitude of the sun is not excessively great. One also checks the drift and current, while the run between observations is computed in three ways: from *lr* difference $= lr_2 - lr_1$, from the time and speed, and from the engine revolutions,

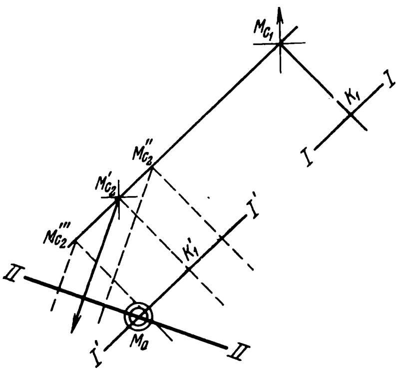

Fig. 204

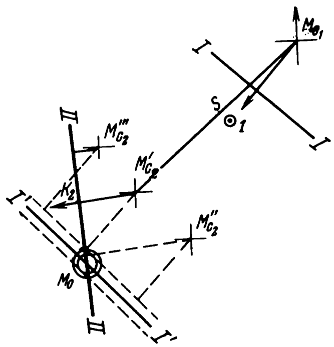

Fig. 205

and the results are intercompared. To reduce graphical errors, replace plotting on small-scale charts by dead reckoning, particularly in high latitudes.

In the interval between observations do not improve on the D.R. position in any other way, for then the first line will be advanced incorrectly.

In special cases it is possible to eliminate the effect of systematic errors in the run and course on the observed position and reduce the effect of random errors by observing the sun at specified positions:

(a) If errors in the run are expected, then observe the sun abeam in order to eliminate systematic errors and reduce random errors. Then the first line, situated along the course line, will take up the same position, irrespective of the place M_{c2} (Fig. 204), and errors in the first line due to the errors ΔS will be eliminated.

(b) If errors are expected in the ship's position perpendicular to the course (due to ΔK, current, drift), observe the sun along the course. Then the line *I-I* will occupy a correct position, irrespective of M_{c2} (Fig. 205) and errors in the first line due to errors ΔK will be eliminated.

Analytically, this follows from equations (20.16) and (20.17) for errors Δ_Z and $\varepsilon_{\Delta Z}$ in the advanced line I'. Indeed, when observing the sun abeam $A - K = 90°$, $\cos(A - K) = 0$ and the systematic errors due to the run vanish. When observing the sun along the course, $A - K = 0°$, $\sin(A - K) = 0$ and systematic errors of the line due to the errors ΔK in the course vanish.

IV. OBTAINING THE AREA OF THE PROBABLE POSITION OF THE SHIP

As already mentioned, the first line, after plotting from the first D.R. position M_{c1} (Fig. 206), will be located in the "position band" about I-I, due to errors $\pm\varepsilon_{h1}$. When the line is advanced to point

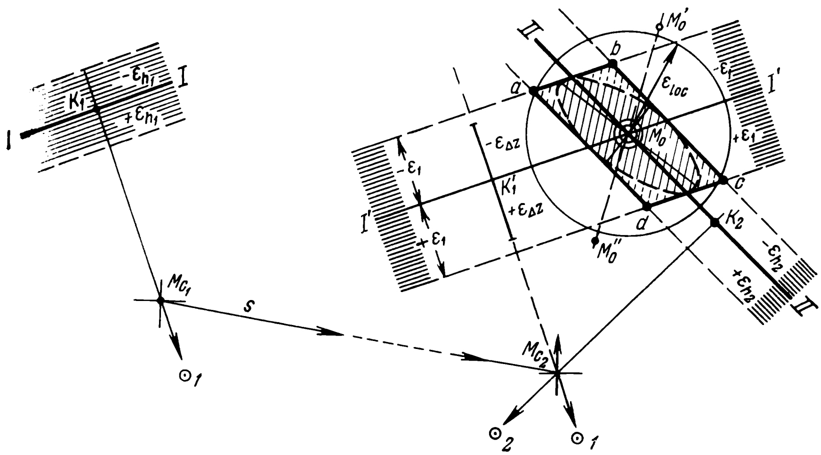

Fig. 206

M_{c2}, these errors will be supplemented by dead-reckoning errors $\pm\varepsilon_{\Delta Z}$, increasing the "position band" about the line I' to $\pm\varepsilon_1$. The second line II-II is also located in the band of the position due to errors $\pm\varepsilon_{h2}$. As a result, the observed position is taken to be at the point M_0, while the actual position of the ship must be somewhere in the area of errors about it. To a first approximation, this area is expressed by a **parallelogram of errors**, *abcd*; more precisely, by an ellipse of errors situated inside the parallelogram of errors and touching its sides (see Fig. 206). This area may also be depicted in the form of a circle of errors of radius $\rho = \varepsilon_{loc}$ obtained from the formula (20.20). To simplify obtaining ρ from (20.20) take ε_h outside the radical and get the ratio $\frac{\varepsilon_{\Delta Z}}{\varepsilon_h}$ of dead-reckoning errors to observa-

tional errors, that is,

$$\rho = \varepsilon_{loc} = \pm \frac{\varepsilon_h}{\sin \Delta A} \sqrt{2 + \left(\frac{\varepsilon_{\Delta Z}}{\varepsilon_h}\right)^2} \tag{20.22}$$

Drawing a circle with this radius from the observed position, we get a circle of errors inside which should be the actual position of the ship with a probability of about 63-68%.

V. ON CHOOSING THE OBSERVED POSITION AND ON TRANSFERRING THE DEAD RECKONING

In the general case, the original errors of dead reckoning considerably exceed observational errors; for this reason, M_0 is the running fix (the point of intersection of the second and advanced first line) and is taken as the observed position (see Fig. 206).

There are cases when the accuracy of dead reckoning is rather high: in the absence of drift and current, with reliable corrections, and so forth, while the precision of observations is contrariwise rather low (poor horizon, unknown dip, etc.). In such cases, the errors due to such causes may prove to be of the same order of magnitude; then do not disregard the weight (significance) of the D.R. position, and the D.R. running fix M_0 may be taken at the midpoint of the straight line between the second D.R. position and the obtained point M_0.

Blunders in the plotted lines are found by techniques analogous to those considered for two stars. To enhance reliability of fixes, it is best to have a second observer who obtains a fix independently of the first one.

A single running fix obtained from two lines obtained at different times is not reliable, and one should not transfer the dead reckoning to a single observation. Only when two or three independently obtained and analyzed positions are available and are in good "agreement" can the dead reckoning be advanced to the observed position.

CONCLUSIONS

1. The most favourable difference of azimuths in a running fix lies between 35° and 60° under average conditions. Do not make a running fix, in the general case, for $\Delta A < 30°$.

2. The most favourable time for sun sights is the interval of 1.5 to 2 hours prior to and following upper transit.

3. To reduce errors in position obtained, strive to reduce systematic and random errors in the lines and improve the dead reckoning between sights.

4. The ship's position should be considered as lying within the area about the observed position.

SEC. 119. ROUTINE AND PRACTICAL SUGGESTIONS FOR OBTAINING A RUNNING FIX (INSTRUCTIONS)

To determine position from sun sights taken at different times, we need the same instruments and aids as for simultaneous observations of stars, with the exception of the globe. No preliminary preparations are needed.

I. TAKING SIGHTS

1. Before starting observations:

(a) Prepare sextant for taking sun altitudes, make a rapid check of sextant errors, and take up a position for observations 10 to 15 minutes ahead of time, but do not keep the instrument in the sun.

(b) Compare deck watch (or other timepiece) with chronometer, or stop watch with chronometer, if the former is used.

(c) Compute u_{ch} at the instant of observations, and also u_{wat} if required.

2. Determine the index correction with a check.

3. Take a series of 3 (or 5) sun altitudes and note the time for each one (using watch or chronometer); it is common practice to measure the altitude of the lower limb of the sun.

4. When measuring the mean of the altitudes (or after the measurement), note and record T_{sh}, log, course, ship's speed, height of observer's eye, temperature and pressure of air (if $h < 30°$). Take compass bearing of the sun (CB_1).

5. If possible, measure dip of horizon or perform extra sun sights "via the zenith".

II. WORKING FIRST SIGHTS

1. Using T_{sh} and lr, take from chart φ_c and λ_c to within $0'.1$.

2. Compute the arithmetic mean of the readings (sr_{av}) and instants (T_{av}).

3. Correct mean reading of sextant sr_{av} with corrections and obtain h.

4. Compute the approximate $T_{gr} = T_{sh} \pm \mathrm{ZD}_E^W$ and exact $T_{gr} = T_{av} + u_{ch}$.

5. Obtain $t_{loc}^{\odot}$ and $\delta_{\odot}$ from the MAE.

6. Compute h_c and A_c from formulas or special tables and obtain $h - h_c$.

7. If there is any necessity to analyze the dead reckoning, plot the first line on the chart (to a scale not less than 1 : 200,000). If there is no necessity, the line is not laid down.

8. If there is an acute necessity for refining the dead reckoning, carry out observations with two or three observers at the same time, then plot all three lines and analyze them.

In this case, the estimated position may be taken either at the determining point K_1 (Fig. 206) if the errors of dead reckoning greatly exceed observational errors or at half distance to K_1 if errors of the same order are expected.

9. After working the sights, again take the sun bearing (*CB*) and note T_{wat}.

10. Using the formula $\frac{CB_2-CB_1}{(T_2-T_1)^{\text{min}}}\cdot 60^{\text{min}}$, find the rate of change of the sun's azimuth in °/hr. Using Table 13 or personal experience, find the most favourable difference ΔA for two lines and compute approximately the time of the second set of observations by tables BAC-58. Note that towards noon the rate of $\frac{\Delta A}{\Delta T}$ increases. The value of $\frac{\Delta A}{\Delta t}$ and the time of observation may also be computed from numerical tables without taking the *CB*.

11. Using the obtained *CB* and known φ_c, $t_{loc}^{\odot}$ and δ, compute the compass correction for the time of the second set of observations.

12. Take measures to enhance accuracy of dead reckoning between observations:

(a) check corrections: ΔK and Δl, drift and current;

(b) obtain the ship's run S in three ways and take the most reliable one or the arithmetic mean;

(c) do the plotting on a large-scale chart; if no such chart is available or the run is very long, obtain φ_2 and λ_2 by dead reckoning.

III. SECOND SET OF SIGHTS AND THEIR WORKING

1. If visibility of the sun or horizon deteriorates (increased cloud cover, approach of fog, etc.) or if there is any other insistent necessity, the second set of observations are carried out without waiting for a computed instant. If ΔA is then less than 20°, work the sights and plot the observations by the technique used "for a small azimuth difference" (see Sec. 132).

2. Under ordinary conditions, the second set of observations are executed after the specified time interval, like the first set.

3. When measuring the mean of the altitudes, note T_{sh} and log. If conditions have changed, obtain other data as well.

4. Working the second set of observations is done in a similar fashion to that for the first set, but with φ_{c2} and λ_c obtained at the second mean instant.

5. Plot first and second lines from second D.R. position; label the point of intersection M_0. If the dead reckoning was done from a position—the determining point K_1—then the line is drawn through the second D.R. position parallel to its first position.

IV. AN ANALYSIS OF THE RUNNING FIX

1. Select the observed position (at the point of intersection of the lines or taking dead reckoning into account).
2. Construct an area of the probable position of the ship (a circle of errors) due to random errors and transfer it along the line of mean azimuth at the expense of presumed systematic errors ($\pm\Delta$). We get the area of the possible position of the ship.
3. Detect possible blunders.
4. Decide on possibility of transferring the dead reckoning to the observed position. Record in log book the results of the running fix, that is φ_0, λ_0, and leeway.

Example 4. On 13.09.62 sun observed in Atlantic Ocean, course 65° true; speed 11 knots about $T_{sh}=$9h 30m; $lr=38.3$; ($\Delta l=-3.5\%$); $\varphi_c=66°11'.0$N; $\lambda=9°32'.0$W.

(1) Corrections: $u_{ch}=-$12m 5s

	sr	T_{ch}
	23°33'.4	10h 40m 39s
	36 .2	41 32
	38 .8	42 13
Mean	23°36'.1	10h 41m 28s

$oi_1=360°33'.0$, $oi_2=359°29'.0$; $S_1=-0'.3$

$i=-1'.0$
$t=+15°$
$B=765$ mm
$l=6.5$ metres

$RCB_{\odot}=348°.5$; $T_{ch}=$10h 43m 50s. Compass declination equals $-20°.2$W.

(2) Working of sights:

T_{sh}	9h 30m
$+$ ZD$_W$	1
T_{gr}	10h 30m 13.09
δ_T	3°42'.1 (−1.0)
$\Delta\delta$	−0 .4
$\delta_{\odot}$	3°41'.7N

T_{ch}	10h 41m 28s
u_{ch}	−12 5
T_{gr}	10h 29m 23s
t_T	331°02'.1 (0.5)
Δt_1	7 20 .6
Δt_2	0 .2
$t^{\odot}_{gr}$	338°19'.7
$-$ λ	9 32 .0
$t^{\odot}_{loc}$	328°50'.9W =31°09'.1E

sr	23°36'.1
$i+s$	−1 .3
h'	23°34'.8
Δ_{tot}	+9 .4
Δ_{ad}	−0 .1
h	23°44'.8
h_c	23 49 .1
$h-h_c$	−5'.0

Computations by tables "BAC-58"

		h_T	$24°18'.0$	A_T	$145°.7$	
φ	$66° + 11'.0$N	Δh_φ	$-9\ .1$	ΔA_φ	$+0\ .1$	
δ	$4\ -\ 18'.3$N	Δh_δ	$-17\ .8$	ΔA_δ	$+0\ .1$	$q=13$
t	$31\ +\ 9'.1$E	Δh_t	$-2\ .0$	ΔA_t	$-0\ .2$	
		h_c	$23°49'.1$	A_c	$145°.7$NE	

(3) Compass correction:

$t^{\odot}_{loc} = 328°51' + 35' = 329°26'W= 30°34'$E.

$$\Delta T = 26' + 9' = 35' \approx 0°.6$$

From the BAC-58 tables:

A_c	$145°.7$
ΔA_t	$+0\ .7\ (1°.1 \cdot 0°.6)$
A	$146°.4$NE
CB	$168\ .5$
ΔK	$-22°.1$
Declin.	$+20\ .2$
Deviation	$-1°.9$

(4) After the sights were worked, RCB_2 was again determined and found to be $352°.5$, $T_{ch} \approx 10$h 59m. Rate of change of azimuth: $\Delta A' = (RCB_2 - RCB_1) \times \times \frac{60^{\text{min}}}{15^{\text{min}}} = 16°$ per hour. From Table 13, considering dead-reckoning errors to be average and observational accuracy high, we get the most favourable difference of azimuths $\Delta A \approx 49°$, $\Delta T \approx \frac{49°}{16°/\text{hour}} \approx 3$h. Second set of observations planned for $T_{sh} = 12$h (2.5 hours later) in order to obtain meridian point.

(5) Second set of observations.

Av. $sr_{\odot}$	T_{ch}
$26°48'.9$	1h 14m 20s

$i + s_2 = -1'.5$; $lr_2 = 67.6$

Computation of second set of coordinates.

(a) Distance run by speed: $V = 11$ knots, $\Delta T = 2$h 32m, from Table 27a, MT-53, $S = 27.9$ miles.

(b) Distance run by log; difference $lr = 29.3$. From Table 28б, $S_{log} = 28.2$.
(c) Distance run by engine revolutions: $S = 28.4$.
We take the mean value: $S = 28.2$.
From Table 24, with course 65 true and $S = 28.2$, we have: $l = 11'.9$N, $Dep. = 25'.56$E and from Table 25, $DLo = 63'.6$E.

T_{sh}	12h 0m	φ_{c1}	66°11′.0N	λ_{c1}	9°32′.0W
ZD_W	1	l	+ 11 .9N	DLo	1° 3′.6E
T_{gr}	13h 0m 13.09	φ_{c2}	66°22′.9N	λ_{c2}	8°28′.4W
ΔT	3°38′.2 ($\overline{1.0}$)	T_{ch}	1h 14m 20s	sr	26°49′.8
$\Delta\delta$	0 .0	u_{ch}	− 12 5	$i+s$	− 1 .5
$\delta_\odot$	3°39′.2N	T_{gr}	13h 2m 15s	h'	26 48 .3
				Δ_{tot}	+ 9 .7
				Δ_{ad}	− 0 .1
				h	26°57′.9
				h_c	27 00 .7
				$h - h_c$	− 2′.8

t_T	16°02′.8 (0.5)
Δt_1	0 33 .7
t_{gr}	16°36′.5
− λ_W	8 28 .4
$t^{\odot}_{loc}$	8° 8′.1W

		h_T	27°44′.6	A_T	171°.0	
φ	66° +22′.9N	Δh_φ	−22′.6	ΔA_φ	0 .0	
δ	4 −20 .8N	Δh_δ	−20 .8	ΔA_δ	0 .0	$q = 4$
t	8 + 8 .1W	Δh_t	−0 .5	ΔA_t	−0 .1	
		h_c	27°00′.7	A_c	170°.9NW = 189°.1	

(6) Plotting (Fig. 207). The results of the determination are:

$$\left.\begin{array}{l} T_{sh}=12\text{h } 2\text{m} \\ lr=67.6 \end{array}\right\}$$

φ_{c2}	66°22′.9N	λ_{c2}	8°28′.4W
l	3 .4N	DLo	9 .4W
φ_0	66°26′.3N	λ_0	8°37′.8W

$$C=312°-5'.1$$

(7) To construct an area of the probable position of the ship we take the dead-reckoning errors during this period at about 1 mile, and $\varepsilon_h \approx \pm 0'.5$;

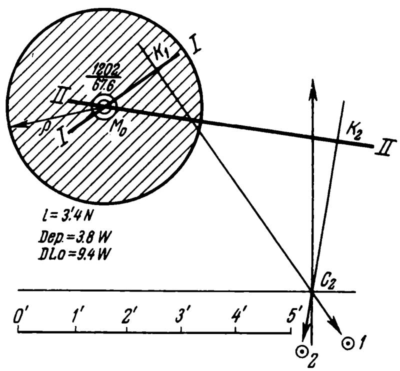

Fig. 207

then from formula (20.22) we have $\rho=3.5\cdot\varepsilon_h=\pm 1'.8$. With this radius we construct a circle of errors.

SEC. 120. COMBINATION METHODS OF OBTAINING FIXES BASED ON ALTITUDE AND NAVIGATIONAL LINES OF POSITION

In navigational practice there are often cases when only one line of position is obtained (radio bearing, the bearing of a distant object, etc.), the second line of position being unobtainable. In such cases, a sun sight can be taken in the daytime and the altitude line of position combined with the navigational line. The observations may be simultaneous or at different times. To obtain the most favourable azimuth difference of 80° to 90° the sun and the object whose bearing is taken must be on a single vertical circle. If the distance to the object is measured, there should be a diffe-

ence of 90° between the vertical circles of the sun and the object. It is advisable to combine with the altitude line the following navigational lines: (a) radio bearings, (b) bearings of a distant mountain peak, (c) distances by vertical angle, (d) distances by radar, (e) reliably identified isobath.

As a rule, the altitude and "navigational" lines are not obtained at the same time, and so the first line should be adjusted to the zenith of the second. This reduction is best done graphically, on a chart, by advancing the first line in the direction of the course by the ship's run between observations $S = \frac{V}{60}(T_2\text{-}T_1)^{\min}$

When plotting a radio bearing or the bearing of a distant mountain, take into account the orthodromic correction. The correction is allowed for if the station or mountain is 20 or more miles from the vessel in the direction E-W. This is particularly important in high latitudes. The observed position is taken at the point of intersection of the lines and is labelled as astronomical (less accurate).

When plotting the area of the possible position of the ship, use the same methods as for two position lines, but bear in mind that the lines of position here are not of equal accuracy, for the navigational line may prove to be more or less precise than the astronomical line. In this case, the area will be elongated more along the navigational line.

SEC. 121. USING A SINGLE ALTITUDE LINE OF POSITION

A single properly located altitude line may prove to be extremely useful to the navigator for improving dead reckoning, as a safety line, or for an approach to a specified point on shore.

I. IMPROVING DEAD RECKONING

(a) If the errors of dead reckoning are very great and the D.R. position is unreliable, it may be advanced to the *determining point* K_1 and considered to be estimated (computed-observed, Fig. 208). Here the area of the probable position will be in the form of an ellipse along the line of position. The semimajor axis may be taken equal to the radius of the circle of errors due to dead reckoning $a = \rho_c$ and the minor axis approximately equal to the band of position in the absence of systematic errors and blunders. It is therefore very important here to reduce the systematic errors (by measuring the dip of the horizon, and so forth). Sometimes the position is refined with account taken of dead reckoning (see Sec. 119, Item 8).

(b) If the log has performed unreliably, and the compass reliably, the estimated position is transferred to point M (Fig. 209), at the intersection of the altitude line of position with the course line.

(c) If the ΔK obtained is appreciably different from the one taken

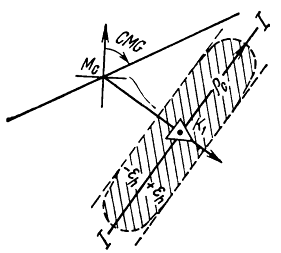

Fig. 208

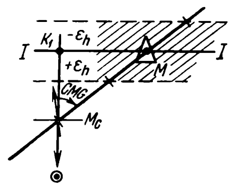

Fig. 209

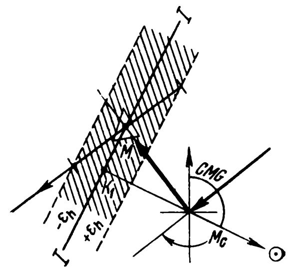

Fig. 210

earlier or if a lateral drift is noticed, the estimated position may be advanced to *point M* (Fig. 210) at the intersection of the altitude line of position with the perpendicular drawn from the D.R. point to the course line.

II. WARNING OF APPROACHING DANGER

(a) When on course near danger spots, observe the celestial body abeam. Then the position line, with direction parallel to the danger, will indicate whether the course is safe or not. If the band of position passes too close to danger, specify a safe distance, turn perpendicular to the plotted line and take up a fresh course, the basis of which will be the displaced position line.

(b) When approaching danger on course, observe the celestial body ahead or astern. The position line, with direction perpendicular to the course and possible errors allowed for, will indicate the distance from the danger. The distance and time of ship's run to turning point are determined in similar fashion.

III. APPROACHING A SPECIFIED SHORE POINT

On approaching some point on shore that is poorly identifiable, or in the absence of shore visibility, observe a celestial body (the sun), the azimuth line of which is located along the shoreline. The altitude line of position I-I will then pass at an approximate right angle to the shore (Fig. 211). Using a parallel ruler, advance the line until it passes through the specified point A. This is the course

that should be reached. Measure the distance S_1 between the first line I and the advanced line I' with allowance for distance (ΔS) run during computation. We head perpendicular to the altitude line and by dead reckoning we reach the course I'-A (in time $T' = \frac{S_1}{V}$ and in difference lr) which we follow up to point A (safe depths).

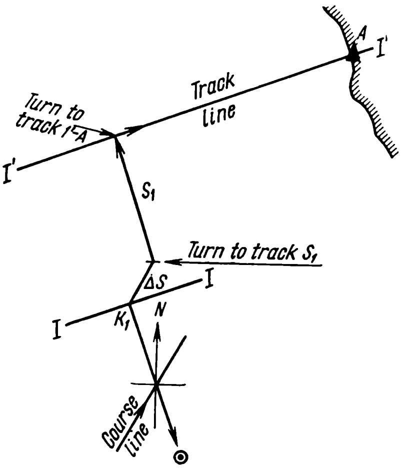

Fig. 211

Generally speaking, when observing a celestial body on course the amount of intercept $h - h_c$ will indicate the leeway along the course; observations abeam will indicate the magnitude and direction of lateral drift of the ship.

When using a single altitude line of position, one can apply the method of precomputation of the elements of the line, whereby h_c and A_c are computed with the coordinates φ_c and λ_c taken at a specified advanced instant of time. The altitude is corrected with corrections taken with sign reversed and is plotted on the chart. This altitude is compared with the altitude observed at about the specified instant.

C H A P T E R 21

METHODS FOR SEPARATE DETERMINATION OF THE LATITUDE AND LONGITUDE OF A SHIP'S POSITION

SEC. 122. ON METHODS FOR SEPARATE DETERMINATION OF COORDINATES

Methods for the separate determination of the latitude and longitude of a ship may be applied only for certain particular positions of celestial bodies. From an analysis of the equation of errors in Sec. 98 it was found that the latitude is determined with least error if the celestial body is located on the meridian of the observer or is very close to him. The longitude will be found with greater accuracy if the body is situated on the prime vertical. Whence follow the methods for separate determination of latitude and longitude at sea:

(1) determination of φ from the meridian altitude of the celestial body;

(2) determination of φ from the ex-meridian altitudes of the body;

(3) determination of φ from the Pole Star (Polaris);

(4) determination of λ from the altitudes of the body near the prime vertical by chronometer (the "method of transporting chronometers").

The basic principle underlying all these methods is the earlier examined relationship between geographical coordinates of the ship's position and the coordinates of its zenith (Sec. 96), which is based on an analogy of the systems of equatorial coordinates of celestial bodies and the geographical coordinates of points on the earth. In determining φ and λ, any celestial bodies may be used and—theoretically—at any altitude. However, each method has its limitations and peculiar errors.

SEC. 123. DETERMINING THE LATITUDE OF A PLACE FROM THE MERIDIAN (GREATEST) ALTITUDE OF A CELESTIAL BODY

Let us assume that the bodies C_1, C_2, and C_3 (Fig. 212) lie on the observer's meridian—at upper (C_1 and C_2) or lower (C_3) transit. Then, as was pointed out at the beginning of this course (Secs. 3 and 9), it is easy to obtain the latitude of any place from the altitude and declination of the body. From Fig. 212, we have for cele-

stial body C_1 (δ of same name as φ) in upper transit

$$\varphi = Z_1 + \delta_1 = (90^\circ - H_1) + \delta_1$$

For C_2 (δ of contrary name to φ) we similarly have

$$\varphi = Z_2 - \delta_2 = (90^\circ - H_2) - \delta_2$$

From these equations we get a formula for determining φ at the instant of upper transit of the celestial body. In the general form it is

$$\varphi = Z \pm \delta \tag{21.1}$$

The sign will be positive if δ and φ (or Z) are of the same name, and negative if of opposite names; then the smaller quantity is subtracted from the larger one. It will be recalled that the name (N or

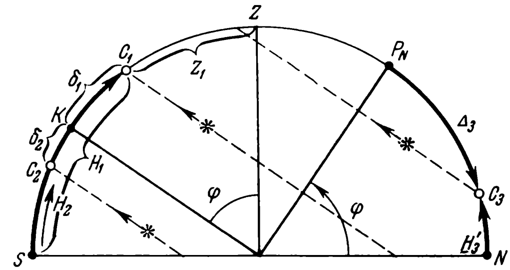

Fig. 212

S) of the altitude H is determined from the point of the horizon above which the altitude was measured, and the name of Z is the reverse of H. The name of the latitude is the same as that of the greater term.

For C_3 at lower transit we get (see Fig. 212)

$$\varphi = H_3' + \Delta_3 = H_3' + (90^\circ - \delta_3)$$

or, in general form,

$$\varphi = H' + \Delta \tag{21.2}$$

where Δ is the polar distance equal to $90^\circ - \delta$.

Here, the name of φ is determined from H' or from δ, which in the given case is of the same name as the latitude.

These same formulas are readily obtainable analytically from the basic equation of the circle of equal altitudes on a sphere, as shown in Sec. 97.

Any body may be used to determine the latitude from the meridian altitude, but at present this method is used only in daytime

observations—of the sun in upper transit and in lower transit too at very high latitudes.

To start observations, it is necessary to compute T_{sh} of transit of the sun on the meridian with the D.R. longitude of the ship. Observations begin 10 to 12 minutes prior to the computed T_{sh} of transit. The altitudes are taken near the meridian by one of the techniques described in Sec. 64. As a rule, the *greatest* altitude is measured and is taken for the meridian altitude. After correcting the altitude and taking δ out of the MAE for the instant of observations, the observed latitude φ_0 is obtained from formula (21.1). The observed latitude is not the true latitude of the place because it includes errors of the method of greatest altitudes and errors of measurement and correction of altitudes, which we shall examine below. The routine and practical suggestions for finding φ are given in Sec. 127 together with a determination based on ex-meridian altitudes.

Example 1. On 12.09.68 we expect to observe the meridian altitude of the sun to determine the latitude of the ship. At noon we get $\lambda_c \approx 22°20'$E.

(1) Calculating time of transit

$T_{tab\,gr}$	11h 56m from MAE
ΔT_{λ}	00
T_{loc}	11 56
$-\ \lambda$	1 29
T_{gr}	10 27
$+$ ZD	2
T_{sh}	12h 27m

(2) At about $T_{sh}=12$h 30m (ZD $=$ 2E) observed greatest altitude $\underline{\odot}$ $sr=59°43'5$S; $e=9.4$ metres; $i+s=+1'.7$; course 94° true; speed 12 knots.

sr	59°43′.5
$i+s$	+1 .7
H'	59°45′.2
Δ_{tot}	10 .1
Δ_{ad}	−0 .1
$H_{\odot}$	59°55′.2S
$Z_{\odot}$	30 4 .8N
$\delta_{\odot}$	4 04 .6N
φ_0	34°09′.4N

T_{sh}	12h 30m
$-$ ZD	2
T_{gr}	10h 30m 12.09
δ_T	4°05′.1 $(\overline{1.0})$
$\Delta\delta$	−0 .5
$\delta_{\odot}$	4°04′.6N

SEC. 124. THE EFFECTS OF ERRORS OF METHOD AND ERRORS OF OBSERVATION ON THE OBSERVED LATITUDE

DISCREPANCIES BETWEEN GREATEST AND MERIDIAN ALTITUDES IN TIME AND ALTITUDE

As already mentioned, when determining the latitude, the greatest altitude of those measured is taken as the meridian altitude. However, this is true only when the celestial body does not have proper motion over the sphere and the observer is stationary.

Referring to Fig. 213a, which depicts the southern part of the sky, the observer is stationary and the body (star C, say) has no proper motion. Its path a_1a_2 will be symmetric about the meridian, and the greatest altitude (h_{max}) will be the meridian altitude H. But if the declination δ of the body (C_1, say, in Fig. 213b) increases, then in the meridian, when the body moves parallel to the horizon ($\Delta h_T = 0$), an increment Δh_δ is added to its altitude, due to changing declination, and for a certain time the altitude will continue to increase. The path of the body (a_3a_4) will not be symmetric about the meridian, and h_{max} will not be equal to H. What is more, the observer himself will be moving over the earth with the ship, thus causing the altitude of the body to change (the change is by the amount Δh_Z), which means an increase if the observer is moving towards the celestial body, and a decrease if he is moving away from the body. For an observer aboard ship (in a northern latitude on a course S), the greatest altitude will be reached when the rate of decrease of altitude due to diurnal motion becomes equal to the rate of increase in altitude due to the motion of the ship; that is, $\frac{\Delta h_T}{\Delta T} = \frac{\Delta h_Z}{\Delta T}$. Obviously, this will occur after the body has crossed the meridian and h_{max} will not be equal to H (Fig. 213c). Similarly, for a northern course, the greatest altitude h'_{max} will come before the meridian.

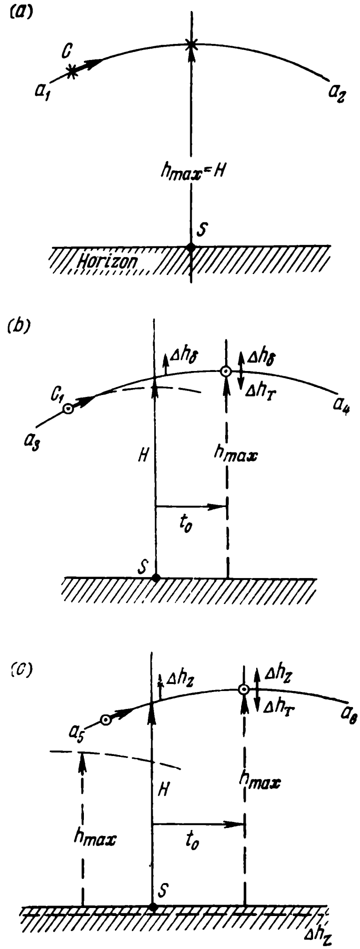

Fig. 213

The first cause—change of declination—acts on the altitudes of bodies that have considerable proper motions (the moon, sun, planets); the second cause—the ship's motion—affects the altitudes of all bodies.

If the combined action of both causes increases the altitude, then the greatest altitude, h_{max}, will come after transit of the body and will be greater than H; but if it reduces the altitude, then h_{max} will occur prior to transit and will be less than H.

Obviously, when in such cases we take h_{max} for H and introduce it into (21.1), we get the observed altitude with an error $\Delta\varphi$, which depends on the speed of the ship and the rate of motion of the celestial body. The instant of maximum altitude will likewise differ from the instant of transit.

(a) *Determining the time interval between the instant of upper transit and the instant of maximum altitude in the general case.*

Since at the instant of upper transit of the body its $t = 0$, the problem reduces to determining the local hour angle t_0 of the body at the instant of maximum altitude h_{max} (Fig. 213*b* and *c*).

In the formula $\sin h = \sin\varphi \cdot \sin\delta + \cos\varphi \cdot \cos\delta \cdot \cos t_{loc}$ the quantities h, φ, δ, t are variable ones dependent on the time T; therefore, after differentiating with respect to these variables we have

$$\cos h \frac{\partial h}{\partial T} = (\sin\varphi \cdot \cos\delta - \cos\varphi \cdot \sin\delta \cdot \cos t)\frac{\partial\delta}{\partial T} + \\ + (\cos\varphi \cdot \sin\delta - \sin\varphi \cdot \cos\delta \cdot \cos t)\frac{\partial\varphi}{\partial T} - (\cos\varphi \cdot \cos\delta \cdot \sin t)\frac{\partial t}{\partial T}$$

By the rules of finding the extreme values of a function we equate $\frac{\partial h}{\partial T}$ to zero, then $t = t_0$. Due to the smallness of t_0, we can take $\cos t_0 \approx 1$, $\sin t_0 = t_0' \cdot \text{arc } 1'$; then

$$\cos\varphi \cdot \cos\delta \cdot t_0' \text{ arc } 1' \frac{\partial t}{\partial T} = (\sin\varphi \cdot \cos\delta - \cos\varphi \cdot \sin\delta)\frac{\partial\delta}{\partial T} + \\ + (\cos\varphi \cdot \sin\delta - \sin\varphi \cdot \cos\delta)\frac{\partial\varphi}{\partial T}$$

Taking the minus sign outside the second parenthesis, combining the expressions in the brackets, and dividing by the factors of the unknown t_0, we get

$$t_0' = \frac{(\sin\varphi \cdot \cos\delta - \cos\varphi \cdot \sin\delta)}{\cos\varphi \cdot \cos\delta \cdot \text{arc } 1' \frac{\partial t}{\partial T}}\left(\frac{\partial\delta}{dT} - \frac{\partial\varphi}{\partial T}\right) \qquad (21.3)$$

In the general form, the derivative $\frac{\partial t}{\partial T}$ will be obtained from the formula (see Sec. 46)

$$t_{loc} = T_{gr} \pm 12\text{h} + (\alpha_{\oplus} - \alpha_{body}) \pm \lambda_W^E$$

$$\frac{\partial t}{\partial T} = 1 + \frac{\partial(\alpha_{\oplus} - \alpha_{body})}{\partial T} \pm \frac{\partial\lambda_W^E}{\partial T} \qquad (21.4)$$

For the sun $\alpha_{\oplus} - \alpha_{\odot} = -\eta$, i.e., it is equal to the equation of time, which changes but slightly ($\Delta\eta \leqslant 0'.3$ per hour) during a short time interval. Analogously, the difference $\alpha_{\oplus} - \alpha_{pl}$ will be small for planets as well. For this reason, to a first approximation,

the second term in (21.4) may be ignored for all bodies except the moon. Then from formula (21.4) in which we take $d\lambda_W$, we have

$$\frac{\partial t}{\partial T} = 1 - \frac{\partial \lambda}{\partial T}$$

Passing to finite increments in (21.3) by dividing by $\cos\varphi \cdot \cos\delta$ and expanding the quantity $1 - \frac{\Delta\lambda}{\Delta T}$ (transferred to the numerator) into a series by the binomial theorem

$$\left(1 - \frac{\Delta\lambda}{\Delta T}\right)^{-1} = 1^{-1} - (-1)\cdot(1)^{-2}\cdot\frac{\Delta\lambda}{\Delta T} + \ldots = 1 + \frac{\Delta\lambda}{\Delta T}$$

we get

$$t_0' = \frac{\tan\varphi - \tan\delta}{\operatorname{arc} 1'}\left(\frac{\Delta\delta}{\Delta T} - \frac{\Delta\varphi}{\Delta T}\right)\cdot\left(1 + \frac{\Delta\lambda}{\Delta T}\right)$$

We take the time interval ΔT equal to 1h $= 900'$ and denote by θ', ψ', $\Delta\lambda'$ the changes $\Delta\delta$, $\Delta\varphi$, $\Delta\lambda$ in minutes of arc during this time. Substituting the value $\operatorname{arc} 1' = \frac{1}{3{,}438}$ and taking these designations, we finally get

$$t_0' = 3.82\ (\tan\varphi - \tan\delta)\cdot(\theta' - \psi')\cdot\left(1 + \frac{\Delta\lambda'}{900}\right) \tag{21.5}$$

If we express t_0 in seconds of time, this formula will yield the coefficient 15.28. On the basis of determinations of coordinates φ, δ, λ and the condition that in (21.5) we have $\Delta\lambda_W$, we get the following rules of sign:

φ is always considered positive

δ has the plus sign if of the same name as φ, and minus if of contrary name

ψ' is plus if l is of the same name as φ, and minus if different

θ' is plus if the body is approaching the elevated pole and minus if it is receding from it

$\Delta\lambda'$ is plus for westerly courses ($\Delta\lambda_W$) and minus for easterly courses ($\Delta\lambda_E$).

The formula (21.5) gives the value of the local hour angle of a celestial body at the instant of **maximum altitude**; for this reason, if t_0 has the minus sign, the maximum altitude will occur prior to transit (t_0^E); if it has the plus sign, then it will occur after transit (t_0^W).

For the sun, planets and stars, the hour angle t_0^{body} may be regarded as the time interval ΔT^{sec} between transit and maximum altitude. It is advisable to carry out the computations in formula (21.5) with a slide rule and with the help of Table 18*a*, MT-63, which yields the values of 3.82 tan *x*.

Example 2-3. On 20.03.68, $\varphi = 65°$N; course 180° true; speed 20 knots. Sun observed. Find t_0. From the MAE and Tables 24, 25, MT-63, we have

$$\delta \approx 0°;\quad \theta' = +1'.0;\quad \psi' = -20'.0;\quad \Delta\lambda = 0$$

Using Table 18, MT-63 and a slide rule, we get

$$t_0' = (8.19-0)\cdot(1'.0+20'.0) = 172' = 2°52' \text{ or } 11\text{m } 28\text{s}$$

Maximum altitude occurred 11m 28s after meridian altitude.

Example 4. On 30.09.68, at $\varphi = 72°$N; course 270° true; speed 20 knots. Find t_0. From MAE and MT-63 we have:

$$\delta = 3°\text{S},\quad \theta' = -1'.0,\quad \psi' = 0,\quad \Delta\lambda = 64'.7.$$

$$t_0' = (11.76+0.20)\cdot(-1'.0-0)\cdot\left(1+\frac{64.7}{900}\right) = -12'.8 = -51\text{s}.$$

Maximum altitude occurred roughly 1m prior to transit.

(b) *Determining the difference between greatest and meridian altitudes; that is, errors in latitude.*

Denote by δ_0, H_0, φ_0 the declination, altitude and latitude at the instant of meridian altitude, and by δ_1, h_{max} and φ_1 these same quantities at the instant of maximum altitude. Then

$$\varphi_0 = 90° - H_0 + \delta_0 \quad \text{and} \quad \varphi_1 = 90° - h_{max} + \delta_1 \qquad (21.6)$$

Then the difference $\varphi_1 - \varphi_0 = H_0 - h_{max} + (\delta_1 - \delta_0)$ will represent the error in latitude due to substitution of H_0 by h_{max}; however, these altitudes refer to different places on the earth and different times, and so it is hard to compare them. Practically speaking, it is required to find the correction to the maximum altitude h_{max} (or φ_1) to obtain the observed latitude at this instant (maximum altitude) and not at some other instant, say at the instant of true noon for H_0.

Here, we may regard h_{max} as an ex-meridian altitude; taking $\delta_1 =$ const and considering the ship stationary, we determine the increment in altitude Δh_{t0} due solely to the diurnal motion of the sphere during the interval t_0 found above.

This reduction of altitude to the meridian is designated as r; a practical formula for it is derived in Sec. 125 below.

Since h_{max} is measured off the meridian, it is always less than the altitude (H) on the meridian for constant φ and δ. In this case, to allow solely for the diurnal motion of the sphere during time t_0, perform the computations with δ and φ unchanged and obtained at the instant of observation h_{max}. The computed correction $\Delta h_{t0} = r$ will be positive and additive to h_{max}:

$$H = h_{max} + \Delta h_{t0} \qquad (21.7)$$

From (21.6) it is evident that the increase in altitude h_{max} will cause a decrease in φ_1; thus the correction $\Delta h_{t0} = -\Delta\varphi$, and for

this reason we will have

$$\varphi_0 = \varphi_1 - \Delta\varphi \tag{21.8}$$

Only when $\delta > \varphi$ and they are of the same name will we have $\varphi_0 = \varphi_1 + \Delta\varphi$.

The observed latitude φ_0 thus obtained will always refer to the instant of measuring altitude h_{max}.

To obtain the correction $\Delta\varphi$, one can apply the method of computing the reduction r (see Sec. 125); that is, from φ and δ obtain $K = 100 \tan\varphi \pm 100 \tan\delta$ (Table 17a, MT-53), and from K and t_0 take the value of r (Table 17б) and equate it to $\Delta\varphi$.

Example 5. Using the conditions of Example 3, we have: $\varphi_c = 65°$N; $\delta_{\odot} = 0°$; $t_0 = +2°52'$. Find $\Delta\varphi$.

(a) $K = 100 \tan\varphi - 100 \tan\delta = 215$ . . . (Table 17a, MT-63).
(b) $r = 2'.0 = |\Delta\varphi|$ . . . . (Table 17б, MT-63).
(c) If $\varphi_1 = 65°7'.0$N was obtained, then $\varphi_0 = \varphi_1 - \Delta\varphi = 65°5'.0$N.

In practical work, the correction $r = \Delta h_{t0}$ may be obtained from a special formula, (21.15), developed in the next section

$$r' = \frac{2\sin^2\frac{t}{2}}{(\tan\varphi - \tan\delta)\ \mathrm{arc}\ 1'}$$

Replacing the quantity $\sin^2\frac{t_0}{2} \approx \frac{t_0^2}{4}\ \mathrm{arc}^2\ 1'$, for t_0 and substituting the value t_0' from (21.5), we have

$$\Delta h_{t0} = r_0' \frac{\left[3.82(\tan\varphi - \tan\delta)\cdot(\theta' - \psi')\cdot\left(1 + \frac{\Delta\lambda'}{900}\right)\right]^2_{\mathrm{arc}^2 1'}}{2(\tan\varphi - \tan\delta)\ \mathrm{arc}\ 1'} \tag{21.9}$$

After simplifications, disregarding $\Delta\lambda$, we get

$$\left.\begin{aligned} r_0' &= 0.00212(\tan\varphi - \tan\delta)(\theta' - \psi')^2 \\ &\text{or} \\ r_0' &= (3.82\tan\varphi - 3.82\tan\delta)\frac{(\theta' - \psi')^2}{1{,}800} \end{aligned}\right\} \tag{21.10}$$

and

$$|r_0'| = |\Delta\varphi|$$

where the signs of the quantities δ, θ, and ψ are determined as indicated for formula (21.5).

Computations by these formulas may be performed with the help of Table 18б, MT-63.

Example 6. On 5 October 1968 at $\varphi_c \approx 58°.1$; course 5° true; speed 12 knots, sights taken of h_{max} of sun and $\varphi = 58°9'.2$N determined. Find the latitude error and φ_0.

(1) From MAE and MT-63

$\delta_{\odot} = 4°47'.0$S

In 1h $\Delta\delta = 0' = -1'.0$

In 1h $l = \psi' = +12'.0$

(2) From Table 18a, MT-63: $3.82(\tan\varphi - \tan\delta) = (6.14 + 0.32) = 6.46$; $\Delta\varphi = -0'.6$.

(3) $\varphi_0 = \varphi - \Delta\varphi = 58°8'.6$N.

CONCLUSIONS

1. Determining φ from maximum altitude involves an error due to the difference between the maximum and meridian altitudes, the magnitude of which at $\varphi > 50°$ cannot be ignored. If the ship's speed exceeds 20 knots, the error is perceptible in lower latitudes as well.
2. Random errors in the measured altitude cannot be reduced, since one altitude is taken. The systematic error in altitude enters completely into the observed latitude.
3. Due to the foregoing reasons, latitude obtained from the maximum altitude is not reliable and each time has to be analyzed for possible errors of method, systematic errors of altitude and for blunders. For this reason, the method of maximum altitude is by far not so simple and advantageous as generally believed.

SEC. 125. DETERMINING LATITUDE FROM THE EX-MERIDIAN ALTITUDES OF CELESTIAL BODIES

The most favourable conditions for determining latitude will occur not only during passage of a body across the observer's meridian, but also for bodies located in the immediate vicinity of the

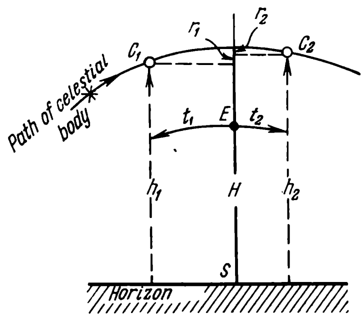

Fig. 214

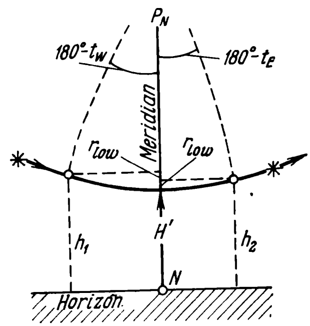

Fig. 215

meridian (see Sec. 98). The altitudes of bodies close to the meridian are called **ex-meridian** altitudes and may be used for determining latitude. Near the meridian, the altitudes of celestial bodies vary but insignificantly and irregularly; for this reason, ex-meridian altitudes differ only slightly from the meridian altitude H and may be reduced to it by adding a small correction r called the **reduction** of ex-meridian altitudes.

From Fig. 214, which depicts the southern part of the sky and the diurnal circle of the body for $\delta = \text{const}$ and a stationary observer,

It is seen that

$$H = h_1 + r_1$$

$$H = h_2 + r_2$$

or, in the general form,

$$H = h + r \tag{21.11}$$

where h_1, h_2 and r_1, r_2 are ex-meridian altitudes and their respective reductions.

From the figure it is seen that:

(1) at upper transit the reduction is positive,

(2) for the same hour angles, reduction east and west has the same magnitude by virtue of the symmetry of the diurnal parallel relative to the meridian.

If we construct a similar drawing for lower transit (Fig. 215), it will be evident that the reduction r_{low} will in this case have a minus sign, or

$$H' = h - r_{low} \tag{21.12}$$

The reduction r is an increment in altitude due to the diurnal motion of a body; in other words, reduction is "reduction of altitude" h to the magnitude of the meridian altitude H. Due to nonuniform motion of the body in altitude about the meridian, the reduction, as already mentioned, depends on the acceleration of the body in altitude, which means it depends both on the first and on the second derivative $\frac{d^2h}{dt^2}$, with respect to which we can find $r = \Delta h_T$.

The reduction r can also be found even if the declination of the body changes, and the ship is in motion. In such a case, take δ, φ_c and λ_c for the instant of measuring the ex-meridian altitude; considering them constant, determine the altitude H at the given place on the meridian. Consequently, in this case too we find the change in altitude Δh_t due to diurnal motion. The quantity φ_0 thus found will refer to the instant the ex-meridian altitude h is taken.

To derive a formula for the reduction r of a body situated near upper transit, use the simple and convenient method of transforming the basic formula

$$\sin h = \sin \varphi \cdot \sin \delta + \cos \varphi \cdot \cos \delta + \cos t \tag{*}$$

Here, considering h the ex-meridian altitude, equal to $H - r$, and replacing $\cos t$ by $1 - 2\sin^2 \frac{t}{2}$, we have

$$\sin (H - r) = \sin \varphi \cdot \sin \delta + \cos \varphi \cdot \cos \delta - 2 \cos \varphi \cdot \cos \delta \cdot \sin^2 \frac{t}{2}$$

or

$$\sin H \cdot \cos r - \cos H \cdot \sin r = \cos(\varphi - \delta) - 2\cos\varphi \cdot \cos\delta \cdot \sin^2\frac{t}{2} \quad (**)$$

Since r is always small, $\cos r$ may be replaced by $1 - \frac{r^2}{2} \times$ $\times (\text{arc } 1')^2$, $\sin r$ by $r \cdot \text{arc } 1'$.

In addition, as we know, $\varphi = Z + \delta$ or $\varphi - \delta = Z = 90° - H$, consequently, $\cos(\varphi - \delta) = \cos Z = \sin H$.

After the substitutions, the formula becomes $\sin H - \sin H \times$ $\times \frac{r^2}{2} (\text{arc } 1')^2 - \cos H \cdot r \cdot \text{arc } 1' = \sin H - 2\cos\varphi \cdot \cos\delta \cdot \sin^2\frac{t}{2}$

Whence, substituting $\sin Z = \sin(\varphi - \delta)$ for $\cos H$, we have

$$r = \frac{\cos\varphi \cdot \cos\delta}{\sin(\varphi - \delta)} \cdot \frac{2\sin^2\frac{t}{2}}{\text{arc } 1'} - \frac{r^2}{2} \cdot \tan H \cdot \text{arc } 1' \quad (21.13)$$

At lower transit of the body, that is, in the lower branch of the meridian, its hour angle t is close to 180° (see Fig. 215), and the reduction r_{low} is subtractive from the ex-meridian altitude. In place of the hour angle, let us take the distance from the meridian $t_1 = 180° - t$ and $h_1 = H' + r_{low}$. Put these values into formula (*); then after similar manipulations and substitution of $\varphi + \delta = 90° + H'$ and $H' = \varphi - \Delta$, we have for lower transit

$$r_{low} = \frac{\cos\varphi \cdot \cos\delta}{\sin(\varphi + \delta)} \cdot \frac{2\sin^2\frac{t_1}{2}}{\text{arc } 1'} + \frac{r_{low}^2}{2} \tan H' \cdot \text{arc } 1' \quad (21.14)$$

This formula differs from (21.13) in signs only. The second term of the reduction (we denote it by r_{II}) in formulas (21.13) and (21.14) is ordinarily very small and should be taken into account only for large H and for the first term $r_1 \geqslant 15'$.

Convert the first terms of (21.13) and (21.14) combined in a single general formula (with different signs) to a more convenient form; to do this, expand the expressions in the denominator and divide by $\cos\varphi \cdot \cos\delta$

$$r = \frac{1}{\dfrac{\sin\varphi \cdot \cos\delta \pm \cos\varphi \cdot \sin\delta}{\cos\varphi \cdot \cos\delta}} \cdot \frac{2\sin^2\frac{t}{2}}{\text{arc } 1'}$$

or

$$r = \frac{1}{\tan\varphi \mp \tan\delta} \cdot \frac{2\sin^2\frac{t}{2}}{\text{arc } 1'} \quad (21.15)$$

Multiplying the numerator and denominator by 100 in order to have whole numbers for the tangents, we finally get

$$r = \frac{1}{100 \tan \varphi \mp 100 \tan \delta} \cdot \frac{200 \sin^2 \frac{t}{2}}{\text{arc } 1'} \pm r_{II} \qquad (21.16)$$

This is the formula used in MT-53 to compute the special Tables 17a, 17b and 17в.

Table 17a contains the values of 100 tan x used to compute the quantity $K = 100 \tan \varphi \pm 100 \tan \delta$. In this formula the minus sign is affixed for φ and δ of the same name and for the upper transit of the celestial body; from the greater value always subtract the smaller one, because only the absolute value of K and not its sign is significant. In formula (21.16) the plus sign is taken if φ and δ are of contrary names or if the transit is lower.

In Table 17б the values of the first term of reduction r_I are given based on K and the hour angle of the body (or $180° - t = t_1$ for lower transit).

Table 17в gives the values of the second term of reduction r_{II} based on the arguments H and r_I. The quantity r_{II} is small and, as already mentioned, is taken into account for $r_I > 15'$ and $H > 45°$; the values of r_{II} are always subtractive from the altitude obtained (irrespective of transit). The quantity r_{II} is ordinarily disregarded.

The computed value of K may sometimes turn out greater than the limiting value of the argument K in Table 17б; then find $K_1 = K : 10$ and use it to take out the value r_1, which likewise should be divided by 10.

Example 7. $\varphi_c = 44°38'$N; $\delta_{\odot} = 16°54'$N; $t = 2°48'$E. Upper transit. Find r.

(1)

100 tan φ	99 ... Table 17a MT-53,
− 100 tan δ	30 ... Table 17a
K	69

(2) $\left.\begin{array}{l} K = 69 \\ t = 2°48' \end{array}\right\} r = 6'.0$ ... Table 17б MT-53

Example 8. $\varphi = 27°23'$S; $\delta_{*} = 19°24'.0$N; $t = 1°54'$W. Upper transit. Find r.

(1)

100 tan φ	52
+ 100 tan δ	35
K	87

(2) $\left.\begin{array}{l} K = 87 \\ t = 1°54' \end{array}\right\} r = 2'.2$

The reduction obtained from Table 17б, MT-53, is added to the corrected ex-meridian altitude, that is,

$$H = h + r_{\mathrm{I}} - r_{\mathrm{II}} \tag{21.17}$$

after which the latitude is found from the usual formula

$$\varphi_0 = (90° - H) + \delta$$

or for the lower transit

$$H' = h - r_{\mathrm{I}} - r_{\mathrm{II}} \tag{21.18}$$

and

$$\varphi_0 = H' + (90° - \delta)$$

Besides the above-described tables of reduction proposed by the Soviet captain Gernet in 1934, there are other tables compiled either from formula (21.13) divided into two terms in a manner similar to that described here, or on the basis of modified formulas. Such, for instance, are the Tables 18, 19, and 20 of MT-33; the tables of the Brown Almanac (British), and others.

SEC. 126. THE EFFECT OF ERRORS IN THE HOUR ANGLE (LONGITUDE) AND IN ALTITUDE ON THE LATITUDE BEING DETERMINED. LIMITS OF OBSERVATIONS

I. THE EFFECT OF A SYSTEMATIC ERROR IN THE HOUR ANGLE

From formula (21.13) it is seen that errors in the hour angle will enter the latitude, via reduction.

Let us determine the increment in reduction Δr for an increment in the hour angle Δt and let us take these increments as errors. To do this, in (21.15), without the second term of the reduction, substitute $\sin^2\frac{t}{2} = \frac{t^2}{4}\operatorname{arc}^2 1'$, which is permissible to within $\pm 0'.1$ for angles approximately up to 11°

$$r' = \frac{1}{\tan\varphi \mp \tan\delta} \cdot \frac{2\frac{t^2}{4}\operatorname{arc}^2 1'}{\operatorname{arc} 1'} = \frac{t^2 \operatorname{arc} 1'}{2(\tan\varphi \mp \tan\delta)} \tag{21.19}$$

Differentiating this formula with respect to r and t, replacing the differentials dr and dt by finite increments, and taking into account that $\Delta r = \mp \Delta\varphi$, we have

$$\Delta r' = \pm \Delta\varphi' = \operatorname{arc} 1' \cdot \frac{t'}{\tan\varphi \mp \tan\delta}\Delta t' \tag{21.20}$$

From expression (21.20) it is evident that the error in latitude depends not only on errors in the hour angle, but also on the hour

angle itself and the altitude of the celestial body (latent in the quantity $\tan\varphi \pm \tan\delta$).

The greater the altitude H of a body and the farther it is from the meridian, the more pronounced are the effects of errors in the hour angle and the greater the latitude error.

Errors in the hour angle are, as we know, due to:

(a) errors in the instant noted by the chronometer, and errors in the chronometer correction,

(b) errors in the computed longitude $\Delta\lambda_c$.

Given proper handling of chronometers, the total error in the instant should not exceed 1s5-2s, or 0′.5. Errors in D.R. position in open sea will always be greater. Therefore, the principal cause of errors Δt is thus the error $\Delta\lambda_c$ in the D.R. longitude; that is, $\Delta t \approx \Delta\lambda_c$.

II. OBSERVATIONAL LIMITS OF EX-MERIDIAN ALTITUDES

Let us determine the limiting value of the hour angle (t_{lim}) for which the measured altitude may be considered ex-meridian and workable by means of reduction. This limit obviously depends on the accuracy with which we desire to obtain the latitude and on the magnitude of possible errors in the hour angle (longitude).

From expression (21.20) we have

$$t' = \frac{\Delta\varphi'}{\Delta t'} \cdot \frac{\tan\varphi \mp \tan\delta}{\text{arc } 1'} \tag{21.21}$$

Let us impose the condition that the errors in the latitude should not exceed 1′. If we take it that in average conditions the error in the D. R. longitude $\Delta\lambda_c \leqslant 7'.0$ and the chronometer error in the instant $\leqslant 0'.5$, then the error in hour angle will be $\Delta t \leqslant 7'.5$ (or $\Delta t^{\min} = 0.5$ min).

Under these conditions, the limiting hour angle t_{lim} will be

$$t'_{lim} = 458'.4\,(\tan\varphi \mp \tan\delta) = 4'.58\ K \tag{21.22}$$

Here, the minus sign will be for upper transit and φ and δ of same name, and the plus sign for φ and δ of different names or lower transit of the celestial body.

If we take the hour angle roughly equal to the time interval (for the sun, this assumption holds within the limits of one hour up to ±1s), we can then obtain the limiting time interval $\Delta T^{\min}_{lim}$ during which the altitude may permissibly be considered ex-meridian (to within the accuracies of $\Delta\varphi$ and Δt given above).

$$t^{\min}_{lim} \approx T^{min}_{lim} = \frac{4'.58}{15}\ K = 0\text{m}.3056\ K = 30\text{m}.56\,(\tan\varphi \pm \tan\delta) \tag{21.23}$$

This formula was used to compile Table 19, MT-53, "Limits For the Observation of Ex-meridian Altitudes". The arguments for entry are φ_c and δ of the celestial body.

Example 9. 5 August 1962 at $\varphi_c = 38°.1N$; $\delta_{\odot} = 17°.0N$. Find the observational limits of ex-meridian altitudes of the sun.

From Table 19, MT-53:

$$t_{lim}^{min} = \Delta T_{lim}^{min} = 14 \text{ min}$$

From formula (21.23): $t_{lim}^{min} = 14.6$ min.

The limiting hour angle t_{lim} will obviously change if the conditions are changed, for instance, when it is necessary to enhance the accuracy of the latitude obtained.

The new value of the limiting hour angle, t_{lim}^{I} expressed by the general formula (21.21) may be obtained from the same table (Table 19) by setting up the proportion

$$\frac{t_{lim}^{I}}{t_{lim}^{min}} = \frac{\frac{1}{15} \cdot \frac{\Delta\varphi'}{\Delta t'} \frac{\tan\varphi - \tan\delta}{\text{arc } 1'}}{30\text{m}.56\,(\tan\varphi - \tan\delta)}$$

whence

$$t_{lim}^{I} = t_{lim}^{min} \cdot 7.5 \frac{\Delta\varphi'}{\Delta t'} \tag{21.24}$$

where t_{lim}^{min} is taken from Table 19, and $\Delta\varphi'$ and $\Delta t'$ are provided by the observer.

The limiting hour angle t_{lim} depends, as we have seen, on the value of K, that is, on the magnitude of the difference ($\tan\varphi - \tan\delta$). Therefore, for sun sights the limits will be less in low latitudes and greater in high latitudes. As is seen from (21.21), the limiting angle, t_{lim}, for lower transit will be greater than for upper and will increase with increasing declination.

By determining $t_{lim}^{min} = \Delta T_{lim}^{min}$ and computing the ship's time of transit of a body T_{tr}, it is possible to determine the time during which observation of ex-meridian altitudes is admissible.

$$\left.\begin{array}{ll} \text{Onset of observations:} & T_{onset} = T_{tr} - \Delta T_{lim}^{min} \\ \text{Termination of observations:} & T_{term} = T_{tr} + \Delta T_{lim}^{min} \end{array}\right\} \tag{21.25}$$

where T_{onset} is the ship's time for the onset of observations of ex-meridian altitudes and T_{term} is the ship's time for the termination of observations.

Example 10. 27 September 1968, at $\varphi = 22°N$; $\lambda = 128°E$. Determine T_{sh} (ZD = 10E) for commencing sights of ex-meridian altitudes of the sun.

(1)

	$T_{tab\ gr}$	11h 51m
	$\Delta T\lambda$	00
	T_{loc}'	11h 51m
	$-\lambda + ZD$	1 28
Transit	T_{sh}	13h 19m

(2)

T_{sh}	13h 19m
$-$ ZD	10
T_{gr}	3h 19m
$\delta_{\odot} \approx$	1°37′.3S

(3) From Table 19 $\Delta T_{lim}^{min} = t_{lim}^{min} =$ $= 13$ min

(4) Onset of observations

T_{sh}	13h 19m
ΔT_{lim}	$-$ 13
T_{onset}	13h 06m

(5) Termination of observations

T_{sh}	13h 19m
ΔT_{lim}	+ 13
T_{term}	13h 32m

III. REDUCING AND ELIMINATING ERRORS IN THE HOUR ANGLE

As a rule, the magnitude of errors in longitude $\Delta\lambda_c$ is not known, and although under ordinary conditions the errors are not great, they can introduce a perceptible error into φ_0.

For an analysis, we apply expression (18.22)

$$\Delta\varphi = \Delta t \cdot \cos\varphi \cdot \tan A$$

where for A we apply circular reckoning (0° to 360°).

To diminish the effect of errors Δt, it is obviously necessary to observe the body as close as possible to the meridian; here, $\tan A$ approaches zero and the error $\Delta\varphi$ diminishes.

The error $\Delta\varphi$ may be eliminated by observing the body in two positions: prior to transit and following transit at equal distances in azimuth from the meridian. Then the azimuth A_1 will be in the second quadrant and $\tan A_1$ will have a minus sign, while A_2 will be in the third quadrant and $\tan A_2$ will have a plus sign. The error $\Delta\varphi$ in these two observations will have different signs and will be eliminated in the half-sum of the latitudes obtained.

In place of equal azimuths we can observe the body at equal altitudes of equal hour angles prior to and following transit.

SEC. 127. DETERMINING THE LATITUDE FROM EX-MERIDIAN ALTITUDES (PRACTICAL ASPECTS)

Determination of latitude from ex-meridian altitudes at sea should be performed on the basis of several altitudes, not one, as is sometimes done.

Two possible ways of working a series of altitudes are:

1. From a series of 3 to 5 altitudes compute the average altitude and the instant; they yield r and φ_0.

2. Work each altitude and instant in a series of 5-11 altitudes. Average the H thus obtained, and the mean altitude yields φ_0.

The first procedure may be applied if few altitudes are taken and in sufficiently rapid succession (within one minute), and the altitudes themselves do not exceed 75° to 80°. Because of its extreme simplicity this method is most convenient for conditions at sea.

The second procedure is utilized for a very large number of measured altitudes and when the altitudes exceed 80°. Practically speaking, this method is applicable in cases when in addition to the latitude of the place the mean square error of the measured altitude and the latitude is computed. The advantage of this procedure is that latitude obtained from a large number of altitudes is computed with greater accuracy, at the same time giving an estimate of the accuracy of φ_0; however, the computations here are considerably more involved than in the first procedure. Working sights by the second procedure involves a definite routine (shown below); to derive errors, all altitudes are reduced to a single zenith.

To determine the latitude from ex-meridian or meridian altitudes requires the same instruments as for determining a position from altitude lines of position (Sec. 114).

Due to the fact that the maximum altitude remains constant for only a short time and there is always the danger of missing it, it is advisable to take altitudes as ex-meridian and confine oneself, if desired, to working the chosen maximum altitude. Let us examine the routine for deriving latitude by the first and second procedures. Reference here is to the sun, but determining φ by other celestial bodies does not differ in any way, though is very rarely done.

I. DETERMINING LATITUDE FROM A SMALL NUMBER OF EX-MERIDIAN ALTITUDES OF THE SUN (FIRST PROCEDURE)

A. Preparation for Observations

Preliminary:

(1) Take from chart φ_c and λ_c for presumed T_{sh} of sun's transit.

(2) Compute T_{sh} of sun's transit with the aid of the MAE and choose the approximate declination of the sun.

(3) From Table 19, MT-53, choose the limits of observation of ex-meridian altitudes, calculate the possible time of onset and termination of observations from formulas (21.25) and set the time for observations (the best is as close to the meridian as possible).

Just before starting observations:

(4) Ready the CH (or ИАС) sextant for daytime observations, make a superficial check of the error; and 10 minutes prior to the set time take the sextant to the place of observation but do not hold it open to the sun.

(5) If a watch or stop watch is to be used, check it with the chronometer.

(6) Adjust the chronometer correction u_{ch} to the instant of observations.

B. Observations

(7) Determine the index correction of the sextant, with a check if possible.

(8) At the set time, rapidly take 3 to 5 altitudes of the limb of the sun and note the time by watch, chronometer or stop watch. The instants of time are recorded either by the assistant observer or by the observer himself by the method of counting seconds given in Sec. 56. When measuring altitudes with the CH sextant do not forget about errors due to backlash. Turn the drum in one direction only, the same as that used to determine i.

(9) If the maximum altitude is desired, continue observations until 3 to 4 diminishing readings have shown that the transit has been passed. The maximum altitude is taken as the meridian altitude. *

(10) Following transit (at about the same altitudes), again measure 3 to 5 altitudes of the sun and note the times. The second set of observations is needed in order to eliminate any error due to inaccuracy of λ_c and also as a check against blunders.

(11) In low latitudes, note over which point (N or S) the altitudes were measured.

(12) Half way through the observations, note T_{sh}, lr, course, and speed of ship If lr has not been noted, compute it from the average T_{sh}. Take φ_c and λ_c at the median instant of observations. Check the eye height of the observer.

(13) If possible, measure the dip of the horizon.

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* This item is included only when determining φ from the meridian altitude.

C. Working Sights

(14) Compute the mean sr_{av} and $T_{ch\ av}$ in each round of sights (before and after the meridian). The observations prior to and after the meridian are worked separately so as to reduce the probability of any blunder.

(15) Correct sr_{av} with corrections. If the dip has been measured with an instrument, introduce its value, and enter Table 8a MT-63, with the eye height $e = 0$.

(16) Make an approximate and exact calculation of T_{gr} and from MAE obtain $t_{loc}^{\odot}$ and $\delta_{\odot}$.

(17) Compute $K = 100 \tan \varphi \pm 100 \tan \delta$ by means of Table 17a and from K and $t_{\odot}$ obtain r_{I} (Table 176); if $r_{\mathrm{I}} > 15'$, then select the second term of the reduction, r_{II}, from Table 17в, and then compute $r = r_{\mathrm{I}} \pm r_{\mathrm{II}}$.

(18) Add the reduction to the corrected altitude at upper transit: $H_{\odot} = h + r$. At lower transit, subtract the reduction: $H'_{\odot} = h - r$.

(19) Compute $Z = 90° - H_{\odot}$ and $\varphi_0 = Z + \delta_{\odot}$ according to the rules for determining latitude from the meridian altitude.

(20) From the two obtained latitudes compute the mean $\varphi_0 = \frac{\varphi_0' + \varphi_0''}{2}$ and adjust it to the mean instant. If conditions do not permit working the second series of altitudes, it is common to confine oneself to φ_0 alone.

Example 11. On 11.07.68 in the Bering Sea we expect to determine φ from the ex-meridian altitudes of the sun. At noon we take from the chart $\varphi_c \approx 54°$N; $\lambda_c \approx 162°45'$E; the ship's clock keeps zone time ZD 12E.

A. Preparation for Observations

(1)		
	T_{sh}	12h 12.07
	− ZD	12
	$T_{gr} \approx$	0h 12.07
From MAE	$\delta_{\odot}$	22°N
From Table 19	ΔT_{lim}	30m

(2) Time of transit

T_{tab}	12h 05m
ΔT_{λ}	00
T_{loc}	12 05
− λ	10 51
T_{gr}	1 14
+ ZD	12
T_{sh}^{tr}	13h 14m

(3) Time of observations

T_{sh}^{tr}	13h 14m
ΔT_{lim}	∓ 30
T_{sh}	12h 44m onset
T_{sh}	13h 44m termination

Plan the first set of observations for 13h8m and the second set for 13h 20m. Prior to observations, prepared sextant and obtained chronometer correction for instant of observations: $u_{ch} = -11$m 52s. Instants of time noted by an assistant.

B. Observations

(1) Index correction

$oi_1 = 359°28'.4$ $\qquad oi_2 - oi_1 = 1°3'.6$

$$i = \frac{-2'.0 + 1'.6}{2} = -0'.2$$

$oi_2 = 360°32'.0$ $\qquad 4R_\odot = 1°3'.2$

(2) First set of observations:

$sr_{\underline{\odot}}$ (to S)	T_{ch}
57°33'.2	1h 20m 56s
33 .5	21 44
34 .0	22 38

At about $T_{sh} = 13\text{h}\,15\text{m}$; $lr = 37.5$; $\varphi_c = 54°17'\text{N}$; $\lambda_c = 162°48'\text{E}$; course 132° true; speed 12 knots; $e = 13$ metres; $s = -36'' = -0'.6$.

(3) Second set of observations:

$sr_{\underline{\odot}}$	T_{ch}
57°32'.6	1h 28m 55s
32 .2	29 48
32 .0	30 25

C. Working Sights

First Set of Observations

T_{st}	13h 15m
ZD	12
T_{gr}	1h 15m 11.07
δ_T	22°7'.7 (0.3)
$\Delta\delta$	0 .0
$\delta_\odot$	22°7'.7N

av. T_{ch}	1h 21m 46s
u_{ch}	— 11 52
T_{gr}	1h 9m 54s
t_T	193°39'.9 (0'.2)
Δt	2 28 .5
$t^\odot_{gr}$	196 8 .4
λ	162 48 .0
$t^\odot_{loc}$	358°56'.4
	=1° 3'.6E

av. sr	57°33'.6
$i+s$	— 0 .8
h'	57 32 .8
Δ_{tot}	+ 9 .0
Δ_{ad}	— 0 .2
h	57 41 .6
r	0 .6
H	57 42 .2
Z	32 17 .8
$\delta_\odot$	22 7 .7
φ'_0	54°25'.5N

100 tan φ	139
100 tan δ	41
K	98

$$\left.\begin{array}{l} t = 1°3'.6 \\ K = 98 \end{array}\right\} r = 0'.6$$

Second Set of Observations

δ_T	$22°7'.7\ (\overline{0.3})$
$\Delta\delta$	$-\ 0\ .1$
$\delta_{\odot}$	$22°7'.6$N

av. T_{ch}	1h 29m 43s
u_{ch}	— 11 52
T_{gr}	1h 17m 51s
t_T	$193°39'.9\ (0'.2)$
Δt	4 27 .8
t_{gr}	198 7 .7
$+\lambda$	162 48
$t_{loc}^{\odot}$	$0°55'.7$W

av. sr	$57°32'.3$
$i+s$	— 0 .8
h'	57 31 .5
Δ_{tot}	9 .0
Δ_{ad}	— 0 .2
h	57 40 .3
r	0 .5
H	57 40 .8
Z	32 19 .2
δ	22 7 .6
φ_0''	$54°23'.8$N

$T_{sh} = 13\text{h}\ 14\text{m}$

$$\varphi_0 = \frac{\varphi_0' + \varphi_0''}{2} = 54°26'.2\text{N} \qquad \left.\begin{array}{l} t = 56' \\ K = 98 \end{array}\right\} r = 0'.5$$

II. DETERMINING LATITUDE FROM A NUMBER OF EX-MERIDIAN ALTITUDES OF THE SUN WITH A DERIVATION OF THE ERROR OF MEASURED ALTITUDE (SECOND PROCEDURE)

A. Preparation for Observations

Same preparation for observations as in first procedure.

B. Observations

(1) Measure 7 to 11 altitudes of the sun and note instants by chronometer.

(2) At mean instant note T_{sh}, lr, course, speed of ship, and compass bearing of sun if sights are taken far away from the meridian.

(3) Measure dip of horizon with a dipmeter if the latter is available.

C. Working Sights

(4) Choose instant at which it is necessary to obtain the latitude. It is most convenient to adjust to a mean instant of time, that is to the third T_3, fourth T_4, fifth T_5 and so forth (bear in mind that this is not the arithmetic mean of the instants).

(5) Compute the declination and hour angle of the sun for this instant.

(6) From *sr* compute for this instant the total altitude correction: $\Delta h = i + s + \Delta_{tot} + \Delta_{ad}$.

(7) From φ_c and $\delta_{\odot}$ compute $K = 100 \tan \varphi \mp 100 \tan \delta$.

(8) Find the difference $A_{\odot} - TC$, and from Table 16, MT-53, find Δh_Z for one minute.

(9) Arrange computation routine as indicated in the example below.

(10) In this form, make the difference $\Delta T \quad T_i - T_5$ between each instant of the chronometer and the mean instant (say T_5).

(11) From Table 2, MAE, convert these differences to degrees and add to (or subtract from) the hour angle for the mean instant; we then have the hour angle for each instant.

Note. There are two other equivalent methods for obtaining hour angles:

(a) Compute the exact time of transit (from λ_c) and take the difference between this chronometer instant and the noted instants of T_{ch} as the hour angles; that is,

$$t = \Delta T = \Delta T_{i\,ch} - T_{tran\,ch}$$

(b) Convert the hour angle at the mean instant (say t_5) to units of time and find the difference $\Delta t = t_5 - (T_5)_{ch}$. Apply this difference to each instant and obtain $t = T_i + \Delta t$ which will be the hour angles at the appropriate chronometer instants. These procedures do not have any particular advantages.

(12) Using each hour angle and the total K, take the reductions from Table 176.

(13) Compute Δh_Z for each altitude; to do this, multiply the Δh_Z obtained above for one minute by the quantities $\Delta T^{\min}$.

(14) Combine the corrections Δh, r and Δh_Z and we have ΔH_{tot}.

(15) Correct each *sr* with the total correction ΔH_{tot} and this gives us the reduced altitudes H_{red}.

(16) Find the arithmetic mean of H_{red}, form the difference $H_{red} - H_{av}$; and then compute v^2 and Σv^2.

(17) Compute $\varepsilon_h = \pm \sqrt{\dfrac{\Sigma v^2}{n-1}}$ and $\varepsilon_0 = \varepsilon_\varphi = \dfrac{\varepsilon_h}{\sqrt{n}}$.

(18) Compute $Z_{av} = 90° - H_{av}$, $\varphi_0 = Z_{av} + \delta_{\odot}$ and $\varphi = \varphi_0 \pm 1\ \varepsilon_\varphi$. The φ_0 obtained refers to the chosen instant (T_5), to which the *lr* should likewise be adjusted.

Example 12. On 3.08.68 in the Barents Sea; course 185° true; speed 12 knots; measured a series of circummeridian altitudes of the sun ($\odot$) at about $T_{sh} =$ 12h 45m (ZD = 4E); $lr = 43.5$; $\varphi_c = 68°11'.6$N; $\lambda_c = 42°0'.4$E; $e =$ 12.5 metres; $i + s = -5'.5$; $u_{ch} =$ 0m 48s; $TB_{\odot} \approx 170°$; readings and instants are recorded in computation form.

No.	T_{ch}	$\Delta T^{\min, \sec}$	$\Delta T^{\circ\prime}$	$t_{loc}^{\odot} = t_{av} \pm$ $\pm \Delta T^{\circ}$	r (table 176)
1	8h 37m 29s	7m 31s	1°53′	10°21′E	25′.4
2	39 34	5 26	1 21 .5	9 50	23 .0
3	41 50	3 10	47 .5	9 16	20 .4
4	43 21	1 39	25	8 53	18 .7
5	45 00	00	00	8 28	17 .1
6	47 12	2 12	33	7 55	14 .9
7	49 4	4 04	1 01	7 27	13 .2
8	51 1	6 01	1 30	6 58	11 .5
9	8h 52m 55s	7m 55s	1°59′	6°29′E	10 .0

Auxiliary computations:

(1)

T_{sh}	124h 5m
− ZD	4
T_{gr}	8h 45m3.08

T_{ch}	8h 45m 00s
u_{ch}	— 48
T_{gr}	8h 44m 12s

δ_T	17°28′.0 ($\overline{0.7}$)
$\Delta\delta$	— 0 .4
$\delta_{\odot}$	17°27′.6N

t_T	298°28′.5 (0′.3)
Δt	11 3 .0
$t_{gr}^{\odot}$	309°31′.5
+ λ	42 0 .4
av. $t_{loc}^{\odot}$	351°31′.9W ≈ 8°28′E

(2)

100 tan φ	250 } Table 17a
100 tan δ	31 } Table 17a
K	219

(3)

$A_{\odot}$	170°
TC	185°
$A-K$	345°
Δh_1 min	+ 0′.20 Table 16

(4)

$i+s$	—5′.5
Table 8 Δ_{tot}	+8 .6
Table 8a Δ_{ad}	—0 .2
Δh	+2′.9

Δh_Z	$\Delta H_{tot} = \Delta h + r + \Delta h_Z$	sr	$H_{red} = sr + \Delta H$	$v = H_{red} - H_{av}$	v^2
1'.5	29'.8	38°42'.0	39°11'.8	+0.9	0.81
1 .1	27 .0	43 .6	10 .6	—0.3	09
0 .6	23 .9	47 .2	11 .1	+0.2	04
0 .3	21 .9	48 .3	10 .2	—0.7	49
0 .0	20 .0	51 .2	39°11'.2	+0.3	09
0 .4	17 .4	53 .6	11 .0	—0.1	01
0 .8	15 .3	55 .0	10 .3	—0.6	36
1 .2	13 .2	38°57'.3	10 .5	—0.4	16
1 .6	11 .3	39°00'.2	39°11'.5	+0.6	0.36
		H_{av}	39°10'.9	$\Sigma v^2 = 2.41$	
		$Z_{\odot}$	50 49 .1N		
		$\delta_{\odot}$	17 27 .6N		
	$T_{sh} = 12\text{h}\ 44\text{m}$	φ_0	68°16'.7N ± 0'.2		

(5) $\varepsilon_h = \pm\sqrt{\frac{\Sigma v^2}{n-1}} = \sqrt{\frac{2.41}{8}} = \pm 0'.55$

$\varepsilon_0 = \varepsilon_\varphi = \pm \frac{\varepsilon_h}{\sqrt{n}} = \pm 0'.18 \approx \pm 0'.2$

$\varepsilon_{prob} = \frac{2}{3}\varepsilon_h = \pm 0'.36$

$\varepsilon_{lim} = 3\varepsilon_h = \pm 1'.65$

Leeway in latitude 5'.1 to N

SEC. 128. DETERMINING LATITUDE FROM POLARIS ALTITUDES

The altitude of the celestial pole above the horizon is, as we know, equal to the latitude of the position; for this reason, if there were a star in each of the celestial poles, it would be a simple task to measure the altitude of such a star and correct it with corrections in order to obtain the latitude of the position.

However, there are no stars in the poles themselves. But quite near the north pole (P_N) there is a rather bright star—α Ursae Minoris, called the Pole Star, or Polaris, because of its proximity to the pole. The coordinates of Polaris are : α = 1h56m (29°) and

$\delta = 89°5'N$ ($\Delta = 55'$). It therefore describes, in its diurnal motion, a parallel with a spherical radius of less than 1°. As a result, the azimuth of Polaris is always close to 0° (N), and it is always in the most favourable position for determining φ. For this same reason the altitude of Polaris is always close to the latitude and can differ from the latter only by a small quantity x (Fig. 216). The problem of determining φ from the altitude of Polaris reduces to finding this correction x, which is equal to the difference between the altitude of Polaris at a given instant and the latitude.

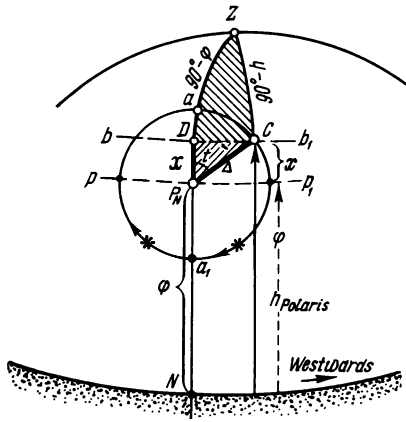

Fig. 216

Let us depict the sphere in the plane of the prime vertical from P_N (see Fig. 216); let the small circle aa_1 be the parallel of diurnal motion of Polaris; points a and a_1 are the upper and lower transits of the star; C is the position of Polaris at a certain instant; P_NC is its meridian, ZC the vertical circle, and bCb_1 is the parallel of altitudes of Polaris.

From the figure it will be seen that:

(1) The altitude of Polaris may be both greater and less than the altitude of the pole, equal to φ (the parallels of altitudes pp_1) and the correction x may be both positive (the section p_1a_1p) and negative (the section pap_1); that is, the sign x is determined by the hour angle t of Polaris.

(2) This correction will have maximum value, equal to $\Delta = 90° - \delta \approx 55'$ at instants of upper transit (a), where $x = -\Delta$, and lower transit (a_1) $x = +\Delta$. At the instants of upper and lower transit, the hour angle of the star will be 0° and 180°, while the local sidereal time at these instants will, from formula $S = t + \alpha$, be $S_{loc} = \alpha_* = 29°$ and $S_{loc} = \alpha_* + 180° = 209°$.

The correction $x = 0$ for positions of the star on the parallel of altitudes pp_1, that is, for $S_{loc} = 119°$ and $S_{loc} = 299°$.

From Fig. 216 we write

$$\varphi = h - x \qquad (*)$$

whence

$$h = \varphi + x \qquad (**)$$

Regarding the triangle P_NCD as a plane triangle (to a first approximation), we get the formula

$$x = \Delta \cdot \cos t \qquad (21.26)$$

which may be used for approximate computations. However, the triangle is really not plane, and so a more precise formula may be derived from the basic equation obtained from the spherical triangle P_NCZ

$$\sin h = \sin \varphi \cdot \sin \delta + \cos \varphi \cdot \cos \delta \cdot \cos t$$

Substituting expression (**) for h and replacing $\delta = 90° - \Delta$, we get

$$\sin (\varphi + x) = \sin \varphi \cdot \cos \Delta + \cos \varphi \cdot \sin \Delta \cdot \cos t$$

Expanding the left-hand side and replacing the sines and cosines of small angles x and Δ with one or two terms of the Taylor's series (that is, $\sin \alpha = \alpha' \text{ arc } 1'$ and $\cos \alpha = 1 - \frac{\alpha^2}{2} \text{ arc}^2\ 1'$), we have

$$\sin \varphi \left(1 - \frac{x'^2}{2} \text{arc}^2\ 1'\right) + \cos \varphi \cdot x' \cdot \text{arc } 1' =$$
$$= \sin \varphi \left(1 - \frac{\Delta'^2}{2} \text{arc}^2\ 1'\right) + \cos \varphi \cdot \Delta' \cdot \text{arc } 1' \cdot \cos t$$

whence, after simplifications and dropping the minute symbols ('), we get

$$x = \Delta \cdot \cos t + \frac{x^2 - \Delta^2}{2} \tan \varphi \cdot \text{arc } 1'$$

The second term is small due to the factor $\text{arc } 1' = \frac{1}{3,438}$, for this reason we take $\tan \varphi \approx \tan h$, and replace x with its first approximation from the formula (21.26). After simplification we have

$$x = \Delta \cdot \cos t - \frac{\Delta^2}{2} \sin^2 t \cdot \tan h \cdot \text{arc } 1'$$

Since $t = S - \alpha$, we can write

$$x = \Delta \cdot \cos (S_{loc} - \alpha) - \frac{\Delta^2}{2} \sin^2 (S_{loc} - \alpha) \tan h \cdot \text{arc } 1'$$

During a single year and from year to year, the quantities Δ and α of Polaris vary mainly due to precession and aberration;

for the sake of simplicity, we take their mean values for the given year Δ_0 and α_0; the changes that this entails are taken into account in an additional term; adding and subtracting $\Delta_0 \cdot \cos (S_{loc} - \alpha_0)$ we have

$$x = \Delta_0 \cdot \cos (S_{loc} - \alpha_0) - \frac{\Delta_0^2}{2} \sin^2 (S_{loc} - \alpha_0) \tan h \cdot \text{arc } 1' - [\Delta_0 \cdot \cos (S_{loc} - \alpha_0) - \Delta \cdot \cos (S_{loc} - \alpha)] \quad (21.27)$$

Then by formula (*), we have in the general form

$$\varphi = h - \Delta_0 \cdot \cos (S_{loc} - \alpha_0) + \frac{\Delta_0^2}{2} \sin^2 (S_{loc} - \alpha_0) \tan h \cdot \text{arc } 1' + [\Delta_0 \cdot \cos (S_{loc} - \alpha_0) - \Delta \cdot \cos (S_{loc} - \alpha)] \quad (21.28)$$

or

$$\varphi = h + \mathrm{I} + \mathrm{II} + \mathrm{III} \quad (21.29)$$

Almanacs (MAE) contain tables of "The Latitude from the Altitude of Polaris" for each year compiled on the basis of formula (21.28). These tables are broken down into three tables:

Table I gives the values of the correction $\mathrm{I} = -\Delta_0 \cdot \cos (S_{loc} - \alpha_0)$, which represents the principal values of x; the argument for entry is the sidereal local time $S_{loc} = t_{loc}^{\Upsilon}$;

Table II gives the values of $\mathrm{II} = \frac{\Delta_0^2}{2} \sin^2 (S - \alpha_0) \cdot \tan h \cdot \text{arc } 1'$, which are corrections for the sphericity of the triangle; the arguments are $S_{loc} = t_{loc}^{\Upsilon}$ and h of Polaris;

Table III gives the values of $\mathrm{III} = [\Delta_0 \cdot \cos (S_{loc} - \alpha_0) - \Delta \cdot \cos (S_{loc} - \alpha)]$, which are the corrections for change of coordinates with time; the arguments are $S_{loc} = t_{loc}^{\Upsilon}$ and the date. For φ computed to within $1'$, quantities II and III of the corrections may be disregarded up to latitudes of $50°N$. These corrections should definitely be included for greater latitudes and for an accuracy up to $0'.1$.

These tables are arranged differently in the almanacs of other countries. For instance, in the British-American Nautical Almanac all three corrections are given in a single table, the first corrections being increased by 1° to obtain positive values. Then 1° is subtracted from the φ obtained.

This method for determining φ may be applied in twilight; and if the horizon is visible, then at any hour of the night, during the entire year and nearly in all northern latitudes from $5°N$ to $75°N$. It is inconvenient to measure the altitude of Polaris any farther north.

I. THE EFFECT OF SYSTEMATIC ERRORS

The effect of systematic errors in the hour angle of Polaris on φ will be practically negligible, as is evident from the variation of the largest correction I with time. Let us differentiate formula (21.26): $\Delta x = \Delta \cdot \sin t \cdot \Delta t \cdot \text{arc } 1'$, whence for $\Delta t = \Delta S = 15'$ and $t = 90°$ we have $\Delta x \approx 0'.2$; that is, even if the error in longitude is 15′, then the error in latitude will be less than the observational accuracy.

The only way to eliminate errors in measured altitudes is by observing another star (in addition to Polaris) in the opposite part of the meridian or near it and by averaging the two latitudes. Since this is rarely done, the latitude obtained from Polaris contains systematic errors in altitudes.

II. THE EFFECT OF RANDOM ERRORS IN THE ALTITUDE OF POLARIS

The effect of random errors may be reduced by taking 3-5 altitudes of a star with subsequent computation of the mean instant and mean altitude. Due to the effect of errors, the parallel of latitude computed from Polaris should be represented in the form of a "band of latitude" like that discussed for altitudes near the meridian.

Practically speaking, determination of latitude from Polaris involves:

(1) measuring 3-5 altitudes of Polaris and recording the instants to within 10 seconds, noting T_{sh}, lr and taking λ_c from a chart and also φ_c for comparison;

(2) computing the arithmetic mean of sextant readings and chronometer times;

(3) computing T_{gr} and from MAE, obtaining $t^{\Upsilon}_{loc} = S_{loc}$. Correct average sr and obtain h_*;

(4) taking from MAE corrections I, II, and III and applying them to h_* with appropriate signs.

Example 13. On 12 September 1968, measured three Polaris altitudes and recorded instants in Black Sea during evening twilight on course 5° true at 12 knots speed at about $T_{sh} = 19\text{h } 50\text{m}$ (ZD = 3E); $lr = +7.3$; $\varphi_c = 45°33'$N; $\lambda_c = 30°20'$E.

sr	T_{ch}
45°23′.5	4h 48m 30s
23 .0	49 20
24 .2	50 40

Corrections: $u_{ch} = +1\text{m } 8\text{s} \approx 1\text{m } 10\text{s}$
$i + s = -2'.7$
$e = 10.6$ metres

(1)

T_{sh}	19h 50m
$-$ ZD	3
T_{gr}	16h 50m 12.09

av. T_{ch}	4h 49m 30s
u_{ch}	$+$ 1 10
T_{gr}	16h 50m 40s
t_T^{Υ}	231°43′.1
Δt	12 42 .1
t_{gr}^{Υ}	244°25′.2
$+\ \lambda$	30 20
$t_{loc}^{\Upsilon} \approx$	274°45′.2

(2)

av. sr		45°23′.6
$i+s$		$-$ 2 .7
h'		45 20 .9
Δ_{tot}		$-$ 6 .8
h_{pol}		45′14′.1
from t_{loc}^{Υ}	I	$+$ 23 .0
from t_{loc}^{Υ} and h	II	$+$ 0 .3
from t_{loc}^{Υ} and date	III	$+$ 0 .1
φ_0		45°37′.5N

Leeway $\Delta\varphi = 4'.5$ along course

From a series of 7-9 altitudes of Polaris it is also possible to derive a mean error of altitude measurement of stars. In this case, each altitude is worked separately for its own instant of time and then reduced to the zenith of the mean altitude. When the latitudes are obtained, the mean is computed and the mean square error $\varepsilon_h = \varepsilon_\varphi$ is obtained, as usual, from the deviations from it.

SEC. 129. RELATIONSHIP BETWEEN A PARALLEL OF LATITUDE AND AN ALTITUDE LINE OF POSITION. THE LIMITS FOR REPLACING AN ALTITUDE LINE WITH A PARALLEL

From equation $\sin h_0 = \sin \varphi_0 \cdot \sin \delta + \cos \varphi_0 \cdot \cos \delta \cdot \cos t$ it is seen that there is a close relationship between the φ_0 being determined and the observed altitude h_0. Graphically, on a chart, h_0 is expressed by a circle of equal altitudes or—on a small section—by an altitude line; the latitude φ_0 is graphically expressed by the parallel φ_0. As we know from the properties of an altitude line, the common point of these two lines will lie on the meridian of the D.R. longitude. Let us consider two problems:

(1) How can we determine φ_0 from an altitude line of position plotted on a chart?

(2) Within what limits can an altitude line be replaced by a parallel of D.R. latitude?

We shall solve these problems graphically.

I. OBTAINING THE LATITUDE FROM AN ALTITUDE LINE OF POSITION

We assume that an altitude line I — I has been laid down on a chart from D.R. (computed) position M_c (Fig. 217). From the properties of an altitude line considered in Sec. 106 it follows that

point D_1, where the altitude line intersects the meridian λ_c, lies on the parallel φ_0'. Hence, to obtain φ_0' it is necessary to find on the chart the point of intersection of the position line with the meridian λ_c and to take the latitude of this point, which will be φ_0'. The greater the error in λ_c and the farther the body from the meridian ($A \neq 180°$ or $0°$), the greater will be the error in φ_0, as is clearly seen from Fig. 217, where point D_1' corresponds to λ_0, and D_1 to

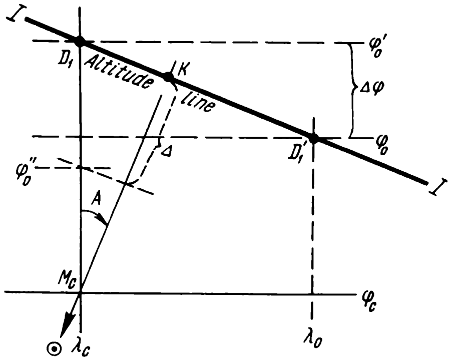

Fig. 217

the erroneous λ_c. The difference $\Delta\varphi$ between the parallels of these points corresponds to the error in φ_0' due to the erroneous longitude. It is clear from this figure that the altitude line of position is independent of φ_c and λ_c, whereas the computed φ_0 depends on the longitude.

Practically speaking, the latitude computed from reduction tables or otherwise may not coincide with φ_0 obtained by the indicated graphic method due to the different magnitudes of errors of the methods and of computations. However, these discrepancies are ordinarily slight.

If there is a systematic error Δ in the observed altitudes (see Fig. 217), the latitude φ_0'' obtained will include this error increased sec A-fold; the same applies to random errors too.

II. THE LIMITS FOR REPLACING AN ALTITUDE LINE OF POSITION BY A PARALLEL OF OBSERVED LATITUDE

Let us assume that the observed position M_0 (Fig. 218) is obtained from two altitude lines of position, the first of which corresponds to the ex-meridian altitude. We can also compute φ_0' from the same altitude; then, as shown above, the parallel of this latitude will pass through point D_1 on meridian λ_c. If M_1, the point of intersection of the parallel φ_0' and the line $II-II$ is taken as the observed position, an error will be committed in the observed position equal

to the segment $M_0M_1 = \Delta$. This error always occurs when combinations are applied of an altitude line of position and a parallel of latitude; these are the so-called particular cases of determining position: "morning—near noon", "near noon—evening", and so forth.

The exact formula expressing the linear error Δ in terms of Δh_1 and Δh_2 is rather cumbersome; a simpler formula may be obtained

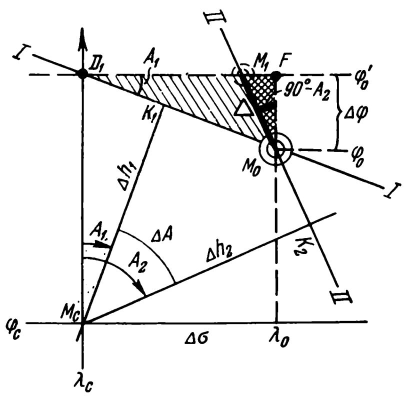

Fig. 218

by applying the error in departure $D_1F = \Delta\sigma = (\lambda_0 - \lambda_c)\cos\varphi$ and the error in the resulting latitude $M_0F = \Delta\varphi$. From the right triangle M_0FD_1 we get a formula for the error in latitude

$$\Delta\varphi = \Delta\sigma \cdot \tan A_1$$

and from the triangle M_0M_1F we have

$$\Delta\varphi = \Delta \cdot \cos(90° - A_2)$$

or

$$\Delta = \Delta\varphi \cdot \operatorname{cosec} A_2 = \Delta\sigma \cdot \tan A_1 \cdot \operatorname{cosec} A_2 \qquad (21.30)$$

In other words, the linear error in the position depends on the possible error in the computed longitude (departure) and on the azimuths of the celestial bodies. It will be seen from this formula and Fig. 217 that Δ is zero in two cases: if $A_1 = 180°$ (0°) or $\Delta\sigma = 0$; that is, if the first body was observed exactly on the meridian or if the error in longitude is zero.

Conclusions

(1) In the general case it is theoretically impossible to replace an altitude line of position by the parallel φ_0.

(2) Practically speaking, it is possible to replace an altitude line by a parallel if the body is located on the meridian or very close to it.

(3) In order to plot position line for a known $\Delta\varphi$, compute the azimuth and construct the line by the "latitude method" (see Fig. 219 and Example 14).

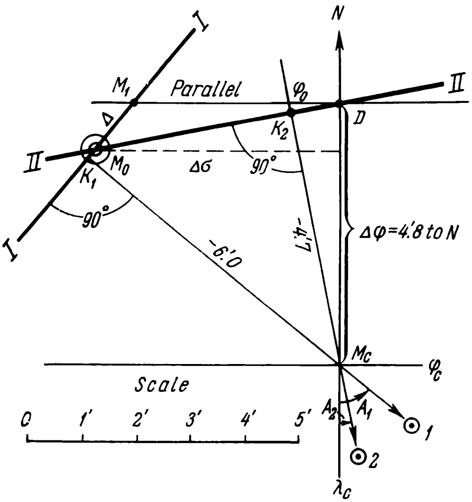

Fig. 219

Example 14. Using the data of Example 12, we have for one altitude at time $T_{sh} = 12$h 45m; $sr = 38°51'.2$; $T_{ch} = 8$h 45 m 00s; $\varphi_c = 68°11'.6$N; $\lambda_c = 42°0'.4$E,

$$T_{gr} = 8\text{h } 44\text{m } 12\text{s}$$
$$t_{loc}^{\odot} = 8°28'.0\text{E}$$
$$\delta_{\odot} = 17°27'.6\text{N}$$
$$h = 38°51'.2 + 2'.9 = 38°54'.1$$

Let us presume that earlier, at $T_{sh} = 9$h 20m, we obtained the first line with $A_{c1} = 50°.5$SE, $h - h_c = -6'.0$. Find: (a) the latitudes by reduction and by the second line; (b) the ship's position at $T_{sh} = 12$h 45m from two lines and from the first line and φ_0; (c) the error Δ due to replacement of the second line by the parallel φ_0.

Solution. 1. Determine the elements of the second altitude line at $T_{sh} = 12$h 45m; the results are: $A_{c2} = 10°.24$SE; $h_c = 38°58'.8$; $h - h_c = 38°54'.1 - 38°58'.8 = -4'.7$.

2. For the ex-meridian altitude at time $T_{ch} = 8$h 45m 00s from Example 12 we have $r = 17'.1$; $H = 39°11'.2$; we obtain

$$\varphi_0 = 50°48'.8\text{N} + 17°27'.6 = 68°16'.4\text{N}$$

$$\Delta\varphi = \varphi_0 - \varphi_c = 4'.8 \text{ to N}$$

3. We plot the line (Fig. 219).

4. The graphical construction yields the following:

(a) as we see, the latitude of point D of the altitude line practically coincides with φ_0 from item 2;

(b) the ship's position given by two lines will be at point M_0;

(c) the ship's position given by the first line and φ_0 will be at M_1;

(d) the linear error in position Δ taken from the figure is: $\Delta = 1'.2$;

(e) computing the error Δ from formula (21.30) with the following data (in the meaning of the formula): $A_{c2} = A_1 = 10°.24$, $A_{c1} = A_2 = 50°.5$, $\Delta\sigma = 4'.5$ (from figure), we have $\Delta = 1'.07 \approx 1'.1$;

(f) the observed coordinates of M_0 (based on two lines) are

$$T_{sh} = 12\text{h } 45\text{m} \begin{cases} \varphi_0 = 68°15'.6\text{N} \\ \lambda_0 = 41°48'.2\text{E} \end{cases}$$

(g) erroneous coordinates of point M_1:

$$T_{sh} = 12\text{h } 45\text{m} \begin{cases} \varphi_1 = 68°16'.4\text{N} \\ \lambda_1 = 41°50'.4\text{E} \end{cases}$$

SEC. 130. ESSENTIALS FOR DETERMINING LONGITUDE AT SEA FROM ALTITUDES OF CELESTIAL BODIES AND A CHRONOMETER

In Sec. 41 it was established that the times or hour angles at different meridians differ by the differences of their longitudes and if one of the meridians is the prime meridian, then the difference is the magnitude of longitude of the second meridian, that is,

$$T_{loc} - T_{gr} = \lambda_E$$

or

$$\lambda_E = t_{loc} - t_{gr} \tag{21.31}$$

where t_{loc} and t_{gr} are necessarily west hour angles of the body. If $t_{loc} > t_{gr}$ by an amount less than 180°, the longitude is east; but if the difference is greater than 180° or if $t_{loc} < t_{gr}$, then the longitude is west.

It is this latter equality that is ordinarily used to determine the longitude. The quantity t_{gr} is obtained from the MAE by the chronometer time and is directly dependent on its readings; for this reason, it is often called the "method of transporting chronometers".

The local hour angle t_{loc} may be computed from the astronomical triangle of the celestial body from the following data: the measured altitude, the declination of the body and the computed latitude. Using the formula of the cosine of a side, we have

$$\sin h = \sin\varphi \cdot \sin\delta + \cos\varphi \cdot \cos\delta \cdot \cos t_{loc} \tag{*}$$

whence

$$\cos t_{loc} = \sec\varphi \cdot \sec\delta \cdot \sin h - \tan\varphi \cdot \tan\delta \tag{21.32}$$

By substituting, in this formula, $\cos t_{loc} = 1 - 2\sin^2 \frac{t_{loc}}{2}$, we have a second formula after some manipulations:

$$\sin^2 \frac{t_{loc}}{2} = \frac{\cos(\varphi-\delta)-\sin h}{2\cos\varphi\cdot\cos\delta} = \frac{\cos(\varphi-\delta)}{2\cos\varphi\cdot\cos\delta}\left[1-\frac{\sin h}{\cos(\varphi-\delta)}\right] \qquad (21.33)$$

Formula (21.32) is used if $t_{loc} > 60°$, and formula (21.33) if $t_{loc} < 60°$.

When computing t_{loc} by (21.33) bear in mind that after taking logs the term in brackets, that is $\log\left[1-\frac{\sin h}{\cos(\varphi-\delta)}\right]$, is β taken from the Gauss Tables for differences of $Arg = \log\cos(\varphi-\delta) - \log\sin h$.

Take the calculated t_{loc} and convert it to west hour angle, then we have

$$\lambda = t_{loc} - t_{gr}$$

SEC. 131. PARTICULAR CASES OF DETERMINING POSITION FROM AN ALTITUDE LINE OF POSITION AND A BODY NEAR THE MERIDIAN (φ) OR THE PRIME VERTICAL (λ)

Particular cases for determining position, by which are meant determinations based on an altitude line of position and the parallel φ_0 or the meridian λ_0, obtained analytically, are at present applied to the sun only. The methods for determining position from sun altitudes near the meridian or off it (that is, by latitude and position line) are known as "morning—noon", "morning—near noon", "noon—evening", "near noon—evening".

The first two methods consist in advancing the morning altitude line of position $I—I$ by dead reckoning to the meridian D.R. point M_c'' (Fig. 220*a*) and combining it with the latitude obtained from the meridian or ex-meridian altitudes. The latter two methods consist in the fact that the parallel of latitude φ_0 obtained at noon or near noon is advanced by dead reckoning to the instant when the second line $II—II$ is determined and is combined with it (Fig. 220*b*). Here, dead reckoning from noon may be done in two ways:

(a) From point D_1 with latitude φ_0' and the original λ_c'. In this case, when laying down the second line II-II, the observed position lies at the point of intersection with the parallel φ_0'', the estimated (computed-observed) point D_2.

(b) From point M_c' dead reckoning is performed as usual; here, the parallel φ_0' is not laid down, but the difference $\Delta\varphi = \varphi_0' - \varphi_c'$ is found.

After determining and plotting the elements of the second line, the quantity $\Delta\varphi = \varphi'_0 - \varphi'_c$ is laid down from the second computed point M''_c and we have the parallel φ''_0 (the actually computed-observed parallel).

From the analysis given in Sec. 129 it follows that in the cases under consideration a plotted parallel φ_0 may replace an altitude line of position provided that we have measured the meridian altitude of the sun or an altitude very close to the meridian.

Due to the fact that at sea one always measures the maximum altitude of the sun, which, especially at high speeds and when sailing in high latitudes, may not coincide with the meridian altitude,

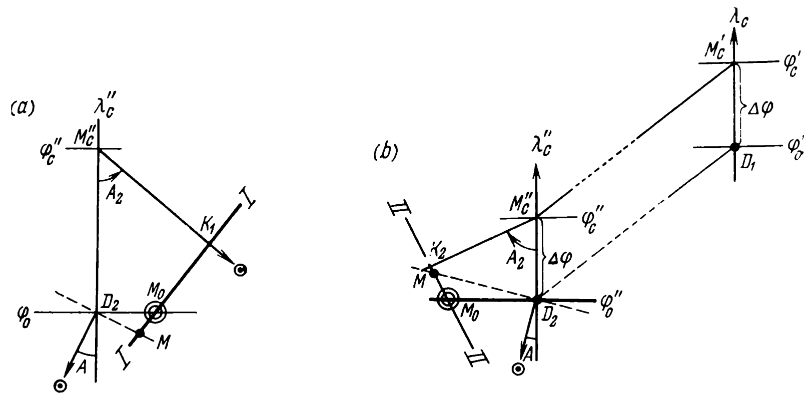

Fig. 220

it may be that in this case too the parallel will deviate considerably from the altitude line. For this reason, *substitution of a parallel of latitude for an altitude line of position is always approximate.* In all these cases it is best, after determining φ_0, to find the difference $\Delta\varphi = \varphi_0 - \varphi_c$ and the azimuth: $A^\circ = t^\circ \cdot \cos\delta \cdot \sec h$; after laying down the azimuth line through point D_2 (φ''_0, λ''_c) at a distance $\Delta\varphi$ from the D.R. point, draw through D_2 a line of position perpendicular to the azimuth line up to intersection with the *II-II* line (or *I-I* line), as shown in Fig. 220*a* and *b*. We get more exact observed positions M that do not coincide with M_0.

To conclude, it should be noted that in place of the particular cases of determinations via sun altitudes "morning—near noon", "near noon—evening", and others, it is much better to apply the general case of determining position from position lines. At night it is also better to work the altitudes of observed stars by the altitude-line method which is more universal, while the computations are routine.

Example 15. On 7.07.68 at about $T_{sh} = 12\text{h}\ 20\text{m}$ (ZD = 0); $lr_1 = 17.5$; course 148° true; speed 12 knots; $\varphi_c = 60°7.0\text{N}$; $\lambda_c = 3°2'.0\text{E}$ observed three ex-meridian altitudes of ☉: av. sr 52°0′.5 (S); $T_{ch} = 1\text{h}\ 0\text{m}\ 55\text{s}$; $u_{ch} =$ 0h 40m 4s; $i + s = -4'.\overline{0}$; $e = 10.4$ metres.

Determine φ_0 and λ_0 applying the special case "near noon — evening".

(1)

T_{sh}	12h 20m
ZD	0
T_{gr}	12h 20m 7.07
δ_T	22°33′.3 ($\overline{0.3}$)
$\Delta\delta$	—0 .1
$\delta_☉$	22°33′.2N

$-\,T_{ch}$	1h 00m 55s
u_{ch}	0 40 4
T_{gr}	12h 20m 51s
t_T	358°47′.6 (0.2)
Δt	5 12 .7
$t_{gr}^{☉}$	4 0 .3
λ_E	3 2 .0
$t_{loc}^{☉}$	7° 2′.3
	≈7° 2′W

(2)

sr	51°57′.3
$i+s$	—4 .0
h'	51°53′.3
Δ_{tot}	+ 9 .6
Δ_{ad}	— 0 .2
$h_☉$	52° 2′.7N
r	19 .6
$H_☉$	52°22′.3S
Z	37 37 .7N
δ	22 33 .2N
φ_0	60°10′.9N
φ_c	60 7 .0
$\Delta\delta$	3′.9 towards N

(3)

100 tan φ	174
100 tan δ	42
K	132

$r = 19'.6$

Supplementary computation of A_1 by BAC—58 tables

A_T	169°.5
ΔA_φ	0 .0
ΔA_δ	—0 .1
A_c	169°.4NW
	=190°.6

At $T_{sh} = 15\text{h}\ 53\text{m}$; $lr_2 = 61.0$ ($\Delta l = 0\%$) measured three altitudes of ☉: av. $sr = 33°36'.8$; $T_{ch} = 4\text{h}\ 34\text{m}\ 40\text{s}$; $i + s = -3'.4$; other corrections are the same.

(4)

lr_2	61.0
lr_1	17.5
diff. lr	43.5

φ_{c1}	60° 7′.0
l	—36 .9
φ_{c2}	59°30′.1N

λ_{c1}	3° 2′.0
DLo	45 .8
λ_{c2}	3°47′.8E

$TC = 148°$

$Dep. = 23'.1\text{E}$

20	39 .7
3	5 .9
0.1	0 .2
DLo	45′.8

From MAE.

(5)

T_{ch}	4h 34m 40s
u_{ch}	—40 4
T_{gr}	15h 54m 36s

δ_T	22°32′.5 ($\overline{0.3}$)
$\Delta\delta$	—0 .3
$\delta_\odot$	22°32′.2N

t_T	43°47′.3 (0′.2)
Δt	13 38 .9
$t^\odot_{gr}$	57°26′.2
λ	3 47 .8
$t^\odot_{loc}$	61°14′.0W

(6)

$\delta = 22°32'.2$N	T +	63 085
$t = 61\ 14\ .0$W	S	6 353
$x = 40\ 46\ .2$N	T	69 438
$\varphi_2 = 59\ 30\ .1$N		

T −	75 934
S	2 414
P +	73 520
S	9 866
T	83 386

T −	80 118
S	12 891
T	67 227

$90 + (\varphi_2 - x) = 108°43'.9$

$A_c = 76°53'.8 \approx 76°.9$SW

$h_c = 33°45'.8$

(7)

sr	33°34′.0
$i+s$	— 3 .4
h'	33°30′.6
Δ_{tot}	+ 8 .8
Δ_{ad}	— 0 .2
h_0	33°39′.2
h_c	33 45 .8
$h_0 - h_c$	— 6′.6

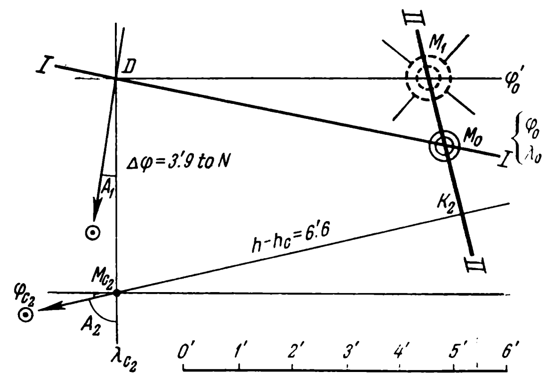

Fig. 221

(8) Plot from second D. R. position M_{c2} (Fig. 221). Replace parallel with a line.

φ_c	59°30′.1	λ_{c2}	3°47′.8
l	+2 .7	DLo	+14′.8E
φ_0	59°32′.8N	λ_0	4° 2′.6E

$T_{sh} = 15$h 53m; $lr = 61.0$
Leeway $C = 67° - 6'.5$

Note. If the parallel φ_0' is taken instead of constructing at point D azimuth A_1 of the circummeridian altitude h_1, then the position will be at point M_1 and will be in error by 1.2 miles.

CHAPTER 22

SPECIAL METHODS FOR DETERMINING THE POSITION OF A SHIP AND ITS COORDINATES AT SEA

SEC. 132. FINDING THE POSITION FROM SUN ALTITUDES FOR SMALL AZIMUTH DIFFERENCE (BRIEF SUN SIGHTS)

As we know, the most favourable difference of azimuths in sun sights varies from 35° to 70°, which makes necessary a time interval between observations of 1.5 to 4 hours. In certain cases, observational conditions do not allow for a second set of observations, for instance due to impending cloud or fog or if the navigational situation imperatively demands a determination of position. A second round of sights may then be taken even if the azimuth difference is less than 30°, but the accuracy of the running fix will be less and will decline with diminishing difference of azimuths.

Attemps to establish a sufficiently accurate brief method of determining position in the daytime have been made since the end of the eighteenth century. The first proposals involved determining only latitude; later, a joint determination of φ and λ of a position was suggested. Among similar methods developed this century, mention should be made of that of E. Willis for determining φ and λ from the altitude and its mean rate of change advanced in 1927, a method proposed by Professor V. Berg (1938-1941), B. Khlyustin (1945), A. Deitsch (1945), and, finally, V. Kavraisky (1942-1948).

The first three methods are based on the following principle. If two altitudes (h_1 and h_2) of the sun are measured with a certain time interval between the measurements and the instants (T_1 and T_2) are recorded by a chronometer, then on the basis of formula (3.10), Sec. 11,

$$\Delta h = \sin q \cdot \cos \delta \cdot \Delta T$$

we have

$$\sin q = \frac{4\,(h_2 - h_1)'}{(T_2 - T_1)\,s} \sec \delta \tag{22.1}$$

where $h_2 - h_1$ is expressed in minutes (′) and $T_2 - T_1$ in seconds (s).

After obtaining q, we can compute φ and t_{loc} from the astronomical triangle using the formulas

$$\left.\begin{aligned} \sin \varphi &= \sin h \cdot \sin \delta + \cos h \cdot \cos \delta \cdot \cos q \\ \sin t_{loc} &= \sin q \cdot \cos h \cdot \sec \varphi \end{aligned}\right\} \tag{22.2}$$

and, finally,

$$\lambda = t_{loc} - t_{gr}$$

Here δ and t_{gr} are obtained from the MAE using T_{gr}.

In formula (22.1), $\sin q$ is computed from the mean rate of change of altitude, which in the general case does not correspond to the "true" rate $\frac{dh}{dt}$; in addition, the effect of random errors in the measured altitudes h_1 and h_2 increases drastically when their difference is small. For this reason, the above method does not yield sufficiently accurate coordinates.

Attemps have repeatedly been made to eliminate these drawbacks. However, as Professor Kavraisky has shown, transformations of the formulas cannot improve the results of determination of coordinates, because the relationship between the differentials of the variables cannot change due to an identical transformation of the formulas connecting these variables. Consequently, neither does the effect of observational errors depend on the kind of formulas used to work the sights. The sole requirement is that these formulas should be correct and then errors of computation may be disregarded.

Formula (22.1) contains an error due to ignored acceleration of a celestial body in altitude, which increases with increasing ΔT; which means that the formula itself is insufficiently accurate. In addition to observational errors, there may also be errors due to inaccuracies of the very method of working sights. For this reason, when the azimuth difference is small it is best to apply the ordinary method of observations and working of sights; but when plotting (in order to reduce graphical errors of construction, which are particularly noticeable for small angles ΔA) it is best to apply the so-called **equivalent lines of position** (which always intersect at 90°) according to a method proposed by Kavraisky. Then the effect of graphical errors is reduced, but the effect of random observational errors naturally remains the same as for two position lines.

To obtain equations of equivalent lines, we transform the equations of altitude lines of position (18.31):

$$\left.\begin{aligned} \Delta h_1 &= \Delta\varphi \cdot \cos A_1 + \Delta\sigma \cdot \sin A_1 \\ \Delta h_2 &= \Delta\varphi \cdot \cos A_2 + \Delta\sigma \cdot \sin A_2 \end{aligned}\right\}$$

where $\Delta\sigma = \Delta\lambda \cdot \cos\varphi$.

Forming the half-sum and half-difference of these equations and replacing the sums and differences of the functions, we have

$$\frac{1}{2}(\Delta h_2 + \Delta h_1) = \Delta\varphi \cdot \cos\frac{A_2 + A_1}{2}\cos\frac{A_2 - A_1}{2} + \\ + \Delta\sigma \cdot \sin\frac{A_2 + A_1}{2}\cos\frac{A_2 - A_1}{2}$$

$$\frac{1}{2}(\Delta h_2 - \Delta h_1) = -\Delta\varphi \cdot \sin\frac{A_2 + A_1}{2}\sin\frac{A_2 - A_1}{2} +$$
$$+ \Delta\sigma \cdot \cos\frac{A_2 + A_1}{2}\sin\frac{A_2 - A_1}{2} \qquad (22.3)$$

Introducing the notations

$$\frac{A_2 + A_1}{2} = A_{av};\ A_2 - A_1 = \Delta A$$

ang factoring out, we have

$$\Delta\varphi \cdot \cos A_{av} + \Delta\sigma \cdot \sin A_{av} = \frac{1}{2}(\Delta h_2 + \Delta h_1) \cdot \sec\frac{\Delta A}{2} - \Delta\varphi \cdot \sin A_{av} +$$
$$+ \Delta\sigma \cdot \cos A_{av} = \frac{1}{2}(\Delta h_2 - \Delta h_1) \cdot \operatorname{cosec}\frac{\Delta A}{2}$$

If in the latter equation we take $A_{av} + 90°$ and denote the terms on the right-hand side of the formulas by

$$\left.\begin{aligned} n &= \frac{1}{2}(\Delta h_2 + \Delta h_1) \cdot \sec\frac{\Delta A}{2} \\ m &= \frac{1}{2}(\Delta h_2 - \Delta h_1) \cdot \operatorname{cosec}\frac{\Delta A}{2} \end{aligned}\right\} \qquad (22.4)$$

we finally get

$$\left.\begin{aligned} n &= \Delta\varphi \cdot \cos A_{av} + \Delta\sigma \cdot \sin A_{av} \\ m &= \Delta\varphi \cdot \cos(90° + A_{av}) + \Delta\sigma \cdot \sin(90° + A_{av}) \end{aligned}\right\} \qquad (22.5)$$

Formulas (22.5) are equations of two lines of position that are equivalent to altitude lines of position but always intersect at an angle of 90°. The first line is sometimes called the "sum" line, and the second, the "difference" line.

The elements of these lines are the quantities m, n, A_{av} and $90° + A_{av}$; here, n and m are computed from (22.4) in which the quantities $\Delta h_1 = (h - h_c)_1$, $\Delta h_2 = (h - h_c)_2$, A_1 and A_2 are computed in the usual way, but using the most accurate tables and formulas. It is best to use formula $\sin^2\frac{z_c}{2}$ with five-place tables of logarithms.

To compute m and n, the difference $\Delta A = A_2 - A_1$ is obtained to within 1′ with its sign, and $A_{av} = \frac{A_2 - A_1}{2}$ to within 0°.1. The quantity n may be computed from the natural value of $\sec\frac{\Delta A}{2}$ and by slide rule; but m should be computed with logarithms, for which purpose the half-difference $\frac{1}{2}(\Delta h_2 - \Delta h_1)$ is determined to hundredths of a minute.

Lines n and m may be constructed both on paper and on a chart in the following sequence:

(a) at the second instant, the line of average azimuth A_{av} is plotted from the D.R. position M_c (Fig. 222);

(b) lay off along this line the quantity n towards the star if we have $+n$, and away from the star if $-n$ (on the distance scale);

(c) through the determining point K thus obtained, draw a straight line I'-I' perpendicular to the line of A_{av};

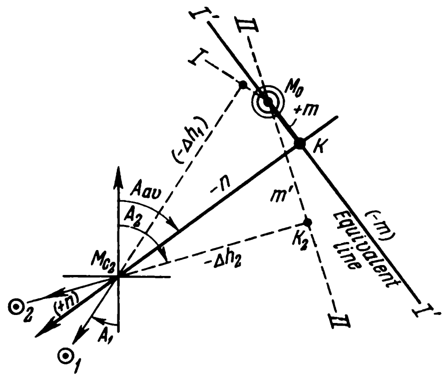

Fig. 222

(d) lay off m on the same scale to the right or to the left. When looking towards the I'-I' line from the D.R. position M_c, m is laid off to the **right** when m and n have the **same** signs, and to the **left** when m and n have **different** signs. At the end of segment m we have the observed position M_0.

From Fig. 222 it is seen that construction of n and m is equivalent to the construction of two altitude lines of position, only the distance to the point of their intersection (m) is determined analytically, which is somewhat more precise than in graphical plotting. This same figure also readily yields expressions for n and m.

In place of computing m and n it is also possible to use the ordinary method of plotting lines, but to a large scale and with precisely plotted azimuths and lines (done with the aid of a protractor, for example).

I. THE EFFECT OF RANDOM ERRORS IN THE DIFFERENCES $h - h_c = \Delta h$ ON THE ACCURACY OF A RUNNING FIX

In the case of a small azimuth difference, it is best to evaluate the area of the probable position of the ship by the construction of an average ellipse of errors. Expressions will be obtained for

the average errors in n and m if each of the terms in equations (22.4) is squared and if we pass to the mean errors ε_n, ε_m and $\varepsilon_{\Delta h}$ and take the arithmetic mean errors of two rows of observations as equal: $\varepsilon_{\Delta h 1} = \varepsilon_{\Delta h 2} = \varepsilon_0$. Then

$$\varepsilon_n = \pm \frac{\varepsilon_0}{\sqrt{2} \cdot \cos \frac{\Delta A}{2}} \quad \text{and} \quad \varepsilon_m = \pm \frac{\varepsilon_0}{\sqrt{2} \cdot \sin \frac{\Delta A}{2}} \tag{22.6}$$

Comparing these formulas with expressions (20.11) for the semiaxes of an error ellipse, we see that $\varepsilon_m = a$, which is the semimajor axis, and $\varepsilon_n = b$, which is the semiminor axis.

From the formulas (22.6) it is seen that the smaller the azimuth difference, the greater is ε_m, which means that the position gradually turns into a "band of position" along one of the lines.

Thus, if from five observations $\varepsilon_0 = \pm 1'.0$ and $\Delta A = 5°$, then $\varepsilon_m = \pm 16'.2$, $\varepsilon_n = \pm 0'.71$; if $\varepsilon_0 = 0'.4$ and $\Delta A = 10°$, then $\varepsilon_m = \pm 3'.3$, $\varepsilon_n = \pm 0'.28$. In the second case the position is rather reliably obtained.

Obviously, to enhance the accuracy of determining a position for small azimuth differences, one should pay special attention to accuracy in measuring altitudes and working them: observe a set of 5-7 altitudes with a check by Table 15, MT-63, and from them obtain mean values; take all measures to improve quality of observations as indicated in Sec. 85 and compute h_c by the most precise method $\left(\text{by } \sin^2 \frac{z_c}{2}\right)$. In addition, try to avoid too small azimuth differences (less than 5°), in which case the position is very unreliable.

When using an artificial horizon (in areas with ice, for example), accuracy in altitude measurements is higher, and for $\Delta A > 5°$ the determined position is rather reliable.

When constructing the area of the probable position of the ship, it is best to double the axes of the ellipse to increase the probability of finding the position in the area obtained.

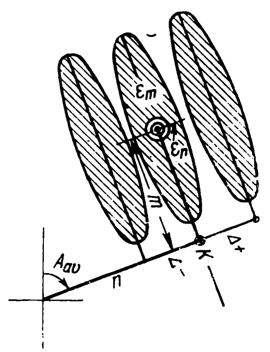

Fig. 223

II. THE EFFECT OF SYSTEMATIC ERRORS

From formulas (22.4) it is seen that recurring errors in intercepts are eliminated in m and remain in n. For this reason, due to the effects of systematic errors, the entire area of the ellipse (Fig. 223) will be displaced along the axis A_{av} by the magnitude of the expected systematic error Δ,

and the area of the possible position of the ship is appreciably increased. Hence we must take measures to eliminate systematic errors.

To improve reliability and accuracy in determining a position for small azimuth differences of the sun, it is best to have an assistant observer.

Conclusions

1. If it is necessary to determine the position by the sun in the case of a small azimuth difference, use the graphical method of plotting n and m.

2. For small azimuth differences, the position obtained is inaccurate due to the effects of random errors of observation, and becomes worse as ΔA diminishes; for $\Delta A \leqslant 5°$ the method is no longer applicable.

Example 1. On 4 March, 1968, in the Pacific Ocean at about $T_{sh} = 12h\ 20m$ (ZD = 10); $TC = 268°$; speed 14 knots; measured 5 altitudes of $\odot$ and noted chronometer time. Av. $sr_{\underline{\odot}} = 41°27'.0$; av. $T_{ch} = 2h\ 23m\ 53s$; $\varphi_c = 41°30'.0N$; $\lambda_c = 143°19'.0E$; $u_{ch} = -2m\ 12s$; $i+s = +0'.8$; $e = 14$ metres.

First set of sights worked:

T_{sh}	12h 20m 4.03
− ZD	10
T_{gr}	2h 20m 4.03
δ_T	6°40'.4 ($\overline{1.0}$)
$\Delta\delta$	−0.4
$\delta_{\odot}$	6°40'.0S

T_{ch}	2h 23m 53s
u_{ch}	−2 12
T_{gr}	2h 21m 41s
t_T	207°00'.6 (0'.4)
Δt	5 25 .2
+ t_{gr}	212°25'.8
λ	143 19 .0
$t_{loc}^{\odot}$	355°44'.8W
	=4°15'.2E

sr	41°27'.0
$i+s$	+0 .8
h'	41°27'.8
Δ_{tot}	+8 .4
Δ_{ad}	+0 .1
h	41°36'.3

$t = 4°15'.2E$ $\delta = 6\ 40\ .0S$ $\varphi = 41\ 30\ .0N$			$\sin^2$ cos cos	7.13895 9.99705 9.87446	sin cos — cosec z	8.87021 9.99705 — 0.12673
$\varphi+\delta = 48°10'.0$	$\sin^2$ α	9.22412 0.00266	II *Arg*	7.01046 2.21100	sin A A_c	8.99399 5°39'.6SE ≈174°20'
	$\sin^2$ z_c	9.22412 48°19'.4			− h_c h	41 40 .6 41 36 .3
					$h-h_c$	−4'.3

Second set of observations:

Due to deteriorating visibility and the necessity to obtain a running fix upon entry into straits, second round of observations made at $T_{sh} = 12\text{h}\ 56\text{m}$; av. $sr_{\odot} = 41°27'.1$; av. $T_{ch} = 2\text{h}\ 59\text{m}\ 54\text{s}.5$. Ship's run $S = 9.0$ miles, with current taken into account, $TC = 268°$; $l = 0'.3$ towards S; $Dep. = 8.99$; $DLo = 12'.00$ to W; $\varphi_{c2} = 41°29'.7\text{N}$; $\lambda_{c2} = 143°7'.0\text{E}$.

Set of sights worked:

T_{ch}	2h 59m 54s.5
u_{ch}	−2 12
T_{gr}	2h 57m 42s.5
t_T	207°00′.6 (0′.4)
Δt	14 25 .8
$t^{\odot}_{gr}$	221 26 .4
λ	143 7 .0
t_{loc}	364°33′.4W = 4°33′.4W

δ_T	6°40′.4 ($\overline{1.0}$)
$\Delta\delta$	−1 .0
$\delta_{\odot}$	6°39′.4S

sr	41°27′.1
$i+s$	+0 .8
h'	41°27′.9
Δ_{tot}	+8 .4
Δ_{ad}	+0 .1
h	41°36′.4

$t = 4°33'.4\text{W}$
$\delta = 6°39'.4\text{S}$
$\varphi = 41°29'.7\text{N}$

$\varphi + \delta = 48°\ 9'.1$

$\sin^2$	9.22121
α	0.00305
$\sin^2$	9.22426
z_c	48°19′.9

$\sin^2$	7.19876
cos	9.99706
cos	9.87449
II	7.07031
Arg	2.15090

sin	8.90007
cos	9.99706
—	—
cosec z	0.12667
$\sin A$	9.02380
A_c	6°3′.0SW = = 186°3′
h_c	4140.1
− h	41 36 .4
$h - h_c$	−3′.7

Computation of line elements:

Δh_2	−3′. 7
Δh_1	−4′. 3
$\dfrac{\Delta h_2 + \Delta h_1}{2}$	−4′. 0
$\dfrac{\Delta h_2 - \Delta h_1}{2}$	+0′.30

A_2	186° 3′
A_1	174°20′
ΔA	+11°43′
$\dfrac{\Delta A}{2}$	5°51′.5
A_{av}	180° .2

$n = -4'.0 \cdot \sec 5°52' \approx -4'.0$

+ 0′.30	log	1.4771
5°51′. 5	cosec	0.9911
	$\log m$	0.4682
	m	2.939≈+2′.9

Computation of semiaxes of error ellipse:

Altitude error: $\varepsilon_0 = \frac{+1'.0}{\sqrt{5}} = \pm 0'.45 \approx \pm 0'.5$

$$a = \varepsilon_m = \pm \frac{\varepsilon_0}{\sqrt{2}\cdot\sin\frac{\Delta A}{2}} = \frac{\pm 0'.5}{1.41\cdot 0.10} = \pm 3'.5$$

$$b = \varepsilon_n = \pm \frac{\varepsilon_0}{\sqrt{2}\cdot\cos\frac{\Delta A}{2}} = \frac{\pm 0'.5}{1.41\cdot 0.995} \approx \pm 0'.4$$

The plotting is shown in Fig. 224.

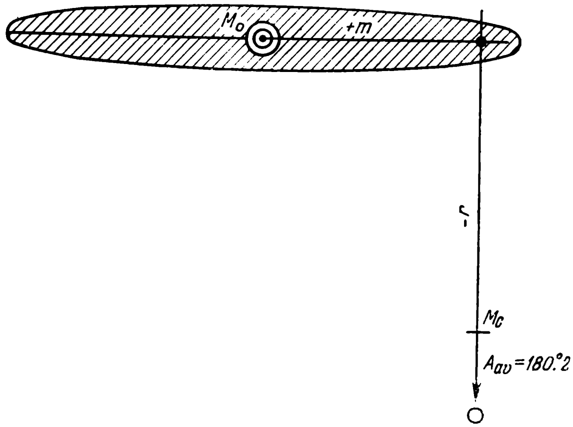

Fig. 224

The results are: $T_{sh} = 12\text{h } 58\text{m}$; $lr = 73.5$;

$$\varphi_0 = 41°33'.7\text{N};\ \lambda_0 = 143°3'\text{E};$$
$$C = 325° - 5'.$$

As we see, the position was determined reliably in the direction of n, which in the given case is by latitude, and very inaccurately along the line m, which in the given case is by longitude.

SEC. 133. DETERMINING A SHIP'S POSITION IN LOW LATITUDES AT SUN ALTITUDES UP TO 85°

In low latitudes, the diurnal motion of the sun has peculiarities that result in certain modifications of the general methods for determining a ship's position.

Let us recall the most significant of these peculiarities that are considered in detail in Secs. 11 and 16:

(a) from sunrise to instants rather close to transit, the sun's azimuth changes at an extremely slow rate;

(b) near times of transit, the azimuth of the sun changes considerably during a brief period of time.

The conditions for observing the sun follow from these peculiarities of motion.

Let us assume (Fig. 225) that the sun was first sighted after sunrise (point a); from the figure it is evident that for the sun's azimuth to change by 30° to 35°, it will be necessary to await its

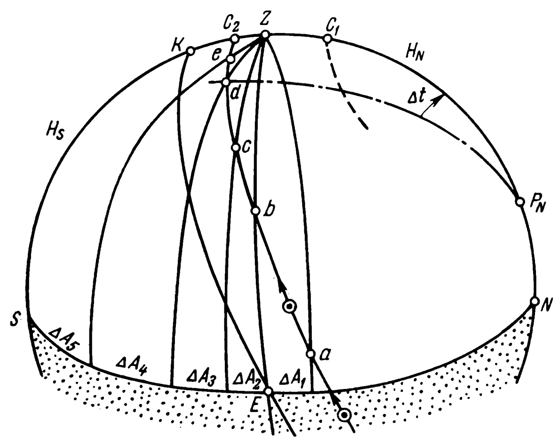

Fig. 225

approach to transit (points d and e), which means a number of hours; during the first 2-3 hours (points a, b, c) the azimuth will change but slightly, less than 10 degrees. It is obvious that the morning line cannot be used because errors in dead reckoning will make it unsuitable practically speaking. On the contrary, close to transit (points d, e) the azimuth will change very considerably ($\Delta A_4 + \Delta A_5$) during a small time interval Δt. Consequently, to determine a position in low latitudes for large sun altitudes, the most advantageous conditions (and, practically speaking, the sole conditions) will be when the sun is near the meridian of the observer.

From these same peculiarities in the sun's motion there follows also the possibility of determining the longitude a relatively short time prior to transit of the sun, for instance at the point d (see Fig. 225). The sun's azimuth will in that case be relatively close to 90° (prime vertical), and the longitude will be determined in permissible conditions.

An approximate value of azimuth change during one minute of time close to transit may be computed from formula (3.18) if we take $\cos q \approx 1$ and $\Delta t = 1\text{m}$, that is,

$$\Delta A^\circ \approx \frac{1}{4} \sec h \cdot \cos \delta \cdot 1 \text{ min} \tag{22.7}$$

This formula is used to compile Table 14, which gives approximate values of change of azimuth in 1 m of time near the meridian in low latitudes (for δ and φ of the same name).

Table 14

$\delta_\odot$ \ $h_\odot$	60°	65°	70°	75°	80°	85°	86°	87°	88°	88°30′	89°	89°30′	89°40′	89°50′
	Change of azimuth in one minute													
0°	0°.5	0°.6	0°.7	1°.0	1°.4	2°.9	3°.6	4°.8	7°.2	9°.6	14°.3	28°.6	43°.0	85°.8
12	0 .5	0 .6	0 .7	1 .0	1 .4	2 .8	3 .5	4 .7	7 .1	9 .4	14 .0	28 .0	42 .0	84 .0
20	0 .5	0 .6	0 .7	1 .0	1 .3	2 .7	3 .4	4 .5	6 .8	9 .0	13 .4	26 .9	40 .4	80 .7
24	0 .5	0 .5	0 .6	0 .9	1 .3	2 .7	3 .3	4 .4	6 .6	8 .8	13 .1	26 .1	39 .3	78 .4

From the table it is seen that there is a sharp increase in the rate of change of azimuth for altitudes greater than 85°; but when the sun passes near the zenith ($h = 89°50'$), azimuth changes by 171° in 2m close to transit, which is almost a change to back azimuth.

In connection with the foregoing peculiarities in the sun's motion in low latitudes, we can apply the following special methods of observation and working sights, depending on the altitude of the sun near the meridian:

1. The ordinary technique of Saint-Hilaire (with certain peculiarities of observation and limitations) applicable for altitudes up to 85°.
2. The "method of short equal altitudes" which may be applied for altitudes of the sun between 75° and 88°.
3. The graphical method of plotting circles of equal altitudes, which may be applied to sun altitudes greater than 88°.

In this section we shall consider only the first method.

PECULIARITIES OF DETERMINING POSITION BY ALTITUDE LINES OF POSITION FOR SUN ALTITUDES 60°-85°

We shall consider the peculiarities of observations and working sights for large sun altitudes.

The most suitable will be observations taken on both sides of the meridian: first line 10 to 40 minutes prior to transit and second

line 10 to 40 minutes following transit. Now if the observations are made on one side of the meridian, for instance, if we wish to obtain the position by noon, the first set of observations are taken 20m—1h—1h.5 prior to transit, and the second set, as close as possible to the instant of transit.

In this case, the second round of sights will be ex-meridian and may be worked both to obtain latitude and an altitude line of position. In the first case, bear in mind that we cannot take the parallel φ_0 for an altitude line; we must also compute the azimuth in the ordinary way or by the approximate formula $A^\circ = t^\circ \cdot \sec h \cdot \cos \delta$ or, finally, using Yushchenko's Azimuth Tables, and then lay down an altitude line of position in place of the parallel as shown above in Sec. 129. If a meridian altitude is obtained, the parallel φ_0 may be taken for a position line.

When obtaining φ_0 in these conditions, one must always note above which point (N or S) the sun transited. For example, if the sun transited at C_1 (see Fig. 225), then H will be to the N, and Z to the S and $\varphi_N = \delta_N - Z_S$. But if the transit was at point C_2, then H will be to the S and $\varphi_N = Z_N + \delta_N$.

When working the sights by the method of altitude lines (which is more advantageous due to the routine nature of the computations), one should bear in mind that h_c in these cases must be obtained from the formula $\sin^2 \frac{z_c}{2}$ and with five-place tables or using the tables TBA-57, but not by $\sin h$ or the tables H.O. No. 214.

In the case of intercepts greater than 15′ and $h > 75^\circ$, the problem should be solved a second time, taking the observed coordinates for the computed ones, or by shifting both lines to the geographic position by the quantities x computed from formulas.

The following observational techniques are recommended when obtaining a running fix from lines of position in the case of large sun altitudes:

After taking 2-3 sun altitudes in the usual way, measure an additional 4-5 sun altitudes "via the zenith".

Working each series of altitudes separately (via h_{av} and T_{av}), we obtain two lines after plotting. In this case, observations of the sun "via the zenith" will be roughly equivalent to observations of a *second body* in back azimuth. If the time interval between observations of these two lines is considerable, the first line should be reduced to the zenith of the second by means of Table 16, MT-53 (Δh_z) or graphically by advancing the line by an amount S, which is the ship's run between observations (Fig. 226); work the sights only with the coordinates φ_{c1}, λ_{c1} taken for M_{c1} for the second instant. Constructing a bisector between them, we get the line I_0-I_0 (see Fig. 226), which is free from systematic errors and more reliable

with respect to random errors. With the second round of observations taken in the same way and worked with φ_{c2}, λ_{c2} for M_{c2}, we obtain a second bisector II_0-II_0, and at the point of its intersection with the first line I_0' advanced in the usual way, we obtain the observed position M_0. Of course, all constructions may be performed directly

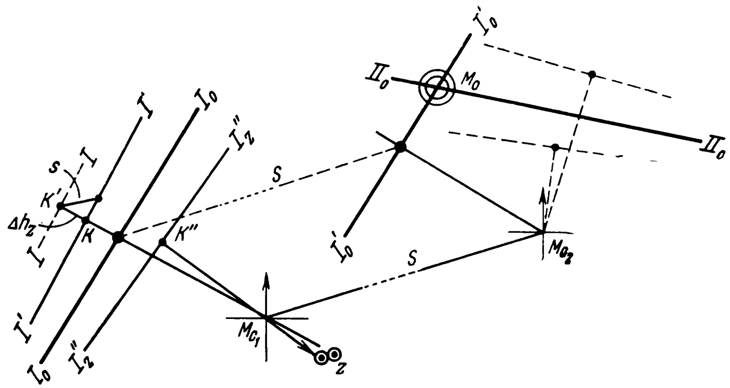

Fig. 226

at the second D.R. position by considering that the line I_0' has been reduced to this place by course and by the run between observations. Such observations may be recommended if a large systematic error is expected, for example due to unusual refraction, etc.

Example 2. On 8 November, 1968, in the Atlantic Ocean it is required to determine the position by the sun.

The altitude of the sun at noon the day before was greater than 70°, and so a determination should be made near upper transit. The time of upper transit in $\lambda \approx 43°$W will be at $T_{sh} \approx$ 11h 36m. The rate of change of azimuth (from Table 20 of this section) is approximately 1° per minute; thus, the observations should be 40m prior to transit and 15m 20m following transit.

(1) At $T_{sh} =$ 10h 45m; $lr_1 = 17.8$; $\varphi_c = 31°12'$S; $\lambda_c = 43°21'$W; first sights taken of three sun altitudes and chronometer times noted: av. T_{ch} 1h 48m 22s; av. $sr_{\underline{\odot}} = 70°59'.7$; $i + s = -1'.8$; $e = 14.5$ metres; $u_{ch} = -$3m 56s; speed 14 knots; heading $= 251°$ $(-2°)$. Dip measured by dipmeter and proved close to tabulated value $(-6'.6)$; and so we work sight by Table 8, MT-53.

(2)

$+T_{sh}$	10h 45m
ZD	3
T_{gr}	13h 45m 8.11

T_{ch}	1h 48m 22s
u_{ch}	− 3 56
T_{gr}	13h 44m 26s

sr	70°59'.7
$i+s$	− 1 .8
h'	70°57'.9
Δ_{tot}	+ 9 .0
Δ_{ad}	+ 0 .2
h	71°07'.1

δ_T	$16°41'.2$ $(\overset{+}{0}.7)$	t_T	$19°\ 3'.1\ (0'.3)$
$\Delta\delta$	$+0\ .5$	Δt	$11\ \ 6'.5$
$\delta_\odot$	$16°41'.7S$	$t^\odot_{gr}$	$30°\ 9'.6$
		$-\ \lambda_W$	$43°21'.0$
		$t^\odot_{loc}$	$346°48'.6W = 13°11'.4E$

(3)

$t = 13°11'.4E$			$\sin^2$	8.12027	$\sin$	9.35828
$\varphi = 31\ 12\ .0S$			$\cos$	9.93215	—	—
$\delta = 16\ 41\ .7S$			$\cos$	9.98130	$\cos$	9.98130
					$\operatorname{cosec} z$	0.49123
$\varphi - \delta = 14°30'.3$	$\sin^2$	8.20241	II	8.03372		
	α	0.22482	Arg	0.16869	$\sin A$	9.83071
					A_c	$42°37'.5$
	$\sin^2$	8.42727				$\approx 42°\ \ .6NE$
	z_c	$18°49'.5$			h_c	$71\ 10\ .5$
					h	$71\ 57\ .1$
					$h - h_c$	$-3'\ \ .4$

(4) At $T_{sh} = 11h\ 50m$; $lr_2 = 34.1$ $(\Delta l = -3\%)$ again measured three altitudes of $\underline{\odot}$: av. $\underline{T}_{ch} = 2h\ 55'm\ 51s$; av. $sr_{\odot} = 74°49'.9$.

Second coordinates by dead reckoning: $\varphi_{c2} = 31°17'.7S$; $\lambda_{c2} = 43°38'.3W$.

(5)

T_{ch}	2h 55m 51s			sr	$74°49'.9$
u_{ch}	$-\ \ 3\ \ \ 56$			$i+s$	$-\ \ 1\ .8$
T_{gr}	14h 51m 55s 8.11			h'	$74°48'.1$
t_T	$34°\ 3'.1\ (0'.3)$	δ_T	$16°41'.9$ $(\overset{+}{0}.7)$	Δ_{tot}	$+\ \ 9\ .0$
Δt	$12\ 58\ .8$	$\Delta\delta$	$0\ .6$	Δ_{ad}	$+\ \ 0\ .2$
$t^\odot_{gr}$	$47°1'.9$	$\delta_\odot$	$16°42'.5S$	h	$74°57'.3$
$-\ \lambda_W$	$43\ 38.3$				
$t^\odot_{loc}$	$3°23'.6W$				

(6) $t = 3°23'.6W$			$\sin^2$	6.94282	sin	8.77225
$\varphi = 31°17'.7S$			cos	9.93171	—	—
$\delta = 16°42'.5S$			cos	9.98127	cos	9.98127
					cosec z	0.58957
$\varphi - \delta = 14°35'.2$	$\sin^2$	8.20726	II	6.85580	$\sin A$	9.34309
	α	0.01892	*Arg*	1.35146	A_c	12°43'.7
						$=12°$ .7NW
	$\sin^2$	8.22618			h_c	75 05 .4
	z_c	14°54'.6			h	74 57 .3
					$h - h_c$	$-8'.1$

(7) The plotting is shown in Fig. 227.
(8) The results are:

$T_{sh} = 11$h 52m $\quad C = 157° - 8'.2$
$lr = 34.6$

φ_c	31°17'.7S	λ_c	43°38'.3W
l	7'.6S	DLo	− 3 .9E
φ_0	31°25'.3S	λ_0	43°34'.4W

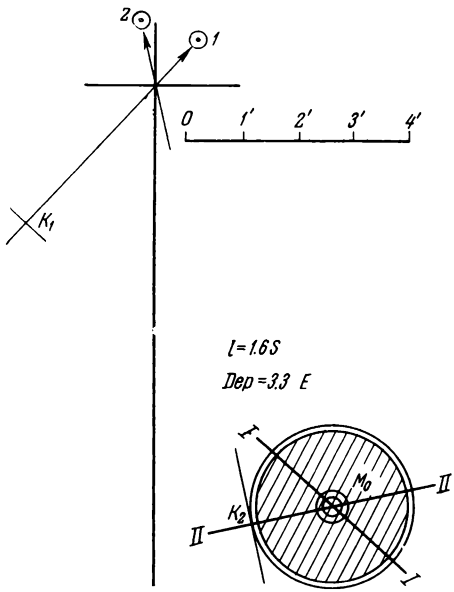

Fig. 227

(9) An analysis of the determination.
We compute the radius of the circle of errors using the formula

$$\varepsilon_{loc} = \rho = \pm \frac{\sqrt{\varepsilon_1^2 + \varepsilon_{h2}^2}}{\sin \Delta A}$$

where

$$\varepsilon_1 = \sqrt{\varepsilon_{h1}^2 + \varepsilon_{\Delta Z}^2}.$$

Taking $\varepsilon_{\Delta Z} = \pm 0'.7$, due to the small run between observations and $\varepsilon_h =$ $= \pm 0'.7$, we have

$$\varepsilon_{loc} = \rho = \pm 1.5 \text{ miles}$$

With this radius we construct the area of the probable position of the ship. In view of the fact that a simultaneous determination by a second observer yielded close results, we advance the dead reckoning to the observed position M_0. The drift is due to imprecise account taken of the Brazil current.

SEC. 134. ESSENTIALS OF THE METHOD OF SHORT EQUAL ALTITUDES (FOR SUN ALTITUDES FROM 75° TO 88°)

For sun altitudes exceeding 75° we can apply a special analytical method for determining longitude jointly with the ordinary technique of determining latitude.

As we know, the coordinates of the subsolar point at a given instant are: $\lambda_a = t_{gr}^{\odot}$, $\varphi_a = \delta_{\odot}$. If at the time when the sun passes across the meridian of the given place, that is, at the instant of "true noon", the watch reading is noted and if we compute $t_{gr}^{\odot}$ from MAE, we obtain the longitude of the place since the geographic position a will be on the meridian of the place and $\lambda_a = \lambda_{loc} = t_{gr}^{\odot}$. This likewise follows from the familiar formula $\lambda_E = t_{loc}^{\odot} - t_{gr}^{\odot}$, which for $t_{loc}^{\odot} = 0$ takes the form $\lambda_E = -t_{gr}^{\odot}$. The latitude at this instant may be obtained from the meridian altitude. This simple principle has long since engaged researchers, but it involves difficulties that complicate and restrict its application. Indeed, (a) when a ship is in motion it is difficult to establish the instant of true noon, (b) the longitude of the place is determined under the worst conditions if the celestial body is located on the meridian or near it. From the formula (18.23), that is,

$$\Delta\lambda = \Delta h \cdot \sec\varphi \cdot \operatorname{cosec} A - \Delta\varphi \cdot \cot A \cdot \sec\varphi$$

it is evident that the slightest error in altitude or latitude will lead to large errors in the longitude being determined (since the azimuth is close to 180°, cot A and cosec A tend to infinity). This latter circumstance would seem to wipe out completely the very idea of such a method. However, by proceeding from the peculiarities of the sun's motion at the altitudes considered in the preceding section, these difficulties were overcome through the use of a method of measuring two altitudes h_1 and h_2 situated symmetrically about the observer's meridian at small hour angles (Fig. 228). In this case, when $h > 75°$, the sun is 35°-50° in azimuth from the meridian, and conditions for determining longitude improve. Now the instant of true noon is determined from the arithmetic mean of the instants of measurements of these two altitudes. Such altitudes have become known as **short equal altitudes,** and the method is known as the **method of short equal altitudes.**

If the ship is motionless and the declination of a celestial body is not changing, the diurnal circle of the body will be symmetric about the meridian (see Sec. 124) and its altitudes will be equal for equal distances from the meridian. In Fig. 228, the path of such a body is shown by the dashed line and the meridian of the position is depicted by the straight line DS. In this case, $\frac{T_2'+T_1}{2} = T_0$, that is, the mean instant is the instant of true noon and

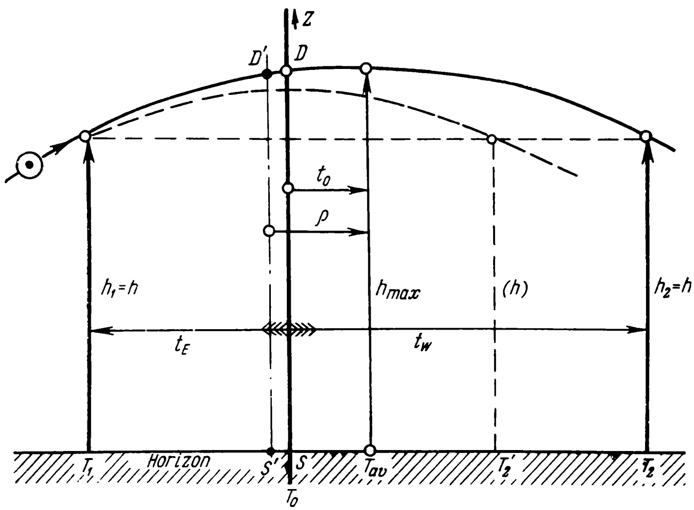

Fig. 228

$\lambda_{loc} = t_{gr}^{\odot}$. Under actual conditions (moving ship and changing declination) the diurnal circle of the sun will not be symmetric about the meridian, and the maximum altitude will not be the meridian altitude. Referring to Fig. 228, let the altitude of the sun at instant T_1 be $h_1 = h$, at T_2, $h_2 = h$, and the maximum altitude h_{max} does not coincide with the meridian but corresponds to an instant close to $T_{av} = \frac{T_2+T_1}{2}$. Consequently, the altitudes $h_1 = h_2$ will be roughly symmetric about the *maximum altitude* if we ignore small variations in the equation of time and the effect of accelerations. However, the time T_{av} does not correspond to the instant of true noon T_0. As we know, the difference between these two instants is equal to t_0, the local hour angle of the sun at the instant of maximum altitude (Sec. 124).

If we refer the entire determination to the time of maximum altitude, then a correction t_0 must be introduced into the instant obtained, T_{av}, or into $t_{gr}^{\odot}$ computed from it, because at the instant of maximum altitude the meridian occupied the position DS (see Fig. 228).

On the basis of the foregoing, to determine longitude at the instant of greatest altitude use the following formulas

$$\left.\begin{array}{l} t_0' = 3.82\,(\tan\varphi - \tan\delta)\,(\theta' - \psi') \\ \left.\begin{array}{ll} T_{av} = \frac{1}{2}(T_2 + T_1);\ T_{gr}^{av} = T_{av} + u; & (t_{gr}^{\odot})_{av} \\ \lambda_W = (t_{gr}^{\odot})_{av} - t_0' & \delta_{\odot} \end{array}\right\} \text{from MAE} \\ \lambda_E = 360^\circ - \lambda_W\ (\text{if } \lambda_W > 180^\circ) \end{array}\right\} \qquad (22.8)$$

The sign of t_0' is found from the rules given in Sec. 124 (or MT-63) and is allowed for in subsequent computations. We ignore any change in longitude in the formula for t_0' due to the small value of $\frac{\Delta\lambda}{900}$ in low latitudes.

The latitude may be determined from one of the altitudes $h_1 = h$ or $h_2 = h$ if they prove to be ex-meridian altitudes; this may be established from Table 19, MT-63, or from the maximum altitude.

In case the latitude is determined from the maximum altitude (h_{max}), we get

$$H = h_{max};\ Z = 90^\circ - H;\ \varphi_0 = Z \pm \delta \qquad (*)$$

This latitude will also refer to the instant T_{av}.

To obtain an approximate calculation of the time of the first round of sights, do the following:

(a) determine the instant of transit of the sun for noon λ_c;

(b) using H and δ, by formula $\Delta A^\circ = \frac{1}{4}\cos\delta \cdot \sec H \cdot \Delta T^{\min}$ or by Table 14 (see Sec. 133) or by any other numerical tables of azimuths obtain ΔA for one minute and compute $\Delta T^{\min}$ for which ΔA (the distance of the celestial body from the meridian) will be of the order of 30°-40° (not less than 20°);

(c) subtract $\Delta T^{\min}$ from T_{sh} of transit to obtain T_{sh} of first altitude sight.

The second altitude is observed after the meridian for the same sextant reading and the same height of eye of the observer.

The accuracy in determining coordinates of a position by the method of short equal altitudes (particularly for several pairs of altitudes) is somewhat higher than by the line of position method based on two sun sights taken at different times due to a lessening of the effect of dead-reckoning errors. To evaluate the area of the

probable position, one may use the formula $\varepsilon_{loc} = \frac{\pm \varepsilon_h \cdot \sqrt{2}}{\sin(A_2 - A_1)}$ where A_1 and A_2 are sun azimuths in measuring h_1 and h_2.

Longitude determined by this method is free from systematic errors of altitude, which is one of the merits of the method. However, the most important advantage of the method of short equal altitudes is the simplicity of computations.

The sequence routine for computing by short equal altitudes is as follows:

1. Compute the time of commencement of observations as indicated above.
2. Five minutes prior to the computed time, perform several altitude measurements by way of practice and checking.
3. Measure h_1 and note T_{ch1}; put the sextant in the shade leaving the setting h_1 unchanged.
4. Using a second sextant, continue measurements until h_{max} is obtained; note T'_{ch} for this altitude and then T_{sh} and lr.
5. Again take the first sextant and wait until the sun arrives at altitude $h_2 = h_1$ and note T_{ch2}. Observe from the same spot as for h_1 and with the same shade glasses.
6. For computations, use formulas (22.8) and (*). The instant $T_{av} = \frac{T_2 + T_1}{2}$ should be approximately equal to T'_{ch} for h_{max}; this is also somewhat of a check.

Example 3. On 9.04.68 in the South China Sea on course 204° true with a speed of 14 knots. Find the position at noon. Since the altitudes of the sun are large (about 85°), it was decided to use the method of short equal altitudes.

(1) Calculation of time of observations.

At noon $\lambda_c \approx 111°15'$E (ZD = 7E).

(a) Time of transit

T_T	12h 02m
ΔT_λ	00
T_{min}	12 02
$-\lambda$	7 25
T_{gr}	4 37
$+$ ZD	7
T_{sh}	11h 37m

(b) Time of observations

From Table 14, Sec. 133, using $H \approx 85°$ and $\delta \approx 7°$ choose rate of change of azimuth $\Delta A \approx 2°.8$ per minute. We choose $\Delta A = 35°$.

$$\Delta T = \frac{35°}{2.8°/\text{min}} \approx 13 \text{ min}$$

1st observation

T_{sh}	11h 37m
$-\Delta T$	$-$ 13
T'_{sh}	11h 24m

2nd observation

$T_{sh} = 11$h 37m

3rd observation (approximate)

T_{sh}	11h 37m
$+\Delta T$	$+$ 13
T''_{ch}	11h 50m

(2) Observations.

1st observation.

$T_{sh} = 11h\ 24m$; $sr_{\underline{\odot}} = 84°7'.0$; $T_{ch} = 4h\ 30m\ 35s$; $i+s = -0'.5$; $e = 14$ metres; $u_{ch} = -6m\ 28s$.

2nd observation.

$T_{sh} = 11h\ 37m$; $lr = 29.4$; $\varphi_c = 12°18'.5N$; $\lambda_c = 111°10'.0E$.
Another sextant was used to measure the maximum altitude $\underline{\odot}$ $sr = 85°7'.5$ to S; $T_{ch} = 4h\ 44m\ 08s$; $i+s = +1'.6$.

3rd observation.

Using first sextant again, we wait until sun arrives at $sr = 84°7'.0$ and note $T_{ch} = 4h\ 57m\ 33s$. Corrections are the same as in the first observation.

(3) Working the sights.

(a) Determining longitude.

Taking data from MAE

T_{ch1}	4h 30m 35s
T_{ch3}	4 57 33
$T_1 + T_3$	8h 88m 8s
$T_{ch\ av}$	4 44 4
u_{ch}	− 6 28
$T_{gr\ av}$	4h 37m 36s 9.04

Calculation of t_0'

Formula:

$$t_0' = 3.82\ (\tan\varphi - \tan\delta)\ (\theta' - \psi')$$

$\varphi_0 = 12°17'.2N$	$3.82 \tan\varphi_0$	0.83	Table 18a, MT-63
$\delta = 7°35'.6N$	$3.82 \tan\delta$	0.51	
	Difference	0.32	

One hour run (by speed) equal 14 miles; true course = 204°.

$$l = \psi - 12'.8$$

For $T_{gr\ av}$ we take out of MAE

$t_T^{\odot}$	239°35'.3 (0'.5)	δ_T	7°35'.0 $(0.\overset{+}{9})$
Δt	9 24 .1	$\Delta\delta$	+0 .6
$t_{gr}^{\odot}$	248°59'.4	$\delta_{\odot}$	7°35'.6N
− t_0	4 .4		
λ_0	248°55'.0W = 111° 5'.0E		

Change of declination in one hour from MAE: $\theta' = +0'.9$

Put data in formula and compute t_0' with slide rule:

$t_0' = +0.32\ (0.9 + 12.8) = +4'.4$

(b) Determining latitude

sr	85° 7′.5S
$i+s$	+ 1 .6
H'	85° 9′.1
Δ_{tot}	+ 9 .3
Δ_{ad}	0 .0
H	85°18′.4S
Z	4 41 .6N
δ	7 35 .6N
φ_0	12°17′.2N

and $\lambda_0 = 111°5'.0E$

100 tan φ	22	Table 17a
100 tan δ	13	
K	9	$r=0.1$
t_0	4′.4	

$\Delta\varphi = -0.1$, which is of no significance.

(c) Time of observations

T_{ch}	4h 44m 08s
u_{ch}	−6 28
T_{gr}	4h 37m 40s
+ ZD	7
$T_{sh} \approx$	11h 38m

$lr = 29.4$

(Time of h_{max} is in good agreement with $T_{gr\ av}$).
Note. Check on discrepancy between h_{max} and H.

SEC. 135. DETERMINING POSITION IN LOW LATITUDES FOR ALTITUDES OF SUN EXCEEDING 88°

For very large altitudes of the sun (> 88°), when it passes close to the zenith, that is for $\delta_{\odot}$ close to φ and of the same name, the above-indicated peculiarities of diurnal motion of the sun are particularly pronounced. As is seen from Fig. 225 and Table 14, Sec. 133, for $h_{\odot} = 88°$ and more, the azimuth changes tens of degrees during a few minutes near transit. For this reason, in order to obtain a sufficient azimuth difference between the lines of position, a few minutes (3-6) suffice in place of several hours in medium latitudes.

Due to the large curvature of segments of circles of equal altitudes at $h > 88°$, their replacement by straight lines will cause additional errors. True, these errors are small and may be compensated for by the introduction of corrections. But in this case a different graphic technique may be used which is much simpler than the Marcq Saint-Hilaire method. The principle of this graphic method is based on plotting circles of equal altitudes on a terrestrial globe as described in detail in Sec. 100.

It will be recalled that if we know the coordinates of the geographic position a of a body (say, the sun) at a given instant (they are $\varphi_a = \delta_{\odot}$ and $\lambda_a = t_{gr}^{\odot}$) and the radius of the circle of equal altitu-

dus $z_{\odot} = 90° - h_{\odot}$, then by laying down on the globe the instantaneous geographic position a and by drawing from this point a circle of radius $z_{\odot}$, we get a line of position of the ship for the given instant. If we plot two or three such circles, then at their point of intersection we have the observed position of the ship on the globe.

In low latitudes, distortions of figures on a chart in a Mercator projection of the globe are only slight; for this reason, we can with sufficient accuracy take the cyclic curve (obtained by projecting a circle of equal altitudes on a chart) to be a *circle*. On the basis of studies by a number of authors, we can take it (to within $0'.2$) that the radius of the circle is equal to the semiaxes of the cyclic curve, while the geographic position coincides with the centre of the curve (circle) on the Mercator chart under the following conditions:

$$\left.\begin{array}{l} \text{(a) for the radius of the curve:} \\ \qquad z' \leqslant 132' \cos \delta \\ \text{(b) for the centre of the curve:} \\ \qquad z' \leqslant 3438' \sqrt{\dfrac{0'.2}{\delta'}} \end{array}\right\} \qquad (22.9)$$

For the sun, the declination of which does not exceed $23°27'$, replacement will be permissible for a radius at $z' \leqslant 120'$ ($2°$) and for centre at $z' \leqslant 41'$ (on the parallel of the tropics). If we admit of an error of $\pm 0'.3$-$0'.5$ when plotting the centre of the curve on the scale of an ocean chart, the condition for substitution of a cyclic curve by a circle in the tropics will be as follows: the zenith distance $z_{\odot}$ must be less than $2°$ or the height of the sun must exceed $88°$.

Analogous boundaries are suggested by practical reasoning as well: a radius of more than 120 miles may go beyond the limits of the chart, and it is difficult to draw circles of large radii.

Formulas (22.9) indicate that in high latitudes this method is very restricted and practically inapplicable to stars. Indeed, for $\delta = 60°$ the curve is replaceable by a circle for $z' \leqslant 66''$ (for the radius) and $z' \leqslant 26'$ (for the centre). Now for all practical purposes it is very difficult to observe stars at sea with altitude $89°$-$89°.5$.

The principle described above for determining position on a globe is applied to a chart as follows. Three sun altitudes are taken at about noon at 2-7 minute intervals, depending on the rate of change of azimuth, and the chronometer times are noted.

From the second (or last) instant we have T_{gr2} and from the MAE we take out $\delta_{\odot}$ and $t^{\odot}_{gr2W}$; note that if $t_W > 180°$, then we take $t_E = 360° - t_W$.

The **parallel of the geographic position** $\varphi_a = \delta_{\odot}$ (Fig. 229) is drawn on the chart in the region of the D.R. position on the assumption that the declination has not changed during the time of observations. On this parallel, from the longitude $\lambda_2 = t^{\odot}_{gr2W}$ (or $t^{\odot}_{gr2E}$ if t_{gr2W} is greater than 180°, see Fig. 229), we plot the geographic position a_2 at the second instant. The geographic positions at the first and third instants are obtained relative to the already obtained position a_2 in the following manner: the longitude λ_1 of the geographic position a_1 at the first instant (in Fig. 229 the longitude is

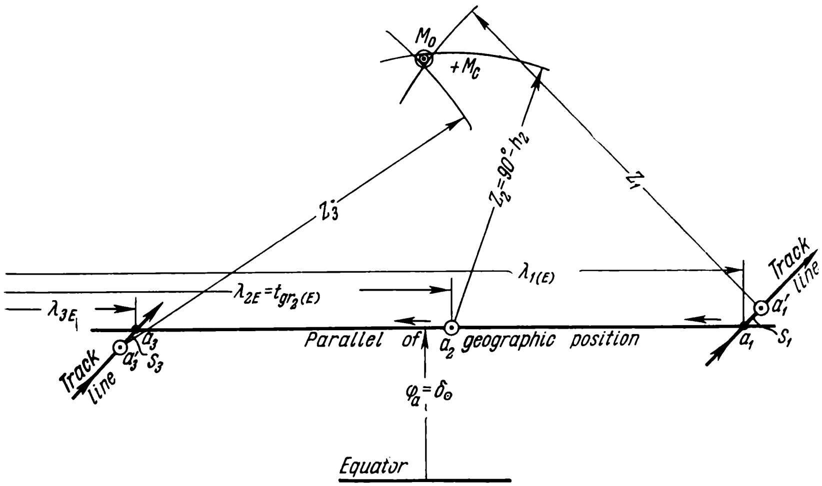

Fig. 229

east) is obtained by subtracting the difference of instants $(T_2 - T_1)°$ from $t^{\odot}_{gr2W}$*, that is $\lambda_1 = t^{\odot}_{gr2W} - (T_2 - T_1)°$, while the longitude λ_3 of the geographic position a_3 at the third instant is obtained by adding the difference of the instants $(T_3 - T_2)°$, that is, $\lambda_3 = t^{\odot}_{gr2W} + (T_3 - T_2)°$. It is of course possible to take t_{gr} from the MAE for all three instants, but the above technique is simpler.

The points a_1 and a_3 thus obtained should be reduced to the zenith of the second series of observations (instead of reducing the line of position). To do this, plot the course line from a_1 forwards, and from a_3 back, and along it plot the run during the time $(T_2 - T_1)$ and $(T_3 - T_2)$, noting that $S_1 = \frac{v}{60}(T_2 - T_1)$ is laid off forwards

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* The hour angle is always west; if it exceeds 180° after subtracting ΔT, then $t_E = 360° - t_W$ and is equated to λ_E. The plot for λ_W is shown in Fig. 230.

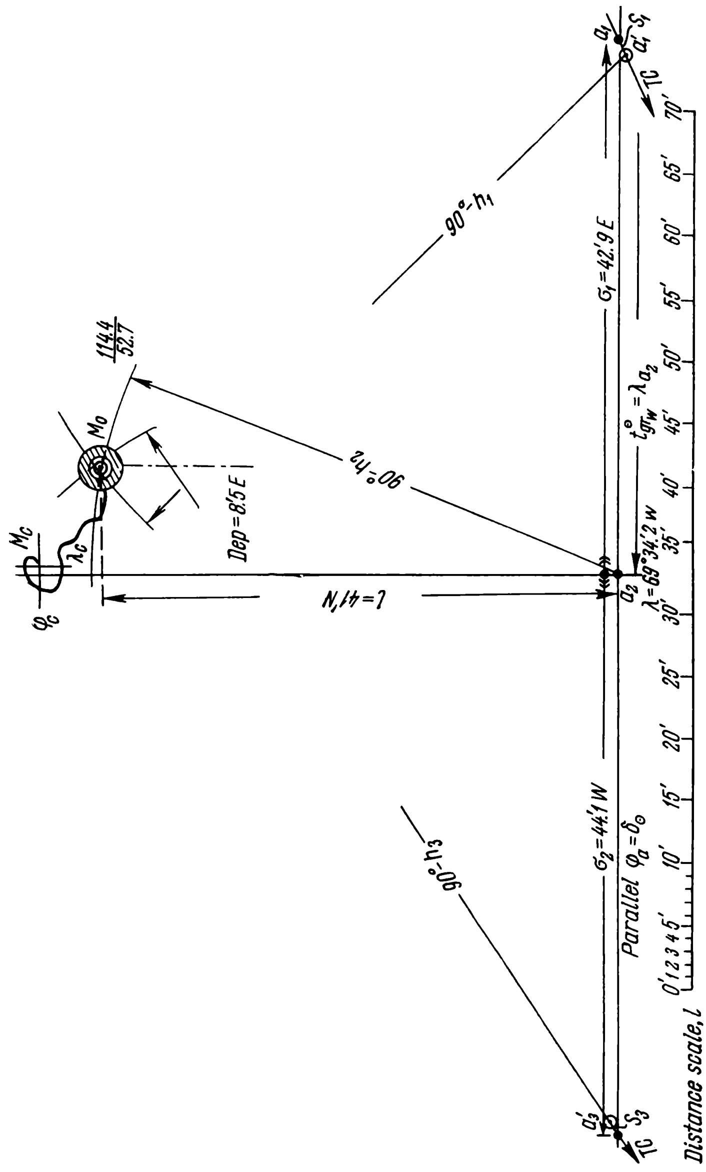
90°-h1
90°-h2
90°-h3
σ1=42'.9 E
σ2=44'.1 W
Dep=8'.5 E
l=41'N
λ=69°34'.2 W
Parallel φa=δ⊙
Distance scale, l
114.4
52.7
Mo
Mc
λc
φc
TC
a1
S1
a2
S3

Fig. 230

along the course line, and $S_3 = \frac{v}{60}(T_3 - T_2)$ is laid off back. The points a_1', a_3' thus obtained and the earlier plotted a_2 are the centres of circles of equal altitudes. The radii of these circles are computed after correcting the measured altitudes by the usual corrections; we have

$$z_1' = 90° - h_1^{\odot}; \quad z_2' = 90° - h_2^{\odot} \text{ and } z_3' = 90 - h_3^{\odot}$$

These distances (in minutes) are taken from the lateral frame of the chart in the region between the D.R. position and the parallel $\varphi_a = \delta_{\odot}$ by means of a drawing compass, chart dividers or a strip of heavy map paper punched with dividers; using radii z_1, z_2 and z_3, draw from centres a_1', a_2, a_3' three arcs of circles of equal altitudes. The arcs are constructed about the D.R. position so as to avoid mistakes in choosing the proper one of two points of intersection. In this case, an objective indication is the direction in which the sun has transited: to N or S of the observer. When the sun transits to the S, the point of intersection will be to the N of the parallel of the geographic position a (see Fig. 229), and vice versa.

The observed position is taken in the centre of the triangle of errors due to the preferential effect of random errors in graphical constructions; it refers to the instant T_2, at which time the log reading should be obtained. The computations in this method are very simple and the position is quickly obtained; there is a difficulty, however, in measuring altitudes close to 90°.

The difficulty of measuring large altitudes lies in the fact that in ordinary swinging of the sextant about the telescope axis the sun's image in the field of view appears to be moving almost parallel to the horizon, since the radius of the arc is very great. Also, the sun's azimuth varies rapidly, thus making its vertical circle move quickly, and it is difficult to put the sextant in the required vertical circle. Thus, measuring large altitudes has its peculiarities:

(a) the common practice of swinging the arc is dispensed with, and the observer with sextant simply rotates slightly about the vertical axis;

(b) the sextant is put into a vertical position "by eye and touch". With some practice, the sextant can be set to within 2° to 4°, which yields an error less than the accuracy of measurement;

(c) the sextant is positioned in the vertical circle of the celestial body by compass. To do this, first compute the approximate azimuth for the first altitude, and also its variations. The original azimuth may be obtained by graphical plotting or computed from the formula

$$\sin A = \cos \delta \cdot \sec h \cdot \sin t \qquad (22.10)$$

where h is the altitude which is taken equal to the meridian altitude, that is, $H = 90° - \varphi_N \pm \delta_S^N$.

Variations of azimuth are computed from the approximate formula

$$\Delta A° \approx \frac{1}{4} \sec H \cdot \Delta T^{\min} \approx \frac{1}{4} \tan H \cdot \Delta T^{\min} \qquad (22.11)$$

or from Table 14, Sec. 133. Most convenient in such cases are numerical azimuth tables.

When measuring the first altitude, the sextant is set in the vertical circle by the computed azimuth with the aid of a compass; its inclination is checked by eye; then the images of the sun and horizon are brought into coincidence. For the second set of observations, the sextant is now rotated in azimuth $A_2 = A_1 + \Delta A$, solely by compass, after which (continuing to hold the sextant in a vertical position) a second measurement is taken by eye; a third measurement is made by compass in similar fashion.

Some captains sailing in the tropics apply the following measuring technique: all altitudes are measured above a single point of the horizon (S or N) and any error in altitude due to inclination of the sextant is taken into account in the form of a correction obtained from the formula

$$\Delta h' = \frac{(\alpha')^2}{13{,}752} \sin 2h \qquad (22.12)$$

where α' is the angle of inclination of the sextant to the meridian of the observer determined by eye ($\alpha' = \alpha° \cdot 60$).

For example, if $h = 88°$, $\alpha \approx 10°$, then $\Delta h' = \pm 1'.8$; for $\alpha = 5°$, $\Delta h = 0'.5$.

From the foregoing it is clear that measuring altitudes greater than 88° requires a great deal of skill and experience; this demands preliminary training and careful checks.

The effect of random errors of altitude in this method is considered as in determinations by three stars located in one part of the horizon (see Sec. 115). The magnitude of the errors when measuring a single altitude in each observation and also the errors due to plotting on small-scale charts will be perceptibly greater than from three-star determinations.

The observed position is taken at the centre of the triangle or at the vertex closest to danger. For average observational conditions, we may take it that the actual position lies in a circle of radius 2 to 3 miles about the observed position.

A routine sequence for determining position from sun altitudes greater than 88°.

A. Preliminary

(1) Compute ship's time of upper transit of sun (with noon longitude).
(2) Compute H from $\delta_\odot$ and φ_c approximately.
(3) From Table 14, Sec. 133, or formula (22.11) determine the rate of change of azimuth per minute. From this rate find the time interval ΔT for which the azimuth difference between lines will be 40°-60°, and compute the instants T_{sh} of onset and end of observations.
(4) Give approximate computation of azimuths for first and last sets of observations on the basis of change of azimuth or from the formula (22.10); it is best to take the second altitude above the point S (or N) at the computed instant of upper transit.

B. Observations

(1) Ready instruments and determine their corrections in the usual way.
(2) From 5 to 7 minutes prior to computed instant of observations, take a trial altitude and compare it with the computed H to see whether you can rely on computations.
(3) At indicated instant, take first altitude and note chronometer time T_1 (orientation by compass).
(4) After interval ΔT_1, take second altitude and time T_2 (if second altitude is H, note over what point, S or N, the measurement was made). Note T_{sh} and log.
(5) After interval ΔT_2 (azimuth A_3 is taken by compass) take third altitude and note time T_3.

C. Working Sights

(1) Obtain T_{gr} at second instant and take out $t_{gr2}^{\odot}$ and $\delta_\odot$.
(2) Compute $\Delta T_1 = T_2 - T_1$ and $\Delta T_2 = T_3 - T_2$ and express them in degrees, using interpolation table of MAE.
(3) Compute $t_{gr1}^{\odot} = t_{gr2}^{\odot}\,\mathrm{w} - \Delta T_1^{\circ}$ and $t_{gr3}^{\odot} = t_{gr2}^{\odot}\,\mathrm{w} + \Delta T_2^{\circ}$.
(4) Compute ship's run S_1 and S_2 during time ΔT_1 and ΔT_2.
(5) Correct altitudes and compute for each altitude $z = 90° - h$.
(6) Perform plotting on chart. To do this:
(a) draw parallel $\varphi_a = \delta_\odot$;
(b) on it lay down points a_1, a_2 and a_3 in longitude $\lambda_1 = t_{gr1}^{\odot}$ and so forth and reduce a_1 and a_3 to zenith of second set of observations;

(c) lay down onto long strip of map paper ("radius") the distances z_1, z_2 and z_3 from the lateral frame of the chart, making punctures with dividers in the centre and at each distance;

(d) bring to coincidence the centre of the "radius" and the geographic position a_1' (by means of dividers) and, putting pencil tip in hole at the distance z_1, draw the first circle of equal altitudes in the region of the D.R. position. Draw the second and third circles in similar fashion;

(e) take the observed position in the centre of the triangle of errors or in one of its vertices and analyze the determination.

Plotting on paper is done in exactly the same way, but the scale should be as large as possible (for instance, 1 mile equals 0.5 cm). The plotting can be done on the reverse of a chart. In this case, in place of the longitudes λ_1 and λ_3 compute the departures: $\sigma_1 = \Delta T_1^\circ \cdot \cos\delta_\odot$ and $\sigma_2 = \Delta T_2^\circ \cdot \cos\delta_\odot$. The meridian $\lambda_2 = t_{gr2}^{\odot}$ is drawn through the middle of the sheet of paper, and the parallel above or below it, depending on where the position is, to the north of the parallel (if the sun has transited to the S) or to the south of it. The observed coordinates are obtained by adding l and DLo to $\varphi_a = \delta_\odot$ and to the meridian selected as the initial one: $\lambda = t_{gr2}^{\odot}$; thus,

$$\left.\begin{aligned} \varphi_0 &= \delta_\odot \pm l \\ \lambda_0 &= t_{gr2}^{\odot} \pm DLo \end{aligned}\right\} \qquad (22.13)$$

Table 25, MT-53, is used to convert the departure obtained into DLo.

Example 4. On 13 July, 1968, in the Atlantic Ocean on course 243° true; speed 13 knots; decided to determine position from sun altitudes. On the basis of the relation $\varphi_c \approx 22°.5$N and $\delta_\odot \approx 22°$ we conclude that the sun altitudes exceed 88°.

(1) Determining time of observations.

(a) Time of transit at $\lambda_c \approx 69°30'$W (from chart at noon).

T_T	12h 06m
ΔT_λ	00
T_{loc}	12h 06m
λ_W	4 38
T_{gr}	16h 44m 13.07
ZD	5
T_{sh}	11h 44m

(b) Time of observations.

ΔA in 1m from Table 20 $\approx 20°$.

$\Delta T \approx 3$m from difference $\Delta A = 60°$.

First observation

$T_{sh1} = 11$h 44m $-$ 3m $= 11$h 41m; $A_1 \approx 60°$SE

Third observation

$T_{sh3} = 11$h 44m + 3m $= 11$h 47m; $A_3 \approx 60°$SW

(2) Observations.

(1) $T_{sh}=11\text{h}\ 41\text{m}$; $sr_{\underline{\odot}}=88°57'.8$; $T_{ch}=$ $=4\text{h}\ 51\text{m}\ 23\text{s}$;

(2) $T_{sh}=11\text{h}\ 44\text{m}$; $sr_{\underline{\odot}}=89°10'.0$ (S); $T_{ch}=$ $=4\text{h}\ 54\text{m}\ 28\text{s}$; $lr=52.7$; $\varphi_c=22°36'.5\text{N}$; $\lambda_c=69°33'.5\text{W}$;

(3) $T_{sh}=11\text{h}\ 47\text{m}$; $sr_{\underline{\odot}}=88°45'.0$; $T_{ch}=$ $=4\text{h}\ 57\text{m}\ 41\text{s}$;

$u_{ch}=-10\text{m}\ 34\text{s}$; $i+s=-1'2$; $e=12.5$ metres.

(3) Working sights.

T_{sh}	11h 44m 13.07
$+$ ZD_W	5
T_{gr}	16h 44m 13.07
δ_T	21° 45'.7 $(\overline{0.4})$
$\Delta\delta$	-0 .3
$\delta_{\odot}$	21°45'.4N

Data for reducing a_1 and a_3 to the zenith of a_2

T_{ch}	4h 51m 23s	4h 54m 28s	4h 57m 41s
u_{ch}	-10 34	-10 34	-10 34
T_{gr}	16h 40m 49s	16h 43m 54s	16h 47m 07s
t_T	$\Delta T_1=3\text{m}\ 5\text{s}$	58°35'.0 (0.2)	$\Delta T_3=3\text{m}\ 13\text{s}$
Δt	$\Delta t_1=0°46'.2$	10 58 .4	$\Delta t_3=0°48'.2$
$t_{gr}^{\odot}$	$\Delta t_1=$ 46'.2E	69°33'.4	$\Delta t_3=48'.2\text{W}$
Departure	$\sigma_1=\Delta t_1\cdot\cos\delta$ $=42'.9\text{E}$	—	$\delta_3=\Delta t_3\cdot\cos\delta$ $=44'.7\text{W}$
Run	$S_1=0'.67\approx$ $\approx 0'.7$	—	$S_3=0'.69\approx$ $=0'.7$

$TC=243°$

	I	II	III
$sr_{\odot}$	88°57'.8	89°10'.0	88°45'.0
$i+s$	-1 .2	-1 .2	-1 .2
h'	88°56'.6	89° 8'.8	88°43'.8
Δ_{tot}	$+9$ .7	$+9$ .7	$+9$ .7
Δ_{ad}	-0 .2	-0 .2	-0 .2
$h_{\odot}$	89° 6'.1	89°18'.3	88°53'.3
$z_{\odot}$	53'.9	41'.7	66'.7

(4) Plotting on paper is shown in Fig. 230.
(5) The results are: $l = 41'$N; $Dep. = 8'.5$E; $DLo = 9'.2$E

$\delta_{\odot}$	21°45'.4N	$t^{\odot}_{gr}$	69°33'.4W
l	+41'.0N	DLo	—9'.2E
φ_0	22°26'.4N	λ_0	69°24'.2W

$T_{sh} = 11$h 44m; $lr = 52.7$; $C = 123° - 9'.5$

(6) Analysis of the determination. Since the position has been obtained from three lines and the triangle of errors is small, the observed position is taken in the middle and we advance dead reckoning to the observed position. About this spot we draw a circle of radius 2'.0; the actual position of the ship is located within the area obtained.

SEC. 136. STAR ALTITUDE CURVES

The problem of determining latitude and longitude of a place from the altitudes of two, three, and four stars may be solved without calculating h_c and A_c: in a purely graphical manner, which is much simpler and faster than the Marcq Saint-Hilaire method.

A special aid called **star altitude curves** represents calculating charts that depict sections of the celestial sphere with grids of the circles of equal altitudes of several bright stars. The underlying concept, mentioned in Sec. 100, consists in the following.

Referring to Fig. 231, let us picture grids of circles of equal altitudes constructed for two (or more) stars C_1 and C_2 on the celestial sphere. The ship's zenith on the sphere at the time of observations of the altitudes of these stars is located simultaneously on two different circles I and II, which correspond to the measured altitudes and, consequently, should be located at the point of intersection of these circles Z_{loc}. Due to the diurnal rotation of the sphere, the zenith of the observer will be constantly moving along the parallel dd_1 among stationary (relative to the sphere) circles of equal altitudes of different stars. Therefore, to determine the instantaneous place of zenith, that is, its place at the given time, it is necessary to have a series of curves in the region of the parallel of the zenith. To do this, grids are built of circles for a number of stars in the zone of the parallel d_1d of the zenith. Depicting this zone of the sphere together with the grid of circles of equal altitudes of the stars in the plane in a Mercator projection, we get a set of star altitude curves for the given zone of latitudes. The coordinates of the zenith on the celestial sphere are connected with the geographic coordinates of the ship's position by the relations (18.1) obtained

in Sec. 96:

$$\delta_Z = \varphi_{loc}$$
$$S_{loc} - S_{gr} = \lambda_{loc}$$

where $S_{gr} = t_{gr}^{\Upsilon}$ is computed for the instant T_{gr} recorded during observation of the stars.

On the basis of these relations, the frames of a star-altitude-curve chart are constructed in coordinates φ and S_{loc}, in which we obtain

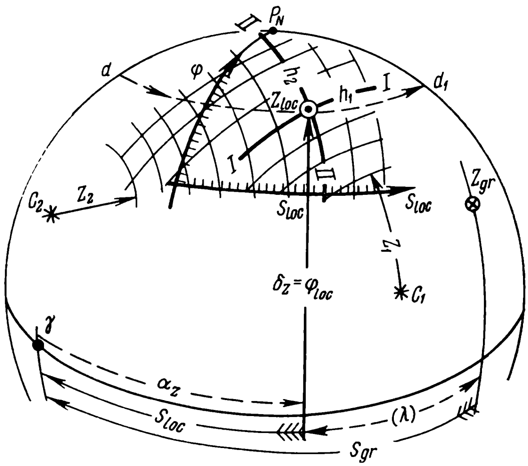

Fig. 231

the place of zenith after working the sights on this chart. The sidereal time S_{loc} on the frames of the chart increases eastwards (to the right), since the "time is greater" on meridians to the east.

Each Soviet star-altitude-curve chart has the curves of three bright stars and also the scales of altitudes of the Pole Star, which can be used to determine the latitude or plot on the chart an altitude line of position for Polaris.

A difficulty in constructing star-altitude curves is the allowance that has to be made for variations of stellar coordinates. Indeed, in the formulas used we can apply only certain instantaneous values of δ_{*} and τ_{*}, which will be those used to build the curves. But δ_{*} and τ_{*} are constantly changing due to precession, nutation, aberration and the proper motion of the stars, and the coordinates φ and $\alpha_Z = S_{loc}$ obtained for a different time will be erroneous.

An investigation of this problem has shown that we can only account for the effects of precession and, sometimes, the proper motion of the stars. We have to disregard the effects on coordinates

of aberration, nutation and, in the general case, of the proper motion of the stars; this results in an error in the zenith of the order of $\varepsilon_{loc} = \pm 0'.5\text{-}0'.6$. Variations of coordinates due to precession may be taken into account by the method of introducing corrections. Professor I. Zhongolovich has suggested taking into account variations of the coordinates φ and S_{loc} due to precession in the same way as the coordinates δ and α of points on the sphere (Sec. 22); that is, by means of formulas (5.6) and (5.7) in the form

$$\left.\begin{aligned} \delta\varphi' &= 0'.334 \cdot \cos S_0 (t - t_0) \\ \delta S' &= (0'.768 + 0.334 \cdot \tan \varphi_0 \cdot \sin S_0)(t - t_0) \end{aligned}\right\} \quad (22.14)$$

where S_0 and φ_0 are the mean values of sidereal time and latitude for the given chart

t_0 is the year for which the stellar coordinates are taken

t is the year the chart is used.

These formulas were used to compile the tables which are given under the altitude curves and which yield (with the argument "year of observations") final corrections $\delta\varphi$, δS which are added with their signs to those obtained from the curve φ' and S'_{loc}.

The most accurate and refined of the star altitude curves are those worked out and compiled at the Institute of Theoretical Astronomy, U.S.S.R. Academy of Sciences. These curves were published by the U.S.S.R. Navy in 1952-1957 for latitudes: 42°-49°N, 49°-56°N, 56°-63°N, 63°-70° and 70°-77°N. In 1957 the U.S.S.R. Academy of Sciences also published star-altitude-curve charts for latitudes 80°-90°N.

Sights are worked by means of star altitude curves in the following way. Let us assume that a series of three altitudes has been taken of three stars chosen from the charts, the instants recorded, and the *lr* and φ_c, λ_c obtained in the usual way for the last instant.

After computing the mean altitudes and instants, we correct the altitudes and choose the quantity t_{gr}^{Υ} from the MAE (or tables in the instructions that accompany the charts) for the last instant of T_{gr}. The chart needed for working the sights is found from the computed value of $S_{loc} = t_{gr}^{\Upsilon} \pm \lambda_{cW}^{E}$ from the table in the chart instructions.

The curves on the chart are given every 10′, and so the intermediate isolines are found by interpolation by eye; for instance, in Fig. 232 the line of the first star for $h = 29°46'.5$ is represented by the segment *I-I* in the region of the D.R. latitude. Plotting in similar fashion the line segments for the second star *II-II* and for the third star *III-III*, we see that all segments are located in different places of one parallel. This indicates that the star altitudes were obtained

at different instants and from different points of the earth; they must therefore be reduced to a single instant and zenith.

These operations are best done graphically—by advancing the lines in course and distance run (to the zenith T_3) and along the parallel (to the instant T_3). It is most convenient to begin by reduction to a single zenith. To do this, indicate arbitrary points A and B

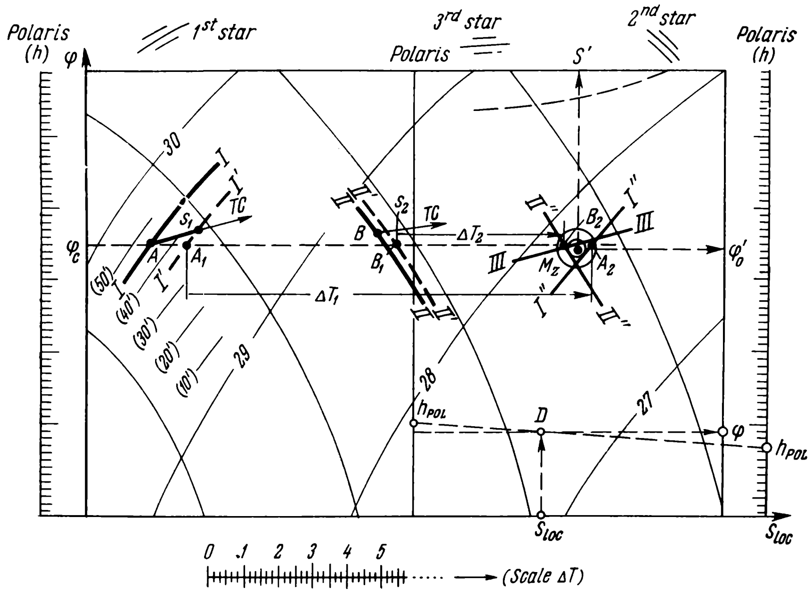

Fig. 232

on the lines I and II in the region of φ_c and from them plot lines of course or TC. Using these lines, plot the ship's runs S_1 and S_2 during the time between observations. These runs are obtained from the time intervals ($\Delta T_1 = T_3 - T_1$ and $\Delta T_2 = T_3 - T_2$) and the velocity from Table 27a, MT-63, or a calculating chart. The direction of the TC is reckoned from the meridian as usual, while the ship's runs S are taken from the lateral frame of the star altitude curve.

The curves obtained, I'-I' and II'-II', are reduced to the zenith T_3 but refer to different instants, T_1 and T_2; they must therefore be reduced to the last instant T_3.

Since the zenith of a stationary observer moves among the curves to the right of the parallel d_1d (see Fig. 231) with the rate of diurnal rotation of the sphere, that is, in proportion to the time, in order to reduce the curves to a single instant, they must be advanced by

the amount of the difference of instants: $\Delta T_1 = (T_3 - T_1)$ and $\Delta T_2 = (T_3 - T_2)$ to the right towards increasing time S_{loc}. The intervals ΔT_1 and ΔT_2 must be expressed in the scale of the chart (in equatorial minutes); for this purpose there is a special scale ΔT at the bottom of the chart in minutes and seconds of mean time.

Taking from this scale the intervals ΔT_1 and ΔT_2 and laying them off from the points A_1 and B_1 to the right along the parallel, we get the points A_2 and B_2, through which the advanced lines I''-I'' and II''-II'' are drawn parallel to the curves of the same star at the points A_2 and B_2.

Now all three lines of position, I'', II'' and III, refer to the instant and zenith of observations of the last star; and so their point of intersection yields the observed zenith. If a triangle of errors has resulted, the site is taken inside it or at the vertex closest to danger, in a manner similar to obtaining a fix from three stars (see Sec. 115).

Take the latitude φ_0' from the lateral frame of the chart, S'_{loc} from the lower (or upper) frame and correct them with the corrections $\delta\varphi$ and δS for precession, which corrections are taken from the table at the bottom of the charts. We then get φ_0 and S_{loc}.

$$\left.\begin{aligned} \varphi_0 &= \varphi_0' + \delta\varphi \\ S_{loc} &= t^{\Upsilon}_{loc} = S'_{loc} + \delta S \end{aligned}\right\} \qquad (22.15)$$

Finally, we get the longitude

$$\lambda_0 = t^{\Upsilon}_{loc} - t^{\Upsilon}_{gr}$$

where t^{Υ}_{gr} is obtained at the beginning of the computation from the instant T_3.

Working the sights of Polaris is a rather simple undertaking. The charts for latitudes from 42° to 63°N have three altitude scales of Polaris; for latitudes 63°-70° there are four scales (two in the middle); there are no scales for Polaris for latitudes exceeding 70°.

To determine the latitude from Polaris we find the computed value of $S_{loc} = t^{\Upsilon}_{gr} \pm \lambda_c$, and in the region of this value (on the scale S_{loc}, Fig. 232) we locate on the two Polaris scales (one of them is constructed in the middle of the chart) the same values of altitude of Polaris: h_{Pol}, which correspond to its corrected altitude. Connecting these points with a straight line, we get a segment of the curve of equal altitudes of Polaris. If a line of position is required, this line $h_{Pol} - h_{Pol}$ is taken as the position line. And if the latitude φ_0 is required, we locate point D at the intersection of the meridian S_{loc} with the line $h_{Pol} - h_{Pol}$. The latitude of point D taken from the lateral frame will be φ_0', which should be corrected by the formula (22.15). We thus see that it is very simple in this

way to obtain a position line of Polaris or the latitude from Polaris; however, we must bear in mind that the third correction is not taken into account here, so that their accuracy will be lower than in the usual method of working sights.

The advantages of star-altitude-curve charts are:

(a) speed of working sights is cut by more than a factor of two;

(b) the possibility of dispensing with the MAE and tables;

(c) simplicity of routine and small number of operations, thus reducing the probability of blunders.

This method, however, has the following essential defects:

(a) the method is not universal, because it is very difficult to construct star-altitude-curve charts for the sun, moon and planets; therefore, conventional methods have to be employed to obtain fixes using these bodies;

(b) the choice of stars is limited;

(c) cases may occur that require a full set of charts for all latitudes, and these may not be available on board ship. A fix will then have to be obtained in the usual way.

Hence, star-altitude-curve charts cannot replace the general method, but can only supplement it. That is the reason why these charts have not come into general usage.

Fixes obtained with the help of star-altitude-curve charts are of somewhat lower accuracy than usual; this is due to the approximation in taking account of changes of stellar coordinates δ and τ, and due to errors of the chart itself. On the whole, the accuracy of working sights depends on the scale of the charts. Working sights by star altitude curves published by the U.S.S.R. Navy in 1952 (these are the most accurate ones) yield possible mean errors in the zenith of the order of $\varepsilon_{loc} = \pm 1'.0\text{-}1'.5$ without allowance for observation errors.

Example 5. On 18 September, 1968, in the Atlantic Ocean, on course 55° true at a speed of 11 knots, decided to obtain a fix in morning twilight from three stars with the help of star altitude curves.

1. Determining time of observations. At $T_{sh} = 4\text{h}$; $\varphi_c \approx 62°.5\text{N}$; $\lambda_c \approx$ $\approx 14°45'\text{W}$.

Sunrise ⊙	T_T	5h 31m
	$\Delta T_{\varphi\lambda}$	0
	T_{loc}	5 31
	$(\text{N}+\lambda)$	−3
Sunrise ⊙	T_{sh}	5h 28m

	Twilight	
	Nautical	Civil
Sunrise ⊙ T_{sh}	5h 28m	5h 28m
Duration of twilight	1 42	0 46
Onset of twil. T_{sh}	3h 47m	4h 42m

We choose the time for beginning observations $T_{sh} = 4\text{h}35\text{m}$. The final time of observations will be determined after choice of stars (Item 2).

2. Choice of stars.

(a)

T_{sh}	4h 35m
ZD	1
T_{gr}	5h 35m 18.9
t_T^{Υ}	72°11′
Δt	8 46
t_{gr}^{Υ}	80°57′
$-\ \lambda_W$	14 45
$t_{loc}^{\Upsilon} \approx$	66°12′ $\approx$ 66°

(b) From Table 2 of chart instructions we select (using $t_{loc} = 66°$)

	h	A
(1) α Ursae Majoris	45°	40°
(2) α Leonis	15	85
(3) α Canis Minoris	25	125

We shall observe the stars in that sequence, proceeding from the visibility of the horizon and the celestial body.

If the t_{loc}^{Υ} obtained is close to the t_{loc}^{Υ} on the right-hand frame of the chart, the time T_{sh} is decreased (increased) so that all curves reduced to a single instant should be on the given sheet of the chart and not on different sheets.

3. Observations

	I. α Ursae Majoris (Dubhe)	II. α Leonis (Regulus)	III. α Canis Minoris (Procyon)
av. s_r ...	48°25′.5	13°12′.0	23°22′.6
av. T_{ch} ...	5h 30m 8s.5	5h 33m 28s	5h 35m 51s
	$i + s_1 = -0.7$	$i + s_2 = -1'.0$	$i + s_3 = -0'.9$

$u_{ch} = +3$m 53s; $e = 7.5$ metres; $t° = +11°$; $B = 760$ mm; $T_{sh} = 4$h 40m; $lr = 28.3$; $\varphi_c = 62°37'$N; $\lambda_c = 14°46'$W.

4. Working sights

	I. Dubhe	II. Regulus	III. Procyon
T_{ch}	5h 30m 8s.5	5h 33m 28s	5h 35m 51s
u_{ch}	3m 53s	3m 53s	3m 53s
T_{gr}	5h 34m 1s.5	5h 37m 21s	5h 39m 44s

Data for reducing to zenith and instant T_3					
	ΔT	5m 42s $\approx$ 5m.7	2m 23s $\approx$ 2m.4		
	ship's run	1′.0	0′.4	t_T^{Υ}	71°38′.3
	TC	55°	55°	Δt	9 57 .6
				$S_{gr} = t_{gr}^{\Upsilon}$	81°35′.9W

sr	48°75′.5	13°12′.0	23°22′.6
$i+s$	—0 .7	—1 .0	—0 .9
h'	48°24′.8	13°11′.0	23°21′.7
Δ_{tot}	—5 .7	—9 .0	—7 .2
h_0	48°19′.1	13° 2′.0	23°14′.5

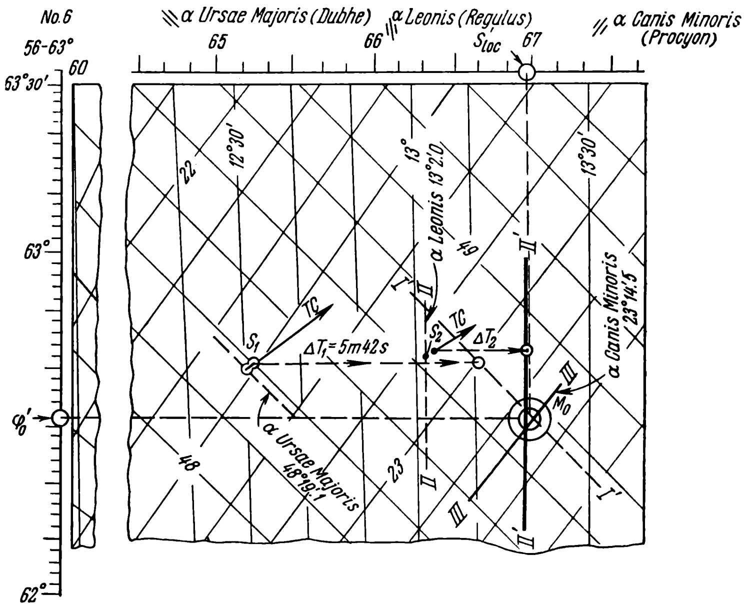

Fig. 233

5. From chart (sheet No. 6 Fig. 233).

φ'	62°31′
$\delta\varphi$	+1 .8
φ_0	62°32′.8N

$S'_{loc}=t^{\Upsilon}_{loc}$	66°58′	
δS	17 .5	—from table at bottom of chart
S_{loc}	67°15′.5W	
S_{gr}	82°08′.4W	
λ_0	14°53′W	

$T_{sh}=4$h 40m, $lr=28.3$

6. Analysis of the observed position is done in similar fashion to a three-star fix by the method of altitude lines of position, but the mean error of a line $\varepsilon_{\Delta h}$ should be increased from 1.5 to 2 times.

CHAPTER 23

DETERMINING TIME OF RISING AND SETTING OF CELESTIAL BODIES AND THE ILLUMINATION OF THE HORIZON

SEC. 137. GENERAL REMARKS

Cases often occur when it is necessary to know the time of rising and setting of some body. Most frequently it is sunrise and sunset. These problems mostly involve determining the natural illumination of the horizon, which means that the time of twilight is also important. Here, high accuracy is not required, it being sufficient to determine the time to within 0h.1; only sunrise and sunset should be known to within 1m.

Two instants are involved in the phenomena of rising and setting of celestial bodies:

(1) **True rising (setting)** of a body: when the centre of the body (sun or moon, say) arrives at the true horizon;

(2) **Apparent rising (setting):** when the upper limb of the body is tangent to the visible horizon.

Determining the hour angle of a body at the instant of true rising and setting was considered in Sec. 9, where these problems were solved as a consequence of the diurnal motion of the body without allowance for variations of the equatorial coordinates and likewise without allowance for the apparent dimensions of the bodies and the parallax.

When determining the instant of apparent rising and setting, it is necessary to know the value of declination of the body at that instant and also the value of dip, the apparent semidiameter and the diurnal parallax of the body, as a result of which the problem becomes involved.

Practically speaking, we need be interested only in the apparent rising and setting of the limb of a body, the true rising (setting) being used only as an intermediate stage in the solution.

In some cases we have to know the natural illumination of the horizon by sun and moon for a given place or locality and over a long period of time. In such instances, it is convenient to build a schedule of illumination for one or two months in advance, which would include, on a single sheet, data on twilight and the risings and settings of these bodies.

SEC. 138. DETERMINING A SHIP'S TIME OF TRUE AND APPARENT SUNRISE AND SUNSET

I. TRUE SUNRISE AND SUNSET

At the instant of true sunrise the centre of the sun will be on the true horizon (Fig. 145, Sec. 94), that is, $h = 0$, and from the triangle ZP_NC_o using formula sin h we have

$$\cos t_{loc}^{\odot} = -\tan\varphi \cdot \tan\delta_{\odot} \tag{23.1}$$

The local hour angle $t_{loc}^{\odot}$ (computed from this formula after the latter has been investigated) of true sunrise (sunset) will be east for sunrise and west for sunset, and if δ and φ are of the same name, then $t > 90°$, and if of contrary name, then $t < 90°$. When taking out of the MAE the declination of the sun, which enters into formula (23.1), we take T_{sh} of sunrise (sunset) obtained a few days earlier; or in a less accurate solution: for sunrise $T_{sh} \approx 6h$, for sunset $T_{sh} \approx 18$h; if greater accuracy is required, the problem is solved by successive approximation.

The computed $t_{loc}^{\odot}$ is converted to west, and then to $T_{gr}^{\odot} = t_{loc}^{\odot} \mp \lambda_W^E$, after which we determine T_{sh}* by reverse entry of MAE. At sea the instant of true sunrise (sunset) may be found in the following approximate fashion: the height of the lower limb of the sun above the apparent horizon at this instant will be approximately 0.7 or 3/4 the vertical diameter of the sun, which on the horizon is less than usual due to refraction and is equal to 26′-28′.

II. APPARENT SUNRISE (SUNSET)

For practical purposes it is much more important to know the instant of the apparent rising (setting) of the upper limb of the sun. At the time of apparent sunset, ships lower the flag and all lights under way and anchor lights are switched on. Beacon and signal lights are put on at the same time. Lights are switched off in the morning at the instant of apparent sunrise.

As may be seen from Fig. 145, Sec. 94, the hour angle t_r and hence the time of apparent sunrise will not be equal to the time of true sunrise. This is due to the circumstance that at the instant of apparent rising of the upper limb of the sun its centre is lower than the true horizon by the amount of negative altitude:

$$DC = h_{\odot} = -d - \rho - R_{\odot} + p \tag{23.2}$$

* See Sec. 47.

The value of the dip of the horizon d depends on the height of the observer's eye, and if the mean height of the bridge is $e = 6.1$ metres, then $d = -4'.4$. On the average the refraction ρ on the apparent horizon will be $\rho = -35'.5$, the mean angular radius of the sun $R_{\odot} = 16'.0$, and its parallax $p = +0'.2$.

Taking these values, we have $h_{\odot} = -55'.7$, which means the centre of the sun lies 55'.7 below the true horizon. If we take $e = 0$, then $h_{\odot} = -50'.3$. This magnitude is the accepted one in the MAE and MT-63.

For this same reason, as will be seen from Fig. 145, *apparent sunrise takes place before true sunrise, and sunset comes later.* The time interval ΔT between these instants depends on the angle of inclination of the parallel to the horizon, that is to say, on the latitude of the position and the declination of the celestial body, as is evident from the same figure.

The hour angle t_r of apparent sunrise may be found in two ways:

(a) by computing t_r from the general formulas for an astronomical triangle for a given $h_{\odot}$;

(b) by computing from the formula (23.1) the hour angle of true sunrise $t_{loc}^{\odot}$, and then introducing into it the correction Δt for a change of altitude $\Delta h_{\odot}$.

In the former instance, the hour angle is determined from the triangle ZCP_N (see Fig. 145) from the formula of the cosine of a side ZC, from which we have

$$\cos t_r = \frac{\sin h - \sin \varphi \cdot \sin \delta}{\cos \varphi \cdot \cos \delta}$$

After transformations considered in Sec. 130 we will have

$$\sin^2 \frac{t_r}{2} = 0.5 \sec \varphi \cdot \sec \delta \cdot \cos (\varphi - \delta) \left[1 - \frac{\sin h}{\cos (\varphi - \delta)}\right] \quad (23.3)$$

where h is computed for given conditions (height of eye, etc.).

In an exact solution of the problem, the quantity δ is taken from the MAE using the earlier (and approximately) known T_{sh} of sunrise (sunset), which is converted to T_{gr}, or by the method of successive approximation. When determining t_r to within 1 minute (in all cases except high altitudes) the declination may be taken for $T_{sh} = 6$h in the morning and 18h in the evening. To avoid a second approximation, the problem may be preliminarily solved in approximate fashion by a globe, and then with exactitude.

Similarly, using formula (23.3) we can also compute the hour angle of sunset (t_s), but if at sunrise the angle t_r obtained is east, then it will be west at sunset.

To determine the *time* of sunrise (sunset) from the determined west hour angle $t_{r(s)}^{\odot}$, we get $t_{gr}^{\odot}$ and then from the MAE by reverse entry we obtain T_{gr} and T_{sh}.

In the second method, the hour angle $t_{loc}^{\odot}$ is computed from the formula (23.1), while the correction Δt is computed from the formula (3.14), that is, $\Delta h = -\cos\varphi \cdot \sin A \Delta t$ in the form

$$\Delta t' = -\Delta h \cdot \sec\varphi \cdot \operatorname{cosec} A \qquad (23.4)$$

If a MAE is not available, $\delta_{\odot}$ is obtained from some kind of tables, charts (see Appendix VII) or by approximate calculation. After obtaining $t_{r(s)}$ in the fashion indicated above the time of sunrise (sunset) is determined relative to the time of solar transit, which is also obtained by computation. Use is made of the following formulas:

(a) local time and ship's time of transit of the sun

$$T_{loc}^{tr} = 12\text{h} + \eta$$

$$T_{sh}^{tr} = T_{loc}^{tr} \mp \lambda_{\text{W}}^{\text{E}} \pm N_{\text{W}}^{\text{E}}$$

where η is taken from tables or from a chart;

(b) time of sunrise: $T_{sh} = T_{sh}^{tr} - t_r$;

(c) time of sunset $T_{sh} = T_{sh}^{tr} + t_s$.

Problems involving computation of $t_{r(s)}$ and determination of the time of sunrise (sunset) by the above-mentioned methods are encountered only in enceptional cases, for instanse, for $\varphi_N > 74°$ (see example 1) or $\varphi_S > 60°$. Ordinarily, the time of sunset (sunrise) is found from MAE tables.

Example 1. On 12 September, 1968, in the Vilkitsky Straits, on course 25° true at a speed of 10 knots, the problem is to find T_{sh} of sunrise and sunset to within one minute.

(1) Determining time of sunrise.

(a) Approximately, we take sunrise at $T_{sh} = 5\text{h}$ (on the previous day, $T_{sh} \approx 5\text{h}.5$).

At $T_{sh} = 5\text{h}$ we have $\varphi_c = 77°57'\text{N}$, $\lambda_c = 103°35'\text{E}$.

Timepiece set to ZD 8E; $e = 14.5$ metres; $t = +5°$; $B = 750$ mm.

(b) Choosing declination and computing h.

$-T_{sh}$	5h 12.09
ZD	8
T_{gr}	21h 11.09

$\delta_{\odot} = 4°17'.4\text{N}$

or

$\delta_{\odot} = 4°17'\text{N}$

d	$-6'.7$
ρ_0	$-36\ .0$
$\Delta_{\rho t}$	$-1\ .2$
$\Delta_{\rho B}$	$+0\ .6$
$R_{\odot}$	$-15\ .9$
ρ	$+0\ .2$
$h_{\odot}$	$-59'.0$

(c) Computing t_r *

$\varphi = 77°57'$N	sec	6.6803		
$\delta = 4°17'$N	sec	0.0012		
$\varphi - \delta = 73°34'$	cos	9.4516	cos	9.4516
0.5	log	9.6990	—	
$h = -59'$	I	9.8321	sin h	8.2346
	α	0.0256		
			Arg **	1.2170
	$\sin^2$	9.8577		

$$t_r = 116°11'\text{E} = 243°49'\text{W}; \quad (t_r \approx 7\text{h}\,45\text{m})$$

(d) Calculating time of sunrise from MAE

	$-\,t_r$	243°49′		
	λ_E	103 35		
	t^r_{gr}	140°14′		
From MAE on 11.09	t_T	135 54 . .	T'_{gr}	21h
From interpol. table	Δt	4°20′ . .	ΔT_{gr}	0h 17m (20s disregarded)
			$+\,T_{gr}$	21h 17m 11.09
			ZD	8
			Sunrise ⊙ T_{sh}	5h 17m 12.09

Taking from MAE $\delta_{\odot}$ for $T_{gr} = 21\text{h}\,17\text{m}$, we see that a second approximation is not required here.

(e) If the problem is solved without the MAE, we obtain from charts (Appendix VII) or tables $\delta_{\odot} = 4°.5$N, $\eta = -4$m. In a solution similar to that given in (c) we obtain $t_r = 7\text{h}\,47\text{m}$.

Transit of Sun ⊙

$+\,12$	12h 0m
η	— 4m
T_{loc}	11h 56m
$(-\lambda + \text{ZD})$	+1 6m
T_{sh}	13h 2m

Sunrise ⊙

(Transit) . . .	$-\,T_{sh}$	13h 2m
	t_r	7 47
(Sunrise) . . .	T_{sh}	5h 15m

* All coordinates are rounded to 1′. Logarithms are rounded to 4 decimal places or are taken from 4-place tables.

** The argument (*Arg*) for taking out α in formula (23.3) is obtained as the difference log cos $(\varphi - \delta)$ — log sin h.

(2) Determining time of sunset on 12.09

(a) A first approximation is obtained from a globe; at $T_{sh} \approx 18h$ we get from a chart $\varphi_c = 77°32'N$; $\lambda_c = 96°27'E$; timepiece set to ZD = 7E.

Sunset ⊙	$-\,t^{\Upsilon}_{loc}$	284°.5 (from globe)		
	λ_E	96 .5		
	t^{Υ}_{gr}	188°.0		
On 12.09 . . .	t^{Υ}_{T}	186 .6 . . .	T'_{gr}	13h
Interpol. table	Δt	1°.2 . . .	ΔT_{gr}	0h 5m
			T_{gr}	13h 5m
			ZD	7
			T_{sh}	20h 5m

(b) At $T_{sh} = 20h\ 5m$, from a chart we have $\varphi_c = 77°25'N$; $\lambda = 95°40'E$.

From MAE at $T_{gr} = 13h\ 5m$;

δ_T	4°2'.2 ($\overline{1.0}$)
$\Delta\delta$	0 .1
$\delta_{\odot}$	4°2'N

(c)				
$\varphi = 77°25'N$	sec	0.6618		
$\delta = 4°\ 9'N$	sec	0.0011		
$\varphi - \delta = 73°16'$	cos	9.4593	cos	9.4593
0.5	log	9.6990	sin h	8.2346
$h = -59'$	I	9.8212	*Arg*	1.2247
	α	0.0251		
	sin²	9.8463		

$t_s = 113°49'$ W (7h 35m)

(d) From MAE	t_s	113°49′			
	$-\lambda_E$	95 40			
	t^s_{gr}	18° 9′			
12.09	t_T	15 57	. . .	T'_{gr}	13h
Interpolation table	Δt	2°12′	. . .	ΔT_{gr}	9m
				T_{gr}	13h 9m
				$+$ ZD	7
			Sunset ⊙	T_{sh}	20h 9m

III. DETERMINING TIME OF APPARENT SUNRISE (SUNSET) FROM MAE TABLES

The Nautical Astronomical Almanac (MAE gives the mean local time of sunrise and sunset for each day of the year for 30 values of latitude (from 60°S to 74°N)* on the Greenwich meridian in a manner similar to that considered above but for height of eye $e = 0$, that is, from sea level. The tables that contain these values also include the duration of twilight and are compiled for 10° intervals of latitude (up to $\varphi = 40°$), 5° intervals (up to 50°) and 2° intervals (up to 74°). Along with the time of sunrise (sunset) is given the diurnal variation (with preceding day); that is, for interpolation with east longitudes; for west longitudes, the sign of the correction should be reversed. To obtain the time of the phenomenon on a local meridian, take the tabulated time T_T obtained from the date and the lower of the two tabulated latitudes closest to φ_c, and interpolate in latitude and longitude; to do this:

(a) form the tabulated differences of time $\Delta\varphi$ along the vertical and take out the diurnal variation Δ_λ with its sign if the longitude is east and with reverse sign if west;

(b) with these differences and the values of $\Delta\varphi = \varphi_c - \varphi_T$ and λ_c, from the interpolation table at the end of the MAE take out corrections ΔT_φ and ΔT_λ for the time of the phenomenon, which have the same signs as the differences Δ_φ and Δ_λ. The corrections thus obtained are applied to T_T

$$T_{loc} = T_T + \Delta T_\varphi + \Delta T_\lambda \tag{23.5}$$

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* Also published in the U.S.S.R. are "Tables of Sunrise and Sunset for Latitudes 74°-20° North".

after which

$$T_{sh}=T_{loc}\mp\lambda^{\mathrm{E}}_{\mathrm{W}}\pm ZD^{\mathrm{E}}_{\mathrm{W}}$$

The corrections ΔT_{φ}, ΔT_{λ} for the sun in low latitudes (up to 54°) and longitudes are small, but in high latitudes and for large longitude they can become very large.

As we know, in high latitudes we observe a nonsetting sun (polar day) and a nonrising sun (polar night). These are indicated in the MAE tables by the symbols: ▭ and ■.

Example 2. Determine the time of sunrise and sunset on 21 August, 1968, in $\varphi=64°47'$N; $\lambda=177°27'$E by legal time.

		Sunrise ⊙	Sunset ⊙
On Greenwich merid.	T_T	4h 09m	19h 55m
Table A	ΔT_{φ}	—4 (—11m)	+4 (+11m)
Table B	ΔT_{λ}	—2 (—3m)	+2 (+4m)
On local meridian	T_{loc}	4 03 21.8	20 01 21.08
	$-\lambda$	11 50	11 50
	T_{gr}	16 13 20.8	8 11
	$+$ ZD + 1	13	13
	$T_{sh}=T_{log}$	5h 13m 21.8	21h 11m 21.08

SEC. 139. DETERMINING THE SHIP'S TIME OF TWILIGHT

When the sun sets, darkness is observed to develop gradually; this time is termed twilight. Twilight in a given place is associated with the passage of the sun's rays through the atmosphere. Indeed,

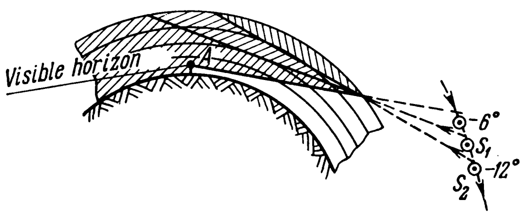


Fig. 234

although for an observer A (Fig. 234) the sun (S_1) has already set, the upper layers of the atmosphere are still illuminated by the sun

and scatter and reflect the sun's rays, thus producing an illumination that diminishes as the sun falls below the horizon. If we construct a curve (Fig. 235) of the amount of light for a given observer in clear weather, it will be seen that illumination on the earth's surface is determined mainly by the angular magnitude of depression of the sun's centre, if we disregard meteorological causes and moonlight. Depending on the degree of light in the atmosphere we generally differentiate between:

(1) **Civil twilight**: the interval of time between the instant of setting of the upper limb of the sun and depression of its centre to 6°

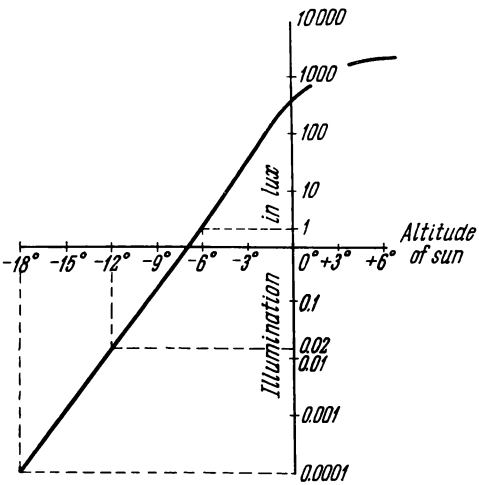


Fig. 235

(sometimes 7°) below the horizon; during this time, illumination is still considerable (diminishing from 700 to 1 lux) and sufficient to see distant objects and take their bearings; the horizon is nicely visible, while towards the end of twilight some of the brightest planets and stars become visible.

(2) **Nautical twilight**: the time interval between the end of civil twilight and depression of the centre of the sun to —12°; the horizon is visible and nearly all navigational stars too; it is no longer possible to take the bearings of objects on the shore. This time is best for star observations above the visible horizon. Towards the end of nautical twilight, the horizon is poorly visible and it is difficult to observe celestial bodies with a marine sextant.

(3) **Astronomical twilight**: the time interval from the end of nautical twilight to depression of the centre of the sun to —18°;

full night has set in by the end of astronomical twilight and all visible stars appear.

The foregoing refers also to morning twilight, but the sequence of events is reversed.

If civil twilight lasts all night, that is, if the centre of the sun does not go beyond 6° below the horizon, we have **twilight nights** (so-called **white nights**) in place of dark nights.

But if the sun does not rise at all and twilight lasts all day long, then we have **twilight days.** In MAE tables, " " " " indicates white nights, and × × × × × indicates twilight days.

The onset and duration of "white nights" are determined in the same way as the polar day, by the expression (for the case of civil twilight)

$$\delta_{\odot} \geqslant 90° - \varphi - d - \rho - R_{\odot} - 6°$$

or approximately

$$89° - \varphi \geqslant \delta_{\odot} \geqslant 83° - \varphi \tag{23.6}$$

where $\delta_{\odot}$ and φ are of the same name. For $\delta \geqslant 89° - \varphi$, a white night turns into a polar day.

For $\delta_{\odot}$ and φ of contrary names we have the dates of the onset (end) of "twilight days" from the formula $91° - \varphi \leqslant \delta \leqslant 97° - \varphi$.

During white nights, one can observe bright stars, but mostly only through a sextant telescope: for this reason, it is best first to compute their h and A from a globe.

If the centre of the sun does not drop beyond 12° below the horizon, nautical twilight will set in in the middle of the night and one can observe fainter stars.

Computing the time of twilight is similar to finding the time of apparent sunrise (sunset). To do this, in formula (23.3) for computation of t_{loc} of the sun at instants of onset (end) of the appropriate twilight one should take $h_1 = -6°$ (civil), $h_2 = -12°$ (nautical) and $h_3 = -18°$ (astronomical). For instance, for nautical twilight we have

$$\sin^2 \frac{t_{naut}}{2} = 0.5 \sec \varphi \cdot \sec \delta \cdot \cos (\varphi - \delta) \left[1 + \frac{\sin 12°}{\cos (\varphi - \delta)}\right] \tag{23.7}$$

To choose declination, as a first approximation, take T_{sh} for onset of nautical twilight in the morning at 4h and in the evening at 20h, or take T_{sh} of the previous day. From the computed t_{loc} (taken from MAE by reverse entry) we have the time T_{gr}, which is converted to T_{sh}. In case it is necessary to refine the solution, take the declination from the T_{gr} obtained and the problem is solved to a second approximation.

The solution by formula (23.7) is similar to that in Example 1 and might be needed only in high latitudes, for which no tables of twilight duration are available in the almanac (that is, for $\varphi_N > 74°$ or $\varphi_S > 60°$).

For smaller latitudes, the same daily tables of MAE have (next to the time of sunrise and sunset and for the same latitudes) columns entitled "duration of twilight", "civil twilight", and "nautical twilight". One must bear in mind that in these tables the duration of nautical twilight signifies the entire time from sunset to depression of the centre of the sun to —12°. And the time of onset of nautical twilight is obtained as the time of ending of civil twilight.

When taking these data from the MAE, they should be interpolated in latitude like the time of sunrise (sunset). Interpolation in longitude may be disregarded due to the impossibility of defining exactly the boundary between these twilights. To obtain the beginning and end of the twilights, the durations thus found should be added to the time of sunset or subtracted from the time of sunrise.

Example 3. Determine the time of onset of nautical and civil twilight on the morning of 28 August, 1968, in $\varphi = 51°13'$N; $\lambda = 160°19'$E (ZD = 12E)

		Nautical	Civil
Sunrise ⊙ for Greenwich	T_T	5h 08m	5h 08m
Table *A*	ΔT_φ	—2 (—3)	—2 (—3)
Table *B*	ΔT_λ	—1 (—2m)	—1 (—2m)
— { Sunrise ⊙ . . .	T_{loc}	5 05	5 05
{ Duration of twilight		1 18	0 35
Onset of twilight	T_{loc}	3 47	4 30
	$(-\lambda + \text{ZD})$	+1 19	+1 19
Onset of twilight	T_{sh}	5h 06m	5h 49m

λ	—10h 41m
ZD	+12
ZD — λ	+1h 19m

SEC. 140. DETERMINING TIME OF APPARENT MOONRISE (MOONSET)

The time of true moonrise (moonset) is computed in exactly the same way as for the sun, but due to the rapid change of lunar coordinates, its declination should be taken for a more exact value of T_{gr}. Therefore, if it is required to obtain the time to within 1 minute, the problem should be solved by successive approximation.

The time of apparent rising (setting) of the upper limb of the moon will differ from the true value by the amount $\Delta t = -\Delta h \sec \varphi \cdot \operatorname{cosec} A$. Let us determine $\Delta h = h_{☾}$ for average conditions, similar to those considered above, but with allowance made for the mean $R_{☾} = 15'$ and the parallax $p_{☾} = 57'$, that is,

$$h_{☾} = -d - \rho - R_{☾} + p_{☾} = -4'.4 - 35'.5 - 15' + 57' = +2' \quad (23.8)$$

Consequently, the height of the centre of the moon above the true horizon at the time of rising of its limb is on the average equal to $+2'.0$ (for height of eye $e = 6.1$ metres).

The magnitude of $p_{☾}$ varies between 53′ and 62′, ρ and d also change considerably; for this reason, the apparent rising of the upper limb of the moon may occur either somewhat earlier or somewhat later than true moonrise. For nautical purposes, it may be considered that the *instant of apparent moonrise* (*moonset*) *is roughly coincident with the instant of true moonrise* (*moonset*).

The hour angle of true moonrise is computed from the formula (23.1) in the form

$$\cos t_{loc}^{☾} = -\tan \varphi \cdot \tan \delta_{☾} \quad (23.9)$$

To pick out the declination of the moon from the MAE, one has to know T_{gr} at the instant of moonrise (moonset). This problem is solved by successive approximation. To a first approximation, the problem may be solved by a star globe on which the moon is indicated and the coordinates $\alpha_{☾}$ and $\delta_{☾}$ are chosen for a very rough time of moonrise (moonset): $T_{loc} = T^{tr} \mp 6\text{h}$ converted to T_{gr}. A second approximation is obtained from the formula (23.9) and an almanac (by reverse entry).

The correction for apparent moonrise is obtained from the formula

$$\Delta t' = -\Delta h'_{☾} \cdot \sec \varphi \cdot \operatorname{cosec} A_{☾}$$

In the solution, obtain $\Delta h_{☾}$ or $h_{☾}$ from (23.8) with $p_{☾}$ and $R_{☾}$ at the given instant and d for the given eye height; the azimuth is computed from the formula for true moonrise, that is,

$$\cos A_{☾} = \sec \varphi \cdot \sin \delta_{☾}$$

If it is required to compute the azimuth of the moon to within 0°.1, formula (17.20), Sec. 94, is used; here we substitute the value of $h_{☾}$ obtained from (23.8) and $\delta_{☾}$ obtained from the MAE by a second approximation.

COMPUTING THE TIME OF MOONRISE (MOONSET) FROM THE MAE

For practical purposes it is sufficient to know the time of apparent rising and setting of the upper limb of the mon to within 0h.1-0h.5; therefore, in place of an exact computation, one can make use of prepared values given in the daily tables of the MAE. These tables are compiled for the very same latitudes as those for the sun, and yield the civil local time of the phenomenon on the Greenwich meridian for every day of the year to within 0h.1. To obtain the time of the phenomenon in a given place, interpolate the chosen instants of T_T in latitude upwards (towards greater latitude) and in longitude (by means of diurnal variations given with their signs alongside the time of the phenomenon); the sign of variation corresponds to interpolation to the preceding 24-hour period, which means it can be applied in case of east longitudes; in case of west longitudes the sign is reversed. Time of moonrise (moonset) is interpolated for $\Delta\varphi$ and λ by means of tables at the end of the MAE.

Example 4. Determine T_{sh} of moonrise on 21 October, 1961, in $\varphi = 64°31'N$, $\lambda = 177°50'E$ at time of ZD = 13E.

	Moonrise	Moonset		
On Greenwich merid. . . . T_T	06h.3	15h.9 (on 21.10)	λ	−11h 51m
Table *A* . . . ΔT_φ	+0 .1 (+0h.2)	0 .0 (−0h.1)	ZD	+13
Table *B* . . . ΔT_λ	−0 .9 (−1 .9)	+0 .1 (+0h.2)		
On local merid. T_{loc}	5h.5	16h.0	(−λ+ +ZD)	+1h 9m $\approx$ 1h.1
−λ + ZD	+1 .1	+1 .1		
T_{sh}	6h.6 21.10	17h.1 21.10		

For certain dates the tables have symbols □ or ■ which indicate that on the Greenwich meridian the moon that day was constantly above the horizon or below the horizon.

In such cases one should take the time of the phenomenon in the succeeding day (for east longitudes) or the preceding day (for west longitudes) and obtain the T_{sh} of the phenomenon. It may turn out that in the given place the moon neither sets nor rises.

APPENDIX I

ESSENTIALS OF PLANE TRIGONOMETRY

1. Values and Signs of Trigonometric Functions of an Angle in Quadrants of a Circle

Functions \ Angles	0°	Quadrant I 0°—90°	90°	Quadrant II 90°—180°	180°	Quadrant III 180°—270°	270°	Quadrant IV 270°—360°	360°
$\sin\alpha$	0	+	+1+	+	0	—	—1—	—	0
$\cos\alpha$	1+	+	0	—	—1—	—	0	+	+1
$\tan\alpha$	0	+	$+\infty-$	—	0	+	$+\infty-$		0
$\cot\alpha$	$\infty +$	+	0	—	$-\infty+$	+	0	—	$-\infty$
$\sec\alpha$	1+	+	$+\infty-$	—	—1—	—	$-\infty+$	+	+1
$\operatorname{cosec}\alpha$	$\infty +$	+	+1+	+	$+\infty-$	—	—1—	—	$-\infty$

2. Formulas for Reducing Functions to an Angle of the First Quadrant

Angles \ Functions	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
$x=\alpha$	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\operatorname{cosec}\alpha$
$x=90°-\alpha$	$+\cos\alpha$	$+\sin\alpha$	$+\cot\alpha$	$+\tan\alpha$	$+\operatorname{cosec}\alpha$	$+\sec\alpha$
$x=90°+\alpha$	$+\cos\alpha$	$-\sin\alpha$	$-\cot\alpha$	$-\tan\alpha$	$-\operatorname{cosec}\alpha$	$+\sec\alpha$
$x=180°-\alpha$	$+\sin\alpha$	$-\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$	$-\sec\alpha$	$+\operatorname{cosec}\alpha$
$x=180°+\alpha$	$-\sin\alpha$	$-\cos\alpha$	$+\tan\alpha$	$+\cot\alpha$	$-\sec\alpha$	$-\operatorname{cosec}\alpha$
$x=270°-\alpha$	$-\cos\alpha$	$-\sin\alpha$	$+\cot\alpha$	$+\tan\alpha$	$-\operatorname{cosec}\alpha$	$-\sec\alpha$
$x=270°+\alpha$	$-\cos\alpha$	$+\sin\alpha$	$-\cot\alpha$	$-\tan\alpha$	$+\operatorname{cosec}\alpha$	$-\sec\alpha$
$x=360°-\alpha$ $x=-\alpha$	$-\sin\alpha$	$+\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$	$+\sec\alpha$	$-\operatorname{cosec}\alpha$

3. Formulas of the Trigonometric Functions of Sums and Differences of Angles (Addition and Subtraction Formulas)

$$\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}$$

4. Double-Angle Formulas

$$\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

or

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

Note. The designation $\sin^2\frac{\alpha}{2}$ is given differently in the manuals of certain other countries:
German: $\operatorname{sem}\alpha$ (semiversus α)
English: $\operatorname{hav}\alpha$ (haversine α, versine α)

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2}{\cot\alpha - \tan\alpha}$$

5. Formulas of the Sums and Differences of Functions

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta) \cdot \cos\frac{1}{2}(\alpha-\beta)$$
$$\sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha+\beta) \cdot \sin\frac{1}{2}(\alpha-\beta)$$
$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta) \cdot \cos\frac{1}{2}(\alpha-\beta)$$
$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta) \cdot \sin\frac{1}{2}(\alpha-\beta)$$

6. Tangent Formulas

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \tan\frac{1}{2}(A-B) \cdot \tan\frac{1}{2}C$$

APPENDIX II

ESSENTIALS OF SPHERICAL GEOMETRY

1. Relationship Between the Arcs of Great and Small Circles

The radius r of a small circle is connected with the radius R of a great circle by a relation that follows from the triangle OO_1A (Fig. 236):

$$r = R \cdot \cos \psi$$

Multiplying by 2π we have

$$2\pi r = 2\pi R \cdot \cos \psi$$

or

$$2\pi r \cdot n = 2\pi R \cos \psi \cdot n$$

which means that the length of an arc of a small circle or a part of it is less than the length of an arc of a great circle or a part of it by the factor $\cos \psi$.

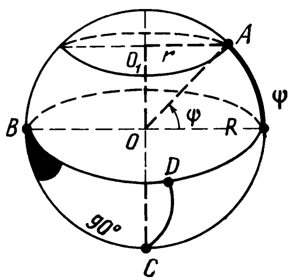


Fig. 236

2. Peculiarities of the Arcs of Great Circles

(a) The arc of a great circle is the shortest distance between any two points on the sphere.

(b) The position of an arc of a great circle is fully defined by two points not lying on the ends of one diameter.

(c) The planes of any two great circles intersect along the diameter of the sphere.

3. The Spherical Angle and Its Measurement

A spherical angle is an angle on the surface of a sphere that is formed by two arcs of great circles.

A spherical angle is measured by the arc of a great circle contained between its two sides and distant 90° from the vertex of the angle (arc CD in Fig. 236 measures angle B).

4. The Spherical Triangle

A spherical triangle is a curvilinear triangle on the surface of a sphere formed by three arcs of great circles that do not intersect at one point.

The sides and angles of a spherical triangle are measured in degrees, hours, or radians; in magnitude, they are ordinarily confined within the limits of 180° (Euler's triangles).

The principal properties of the sides and angles of a spherical triangle are:

(a) The sum of the sides of a spherical triangle lies between 0° and 360°.

(b) The sum of the angles of a spherical triangle lies between 180° and 540°.

(c) Greater sides lie opposite greater angles of the triangle.

(d) The sum of two sides of a triangle is always greater than the third side, and the difference is always less.

A right spherical triangle is one in which at least one of the angles is equal to 90° (all three may be right angles).

A quadrantal spherical triangle is one in which at least one of the sides is equal to 90° (all three sides may be 90°).

A small spherical triangle is one in which the sides are small compared to the radius of the sphere (in nautical astronomy a small triangle of the earth is one which has sides less than 2° for an accuracy of solution to within 1′).

An elementary spherical triangle is one in which one of the sides and the opposite angle are small relative to the radius of the sphere (for the earth, less than 2° for an accuracy of solution to within 1′, and less than 0°.6 for an accuracy to within 0′.1).

APPENDIX III

ESSENTIALS OF SPHERICAL TRIGONOMETRY

1. Basic Formulas of Spherical Trigonometry

(a) **Law of cosines for a side.**

In a spherical triangle the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of those sides multiplied by the cosine of the angle between them.

From Fig. 237 we have

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

There are three such formulas, one for each side.

(b) **Law of cosines for an angle.**

In a spherical triangle, the cosine of any angle is equal to the negative (minus) product of the cosines of the other two angles plus the product of the sines of the same angles multiplied by the cosine of the side between them.

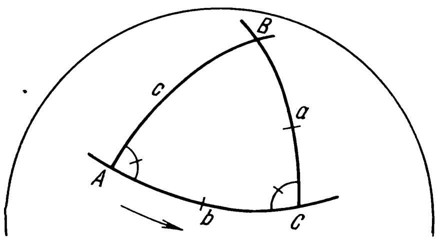


Fig. 237

From Fig. 237 we have

$$\cos B = -\cos A \cdot \cos C + \sin A \cdot \sin C \cdot \cos b$$

There are three such formulas, one for each angle.

(c) **Law of sines.**

In a spherical triangle, the sines of the sides are proportional to the sines of the opposite angles.

From Fig. 237 we have

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}$$

Three such relations may be written.

(d) **Law of cotangents or four adjacent parts.**

In a spherical triangle with four adjacent parts, the *cotangent of the opposite angle multiplied by the sine of the middle angle is equal to the product of the cotangent of the opposite side by the sine of the middle side minus the product of the cosines of the middle parts.*

From Fig. 237, for the parts indicated by dashes (following the arrow) we have

$$\cot A \cdot \sin C = \cot a \cdot \sin b - \cos b \cdot \cos C$$

Six such relations may be written.

2. An Intermediate Formula for Five Parts of a Spherical Triangle

The cosine of an angle of a triangle multiplied by the sine of the adjacent side is equal to the sine of the second side completing the angle by the cosine of the third side minus their product in reverse (the cosine of the completing side by the sine of the third side), *multiplied by the cosine of the angle between them.*

From Fig. 237 for angle B we have

$$\cos B \cdot \sin c = \sin a \cdot \cos b - \cos a \cdot \sin b \cdot \cos C$$

Six relations of this type may be written.

There is a similar formula for the angles of a triangle.

3. Deriving Formulas for Solution of Right Triangles

In the solution of right triangles we can use the basic formulas with subsequent simplification. To speed up obtaining the formulas we can use the *mnemonic rules of Napier*:

(1) In a right spherical triangle *with three adjacent parts, the cosine of the middle part is equal to the product of the cotangents of the opposite parts.*

(2) *If one of the parts is separate, then the cosine of the separate part is equal to the product of the sines of the adjacent parts.*

(3) In this case, a *right angle is not considered a separating part, and in place of the sides, one has to take their complements (to* 90°); for instance, $90° - b$, $90° - c$. From Fig. 238 we get

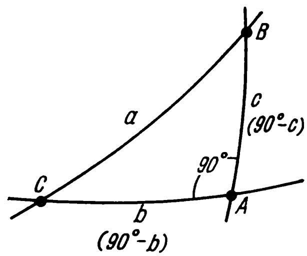


Fig. 238

$$\cos(90° - b) = \cot C \cdot \cot(90° - c) \quad \text{or} \quad \sin b = \cot C \cdot \tan c$$

$$\cos B = \sin C \cdot \sin(90° - b) \quad \text{or} \quad \cos B = \sin C \cdot \cos b$$

4. Semiperimeter Formulas

$$p = \frac{1}{2}(a + b + c)$$

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(p-b)\cdot\sin(p-c)}{\sin p\cdot\sin(p-a)}} \qquad \text{etc.}$$

Assuming that

$$\tan\rho = \sqrt{\frac{\sin(p-a)\sin(p-b)\sin(p-c)}{\sin p}}$$

we get

$$\tan\frac{A}{2}=\frac{\tan\rho}{\sin(p-a)}\qquad \tan\frac{B}{2}=\frac{\tan\rho}{\sin(p-b)}\qquad \text{etc.}$$

5. Formulas for Solving an Elementary Spherical Triangle

(a) A right elementary spherical triangle.

(1) The difference between the hypotenuse and a side (Fig. 239)

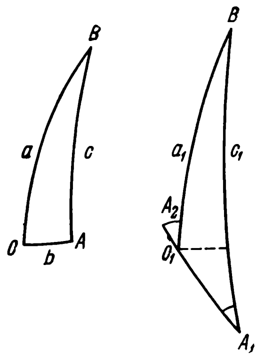


Fig. 239

$$(a-c)'\approx\frac{B^2}{4}\sin 2a\ \mathrm{arc}\ 1'\approx\frac{B^2}{2}\cdot\sin 2c\cdot\mathrm{arc}\ 1'$$

$$a-c<1'\ \text{for}\ B<2^\circ\ (117')$$

$$a-c<0'.1\ \text{for}\ B\leqslant 37'$$

(2) $a=c;\ b=B\cdot\sin a;\ 90^\circ-C=B\cdot\cos c=B\cdot\cos a$

(b) An oblique elementary spherical triangle.

An oblique triangle is solved by means of a right triangle and a small triangle.

$$c=a+b\cdot\cos A_1=a-b\cdot\cos C$$

$$B\cdot\sin a=b\cdot\sin A=b\cdot\sin C$$

$$A_2-A_1=90^\circ-C=B\cdot\cos a$$

(c) A small spherical triangle.

To a first approximation, a small triangle is solved by the formulas of plane trigonometry.

6. An Auxiliary Sphere

When solving problems that deal with finding angles between directions and planes in space, it is very convenient to use a sphere of arbitrary radius, through the centre of which the given and sought directions and planes are drawn. Then:

(a) A bundle of parallel lines in space corresponds to a point on the sphere.

(b) A system of parallel planes in space corresponds to one great circle on a sphere.

(c) A bundle of straight lines forming in space the surface of a circular cone corresponds to a small circle on a sphere.

(d) An angle between two lines or planes in space corresponds to an arc of a great circle on a sphere.

Hence, a consideration of the relations between directions and planes in space reduces to the solution of spherical triangles on the surface of a sphere, which is called an auxiliary sphere. The auxiliary sphere is widely employed in astronomy (celestial sphere), in the study of instruments, crystallography, and elsewhere.

7. Units Used in Measuring Arcs (Angles)

The arc measuring a given angle may be expressed in two systems of units: (a) in parts of the circle, and (b) in parts of the radius of the circle (sphere).

In the former case, the arc (and angle) are expressed in degrees or hours; 1° = 1/360 part of the circle, 1h = 1/24 part of the circle; in the latter case, the arc is expressed **in radians**. The radian (ρ) is an arc equal to the radius R of the circle. Radian measure is also called circular measure.

Conversion from degrees to radians and vice versa is done on the basis of the following relations:

The length of a circumference in degrees is 360°.

The length of a circumference in radians is

$$\frac{2\pi R}{R}=2\pi. \text{ Hence,}$$

$$1 \text{ radian}=360°;\ 2\pi=57°17'44''.8=3{,}437'.7468=206.264''.8$$

$$1°=\frac{2\pi}{360°} \text{ radian} \approx \frac{1}{57°.3} \text{ radian}$$

To indicate that an angle of so many degrees (or minutes, seconds) is expressed in radians, we use the designation arc $\alpha°$; the notation, therefore, "arc $\alpha°=$" is equivalent to the expression "an arc of $\alpha°$ is expressed in radian measure and is equal to...". In this notation we have

$$\text{arc } 1°=\frac{1}{57.3}=0.01745 \text{ (radian)}$$

$$\text{arc } 1'=\frac{1}{3{,}437.74} \approx \frac{1}{3{,}438}$$

$$\text{arc } 1''=\frac{1}{206{,}264.8} \approx \frac{1}{206{,}265}$$

To convert to radian measure, an arc expressed in the same arc units °, ′, ″ is multiplied by the factors arc 1°, arc 1′, arc 1″. For example,

$$\alpha=37°;\ \text{arc } 37°=37°\cdot\text{arc } 1°=0.6458$$

$$\alpha=2°.15'=135';\ \text{arc } 135'=135' \text{ arc } 1'=0.0393$$

In the general form

$$\text{arc } x°=x°\cdot\text{arc } 1°$$

$$\text{arc } x'=x'\cdot\text{arc } 1', \text{ etc.}$$

These expressions are constantly used in nautical astronomy. Numerical conversion may be performed by using Table 38, MT-63.

APPENDIX IV

ESSENTIAL COMPUTATIONAL TECHNIQUES

1. Computing Arcs from Most Favourable Trigonometric Functions

When computing with five-place tables of logarithms, an error Δx in the selected angle for an error in the logarithm of one unit in the fifth decimal place yields

$$\Delta x' = 0.08 \tan x \ \ldots \text{ when computing from } \log \sin x$$

$$\Delta x' = -0.08 \cot x \ \ldots \text{ when computing from } \log \cos x$$

$$\Delta x' = -0.04 \sin 2x \ \ldots \text{ when computing from } \log \tan x$$

$$\Delta x' = -0.08 \tan \frac{x}{2} \ \ldots \text{ when computing from } \log \sin^2 \frac{x}{2}$$

These formulas were used to compile Table 1 of the magnitudes of errors Δx.

Table 1

x	When computnig x from the functions			
	log sin x or log cosec x	log cos x or log sec x	log tan x or log cot x	$\log \sin^2 \frac{x}{2}$
0°	0′.000	∞	0′.000	0′.000
20	0 .029	0′.220	0 .026	0 .014
40	0 .067	0 .095	0 .038	0 .029
60	0 .137	0 .046	0 .035	0 .046
80	0 .449	0 .016	0 .014	0 .066
90	∞	0 .000	0 .000	0 .080

From Table 1 and the formulas it is evident that

(a) angles up to 45° are best computed from $\log \sin^2 \frac{x}{2}$;

(b) $\log \tan x$ is the most favourable for any angle;

(c) angles close to 0° (180°) are best computed from $\log \sin x$, while angles close to 90°, by $\log \cos x$. It is impossible to compute angles close to 90° from $\log \sin x$ and angles close to 0° (180°) from $\log \cos x$.

2. Trigonometric Functions of Small Angles

When considering elementary triangles and in other cases, we frequently encounter the trigonometric functions of small angles. In such cases, we can simplify work and the formulas by replacing the function by its expansion in a Taylor's series. Applying the Taylor's series in the form

$$f'(x)=f(0)+\frac{f'(0)}{1}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^n(0)}{n!}x^n+\ldots$$

to trigonometric functions, we get

$$\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\ldots$$

If we express x in minutes of arc in place of radians, as in the above formulas, it is necessary to multiply the series by the conversion factor arc 1′ in order to retain the dimensions of both sides of the series:

$$\sin x'=x' \operatorname{arc} 1'-\frac{(x')^3}{3!}\operatorname{arc}^3 1'+\frac{(x')^5}{5!}\operatorname{arc}^5 1'-\ldots$$

$$\cos x'=1-\frac{(x')^2}{2}\operatorname{arc}^2 1'+\frac{(x')^4}{24}\operatorname{arc}^4 1'-\ldots$$

$$\tan x'=x' \operatorname{arc} 1'+\frac{1}{3}(x')^3\operatorname{arc}^3 1'+\frac{2}{15}(x')^5\operatorname{arc}^5 1'+\ldots$$

Replacement of a function by a certain number of terms of a series depends on the magnitude of the angle x and the accuracy required in the result. The magnitude of the error due to substitution by one or several terms of a series is equal to the sum of the suppressed terms of the series or, approximately, to the largest term. The following is a table of the limits for substitution by one or two terms of a series to within 1′ and 0′.1.

Substituted function	Substitution	Limiting angles for accuracy: 1′.0	0′.1
$\sin x$	$x' \operatorname{arc} 1'$	7°	3°
	$x' \operatorname{arc} 1'-\frac{x^3}{6}\operatorname{arc}^3 1'$	29	13 .5
$\cos x$	1	1°.5	0°.5
	$1-\frac{x^2}{2}\operatorname{arc}^2 1'$	16 .5	7 .5
$\tan x$	$x'\cdot\operatorname{arc} 1'$	5°.5	2°.5

3. On Linear Interpolation and Extrapolation

Interpolation is the finding of values of a function that corresponds to intermediate values of the argument (independent variable).

If a function on a given interval varies in proportion to the variation of the argument (linear function), then interpolation is termed *linear* or *simple* and is handled in the form of a proportion. The linear character of a function becomes evident after the formation of its first differences Δ (or "tabular differences"), which in this case remain constant or, to be more precise, do not change more than a few units. In nautical astronomy linear interpolation is used exclusively.

For example, we have the table

Argument	Function	First differences
x_1 $\Big\}\ \Delta x$	y_1 $\Big\}\ \Delta y$	
x	y	
x_2	y_2	$\Delta_1 = y_2 - y_1$
x_3	y_3	$\Delta_2 = y_3 - y_2$
x_4	y_4	$\Delta_3 = y_4 - y_3$

Forming the first differences Δ, we see that

$$\Delta_1 \approx \Delta_2 \approx \Delta_3 \approx \ldots$$

If the value of the argument x is given, then the value of the function y is found from the formula for linear interpolation

$$y = y_1 + \Delta y = y_1 + \frac{x - x_1}{x_2 - x_1} \Delta_1$$

The value of the term Δy is ordinarily found from the proportion

$$(x_2 - x_1) \ \ldots \ \Delta_1$$
$$(x - x_1) \ \ldots \ \Delta y$$

whence

$$\Delta y = \frac{(x - x_1)}{(x_2 - x_1)} \Delta_1$$

If it is required to find from tables the value of the argument x from the value of the function y, the problem is solved by reverse entry and the value of the increment of the argument, Δx, is determined from the proportion

$$\Delta_1 \ \ldots \ (x_2 - x_1)$$
$$(y - y_1) \ \ldots \ \Delta x$$

whence

$$\Delta x = \frac{y - y_1}{\Delta_1} (x_2 - x_1) .$$

and

$$x = x_1 + \Delta x$$

It is important to develop skill in rapid mental solution of these proportions because interpolation is required in practically every problem in navigation and astronomy.

Extrapolation is the finding of a value of a function for an argument (independent variable) that goes beyond the limits of the given table. In nautical astronomy, this problem occurs when determining chronometer corrections for the instant of observations and in certain other problems. Values obtained by linear extrapolation should be treated with caution.

APPENDIX V

ESSENTIALS OF ERROR THEORY

Terminology and the Classification of Errors

An **error** of measurement is the difference between the approximate and true values (a_i and a, respectively) of the quantity being measured

$$a_i - a = \delta_i$$

where δ is the error of measurement. An error is characterized by magnitude and sign.

In certain cases, to eliminate an error from the result of a measurement, a correction Δ is introduced, which is the same as the error but with sign reversed

$$\Delta_i = a - a_i = (-\delta_i)$$

Errors are associated with every measurement (independently of the observer) due to imperfect methods, instruments, human sense organs, and so forth.

A **blunder** is an incorrect observation or slip in computation committed by an observer due to lack of attention. Blunders are the greatest danger in navigation.

Systematic errors are those of which both the nature and origin are known. If the magnitude and sign of an error remain unchanged throughout a series of observations, the error is termed recurring; the regularity of its variation may not be known. In nautical astronomy, systematic errors are usually understood to be recurring errors. Systematic (recurring) errors are eliminated either by the introduction of corrections or by observational methods.

Random errors are those brought about by multifarious and contradictory causes. Their magnitudes and signs may change in every observation.

Let us consider certain rules for evaluating and reducing the effects of random errors.

All the formulas and methods given below are those obtained for the normal law of distribution of random errors (Gauss' law).

(a) The most suitable value of a measured quantity is the arithmetic mean a_0 of the individual measurements a_i

$$a_0 = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \tag{1}$$

(b) To evaluate the accuracy of an observational result, apply the **mean square** error ε which is a conventional quantity obtained from the formula

$$\varepsilon = \pm \sqrt{\frac{\Sigma \delta_i^2}{n}} \tag{2}$$

where $\delta_i = a_i - a$ are the random errors of separate measurements
n is the number of measurements
a is the true value of the measured quantity.

(c) The arithmetic mean of a number of observations has a mean error ε_0 less than each separate measurement by the factor $\sqrt{n}$, or

$$\varepsilon_0 = \pm \frac{\varepsilon}{\sqrt{n}} \tag{3}$$

(d) The mean square error of a series of observations may be found (when the true value a is not known) from the formula

$$\varepsilon = \pm \sqrt{\frac{\Sigma v_i^2}{n-1}} \tag{4}$$

where $v_i = a_i - a_0$
a_i are the separate values of the measured quantity
a_0 is their arithmetic mean
n is the number of measurements.

(e) The mean error of different derivations. Suppose we are given an expression $z = x \pm y$. If the quantities x and y are obtained from independent observations with errors ε_x and ε_y, the error ε_z in their sum (difference) will be

$$\varepsilon_z = \pm \sqrt{\varepsilon_x^2 + \varepsilon_y^2} \tag{5}$$

Similarly, if $z = x \pm y \pm u \pm t \ldots$, then

$$\varepsilon_z = \pm \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_u^2 + \varepsilon_t^2 + \ldots} \tag{6}$$

Here, if $\varepsilon_x = \varepsilon_y = \varepsilon_u = \varepsilon_t = \varepsilon$, then

$$\varepsilon_z = \pm \varepsilon \sqrt{S} \tag{7}$$

where S is the number of terms.

If $z = k \cdot x$, then

$$\varepsilon_z = k \cdot \varepsilon_x \tag{8}$$

If $y = f(x)$, then

$$\varepsilon_y = f'(x_0) \cdot \varepsilon_x \tag{9}$$

where x_0 is a certain finite value of x obtained from measurements.

APPENDIX VI

APPROXIMATE METHODS OF ORIENTATION

1. Approximate Methods of Measuring Angles

Natural approximate units are the moon, whose diameter is about 0°.5, and Ursa Major, the distance between the separate stars of which is shown in Fig. 53.

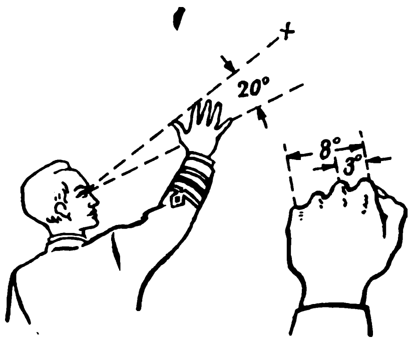


Fig 240

The angular distance between the end knuckles of one's extended hand (Fig. 240) is about 8° (between adjacent knuckles it is about 3°). The distance between the wide-open fingers of one's extended hand as shown in Fig. 240 is about 20°.

2. Approximate Determination of Directions

In the daytime, south can be found approximately by means of the sun and a watch. Hold the watch approximately in the plane of the equator and point the hour hand towards the sun. Dividing in half the angle between this hand and the number 12 if the watch is set to zone time, or one, if set to legal time, we get an approximate southerly direction.

At night, north may be found from the Pole Star.

If one has made a preliminary determination by a star globe of the azimuths of rising (setting) of individual bright stars for a given latitude, it is possible to find one's position from the places of rising (or setting).

3. Approximate Determination of Latitude

In the daytime, latitude may be approximately determined at noon by the sun on the basis of the formula $\varphi = 90° - H_{\odot} \pm \delta_{\odot}$, where $H_{\odot}$ should be at least approximately measured and $\delta_{\odot}$ approximately computed. At night, latitude is readily determined roughly by the altitude of the Pole Star: $\varphi \approx h_{Pol}$.

4. Approximate Determination of Time

In the daytime, the time is approximately found from the hour angle of the sun, for which one must know where south is.

At night, time is found from the position of Ursa Major relative to the meridian. Describe mentally a circle about the Pole Star and graduate it into hours

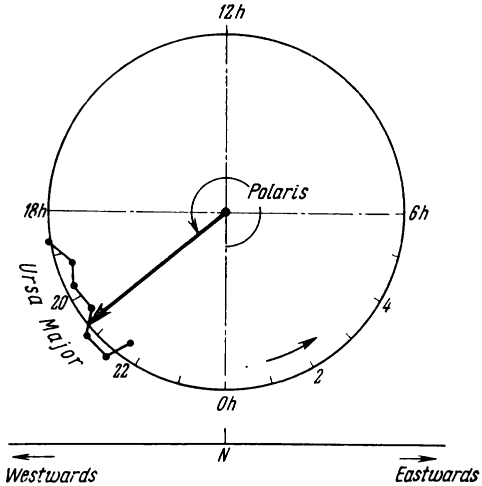


Fig. 241

as shown in Fig. 241, the hour hand being in the direction Polaris—γ-δ Ursae Majoris (the inner part of the ladle). This hand will indicate the number of hours of local sidereal time S_{loc}. A distance of $15° = 1h$ is approximately equal to 1.5-2 fists, as shown in Fig. 240.

In Fig. 241, $S_{loc} \approx 20h.5$. We calculate $\alpha_{\oplus}$ approximately. Suppose the date is August 22. Reckoning from 23.09, we have $\alpha_{\oplus} = 12h - 32 \cdot 4m = 9h.9$; we get $t_{\oplus} = S_{loc} - \alpha_{\oplus} = 10h.6$ and $T_{loc} = 12h + t_{\oplus} = 22h.6$. Introducing a correction for the longitude, and, if needed, for the legal hour, we get T_{sh}. If, for example, $\lambda_E = 39°$, then $\Delta T = 21° \cdot 4$ min/° $= 1h.4$ (ZD$=4E$) and $T_{sh(leg)} \approx$ $\approx 22h.6 + 1h.4 = 0h$.

CHART OF CONSTANT EPHEMERIDES OF THE SUN ($\delta_\odot$, η, $\alpha_\oplus$)

The chart in Fig. 242 is constructed for the second year after the leap year (1966) and makes it possible to obtain the coordinates $\delta_\odot$, η, $\alpha_\oplus$ on the Greenwich meridian from the arguments: Greenwich date and T_{gr}. To find $\delta_\odot$ and η, locate the date, count the number of hours between dates at a glance and indicate a point on the curve. Using dividers, find the distance from the point to the horizontal line; on the right lateral frame we get the declination (up to $\pm 0°.1$-$0°.2$); the distance from the point to the vertical line yields the equation of time on the bottom frame (up to $\pm$5-10s). The magnitude of right ascension of the mean sun $\alpha_\oplus$ is taken from the curve opposite the value of T_{gr} (up to $\pm 0°.2$-$0°.5$); lowest accuracy is obtained from May to August). Coordinates obtained from this chart may be used for determining a compass correction and when working with a globe.

1. Obtaining $\delta_\odot$ and $t^\odot_{gr}$.

To obtain $t^\odot_{gr}$, use formula $t^\odot_{gr} = T_{gr} \pm 12h - \eta$.

Example 1. 12 October, 1966, $T_{gr} = 17h\ 45m\ 40s$.
From the chart we get: $\delta_\odot = 7°.4S$, $\eta = -13m\ 30s$.

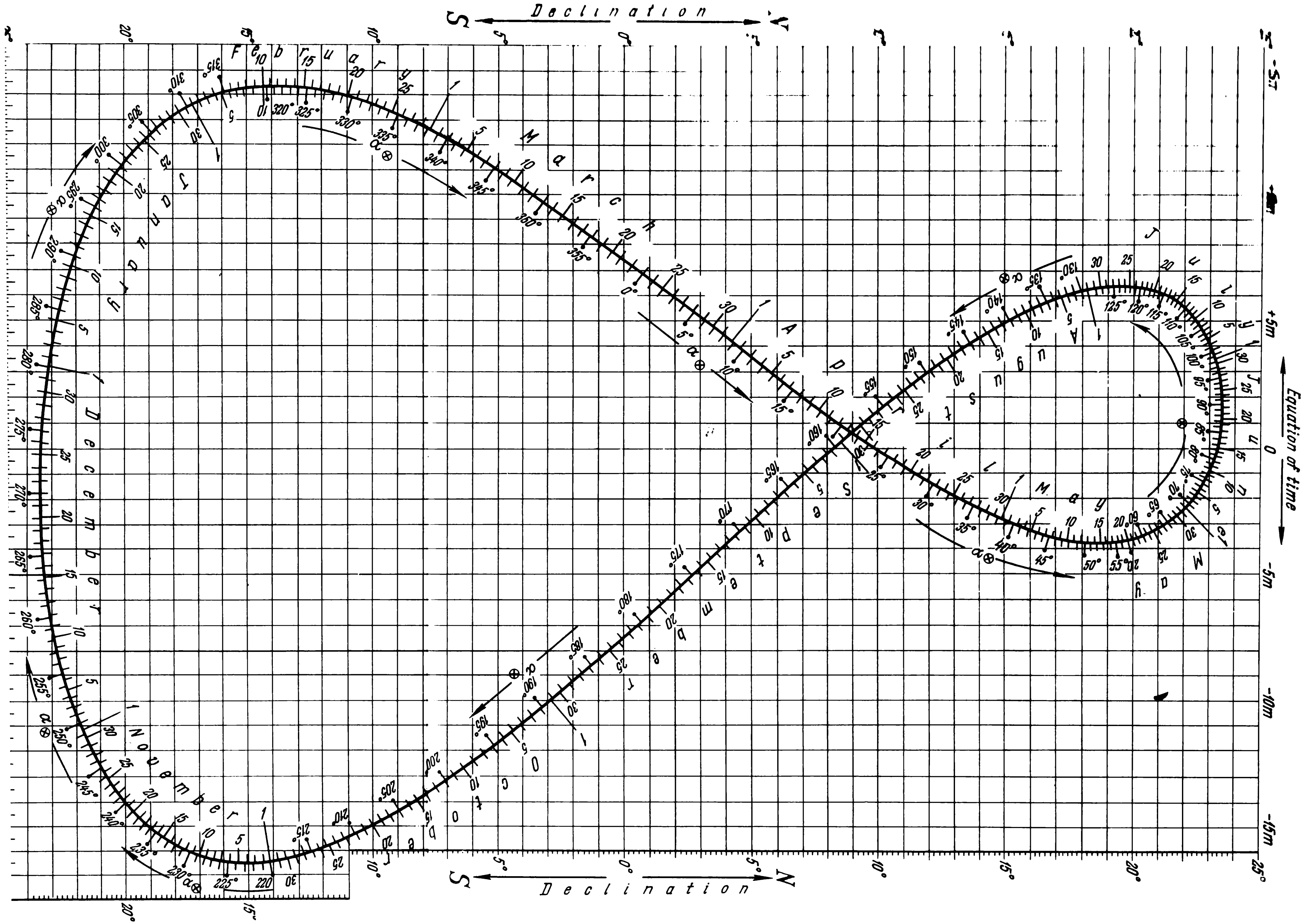
$-$ T_{gr}	17h 45m 40s
12	12
$-$ $t^\oplus_{gr}$	5h 45m 40s
η	+13 30
$t^\odot_{gr}$	5h 59m 10s = 89°48′W

From the MAE, $\delta_\odot = 7°24'.2S$; $t^\odot_{gr} = 89°46'.6$.

2. Obtaining $S_{gr} = t^\Upsilon_{gr}$.

To obtain S_{gr}, use the formula $S_{gr} = T_{gr} \pm 12h + \alpha_\oplus$

Example 2. 13 August, 1966, $T_{gr} = 6h\ 21m$.
From the chart: $\alpha_\oplus = 141°.4$.


S Declination N
Equation of time
+5m
0
-5m
-10m
-15m
January
February
March
April
May
June
July
August
September
October
November
December

T_{gr}	6h 12m
12	12
$t_{gr}^{\oplus}$	18h 21m $= 275°.2$
$\alpha_{\oplus}$	141 .4
S_{gr}	56°.6 (From MAE $S_{gr} = 56°30'.1$)

When using the chart for any other year, before entering the chart introduce a correction of the date into T_{gr} from Table 6, Ch. 9, taking the "compilation year" as the second after the leap year (second column from the right) and picking out the "year of use" on the left. For example, for 1969 the date correction will be $\Delta d = +6\text{h}\ 34\text{m}$, for 1971 $\Delta d = -5\text{h}\ 49\text{m}$. This correction is added to T_{gr} only for entering the chart and does not participate in subsequent computations. For rough computations, no corrections need be introduced.

APPENDIX VIII

ANSWERS TO PROBLEMS

Chapter 2

2. $h=30°$
   $A=25°$NW
3. $\delta=45°$N
   $t=75°$E
4. $h=30°$
   $\delta=10°$S
5. $\delta=10°$N
   $A=65°$SW
6. $A=60°$NE
   $t=55°$E
7. $h=25°$
   $t=65°$W
8. $t=65°$W
   $A=60°$SW
9. $t=30°$E
   $h=60°$
   $A=60°$SE
10. $\delta=35°$N
    $t=85°$W
    $t^{\Upsilon}=100°$
11. $\delta=35°$S
    $t=50°$E
    $\tau=70°$
12. $h=-25°$
    $A=65°$NE
    $\alpha=150°$
15. $h=11°33'.4$
    $A=9°13'.3$NW
16. $h=-12°2'.4$
    $A=17°15'.6$SE
17. $t=76°56'.0$W
    $A=75°6'.8$NW
18. $t=13°47'.5$E
    $\delta=15°50'.5$S
19. $h=11°16'.3$
    $\delta=21°8'.2$N

Chapter 3

I

	a	b	c	d	e
1	Yes, 70°NE (NW)	Yes $h=25°$ $t=70°.0$ (W)	No	—	$H=62°$S
2	Yes, 75°SE (SW)	No	No	—	$H=36°$S
3	Does not set ($\delta_N > 90° - \varphi_N$)	No	No	—	$H=71°$N
4	Does not rise ($\delta_S > 90° - \varphi_S$)	No	No	$h_{elong}=50°$ $A_{elong}=40°$NE (NW)	—

II					
1	Yes, 70°SE (SW)	No	Yes ($\delta_s = \varphi_s$)	—	$H = 90°$
2	Yes, 80°SE (SW)	Yes $h = 25°$ $t = 65°$	No	—	$H = 78°.3$N
3	Yes, 70°NE (NW)	No	No	—	$H = 50°.6$N
4	Yes, 30°SE (SW)	No	No	$h_{elong} = 25°$ $A_{elong} = 40°$SE (SW)	$H = 57°.3$S
III					
1	Nonrising ($\delta_S \geqslant 90° - \varphi_N$	No	No	—	$H = 0°$
2	Nonsetting ($\delta_N >$ $> 90° - \varphi_N$	Tangent	Yes	—	$H = 90°$
3	ditto	No	No	$h_{elong} = 80°$ $A_{elong} = 70°$NE (NW)	$H = 87°$N
4	ditto	Yes $h = 50°$ $t = 60°$	No	—	$H = 72°.7$S

(7) $t = 61°43'.9$E
$A = 42°35'.6$NE

(8) $t = 102°35'.8$W
$A = 75°31'.5$NW

(9) $h = 22°5'.7$
$t = 76°59'.6$E

(10) $h = -40°21'.9$
$t = 130°17'.1$W

Chapter 4

8. Intermediate solution: $\delta_{\odot} \approx 15°$S; $H = 75°$ to S; $A = 75°$SE.
9. Intermediate solution: $\delta_{\odot} \approx 21°$N; $H = 69°$ to N; $A = 69°$NW.
10. 6.04 and 8.09.
11. 3.04 and 10.09.
12. Will not pass through zenith, nor will it cross prime vertical.
13. From 1.05 to 14.08 and from 3.11 to 9.02.
14. Polar day from 22.05 to 24.07; polar night from 20.11 to 23.01.
15. $\varphi = 77°5$S.
16. $\varphi = 67°.5$N and S.
17. $\alpha_{\odot} = 34°$.

Arcturus and Spica to the south, the constellation Gemini to the west, and the constellations Aquila and Lyra to the east.

18. $\alpha_{\odot}=295°$; $\alpha_{*}=115°$; Procyon; α and δ Geminorum.

19. α Aquilae; $\delta_{*}=8°.8$N; $H_{*}=58°.8$ to S.

Chapter 8

(5) $S=0$h (24h); $t_{*}=130°33'$W. (6) $S=12$H; $t_{*}=264°8'.2$W. (7) $S==21$h 12m 38s. (8) S=10h 51m 32s. (9) $S=8$h 28m. (10) $t_{\odot}=77°5$; $t_{☾}==305°.5$. (11) $S=18$h 40m; $t_{*}=167°$; $t_{\odot}=84°$. (13) $T=2$h 2m.5 19.01. (14) $t_{\odot}==28°15'.8$. (15) $T=11$h 44m. (16) $T_{UT}=12$h 14m; $T_{LT}=0$h 14m 16.02. (19) $T=22$h 36m 8.02. (20) $S=21$h 20m. (21) $S=5$h 38m 47s. (28) $T_{leg}==10$h 7m 26s 5.04. (29) $T_{loc}=10$h 52m 11s 14.05. (30) $T_z=2$h 33m 15s 11.06. (31) $T_{gr}=2$h 35m 3.09. (32) $T_{loc}=4$h 56m 10s 8.07. (33) $T_{loc}=19$h 2m 25.11; $T_{gr}=0$h 9m 26.11. (34) $t^{*}_{loc}=70°$; $\lambda=121°$E. (35) $S=16$h 47m.

NAME INDEX

SUBJECT INDEX

B

C

D

N

O

P

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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